

Zero Surgery Obstructs Persistent Free Subgroups of the Figure-Eight Knot Group

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20 October 2026

Abstract

The figure-eight knot is hyperbolic and has no finite rational surgery, but no rank-two free subgroup of its knot group injects under every rational Dehn filling. Indeed, the zero-filling kernel is the second derived subgroup, and the quotient is metabelian. A nontrivial length-eighteen word witnesses the obstruction in every free subgroup. This elementary consequence of classical surgery facts answers the universal question in Oberwolfach Report 8/2020 negatively. Existence of persistent nonabelian free subgroups for other hyperbolic knots remains unaddressed.

1 The question and its negative answer

Let $K \subset S^3$ be a nontrivial knot, let $E(K)$ be its exterior, and write $G(K) = \pi_1(E(K))$. Fix a meridian–preferred-longitude pair (μ, λ) . For a rational slope $r = p/q$, with p, q coprime and $q \neq 0$, put

$$N_r = \langle\langle \mu^p \lambda^q \rangle\rangle_{G(K)}.$$

Dehn filling gives an epimorphism

$$\pi_r : G(K) \longrightarrow \pi_1(K(r)), \quad \ker \pi_r = N_r.$$

Here $0 = 0/1$ is the longitude slope. The meridian slope $\infty = 1/0$ is not among the rational slopes.

The final question in Motegi’s report, joint with Ito and Teragaito [5, p. 493], asks whether every hyperbolic knot with no finite surgery admits a subgroup $H \cong F_2$ satisfying

$$H \cap \bigcup_{r \in \mathbb{Q}} N_r = \{1\}. \tag{1}$$

Equivalently, every restriction $\pi_r|_H$ must be injective. Such a subgroup is called *persistent* in [3, Section 7.1].

For a group J , write $J' = [J, J]$ and $J'' = [J', J']$, using the convention $[a, b] = aba^{-1}b^{-1}$.

Theorem 1. *Let $K = 4_1$ be the figure-eight knot and $G = G(K)$. Then K is hyperbolic and admits no rational finite surgery, but it has no rank-two free subgroup satisfying (1). In fact,*

$$N_0 = G'', \quad G/N_0 \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and every rank-two free subgroup $H \leq G$ satisfies

$$\{1\} \neq H'' \leq H \cap N_0.$$

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The theorem is a counterexample to the universal formulation. It is not a claim that every hyperbolic knot fails to have a persistent F_2 . In particular, it leaves open the search for hyperbolic examples discussed around [3, Question 7.1]. It also does not address a modified condition that omits zero surgery.

2 The figure-eight filling kernel

We recall the standard inputs with their sources. The figure-eight complement is a once-punctured-torus bundle, with fiber group $F = \langle x, y \rangle$ and monodromy

$$\phi(x) = xy, \quad \phi(y) = yxy. \quad (2)$$

Its knot group has the presentation

$$G = \langle x, y, t \mid txt^{-1} = xy, tyt^{-1} = yxy \rangle = F \rtimes_{\phi} \mathbb{Z}. \quad (3)$$

Morifuji records this presentation and its homology matrix A [4, p. 208]. Boyer–Luft–Zhang identify the longitude with $[x, y]$ [2, pp. 682, 686]. Orientation or conjugacy choices do not affect its normal closure. The matrix has determinant one and trace three, so this punctured-torus bundle is hyperbolic.

Lemma 2. *The figure-eight knot has no rational finite surgery.*

Proof. The knot has genus one. By Boyer–Gordon–Zhang [1, Theorem 1.5(2)], every exceptional nonmeridional filling of a hyperbolic genus-one knot is either toroidal or small Seifert with hyperbolic base orbifold. In the toroidal case its fundamental group contains $\mathbb{Z}^2$, coming from an incompressible torus. In the Seifert case it surjects onto the infinite fundamental group of the hyperbolic base orbifold. Nonexceptional fillings are hyperbolic and likewise have infinite fundamental group. Thus no rational filling group is finite. The same standard fact about the figure-eight is explicitly recalled in the proof of [3, Lemma 2.4]. $\square$

Proposition 3. *For the presentation (3), $G' = F$ and $N_0 = F' = G''$. The quotient by N_0 is $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$.*

Proof. Abelianizing the first relation in (3) forces $y = 0$; the second then forces $x = 0$. There is no relation on t . Consequently $G_{\text{ab}} \cong \mathbb{Z}$, and its abelianization map is the bundle projection, whose kernel is F . Hence $G' = F$.

Since F is free on x, y , the quotient $F/\langle\langle [x, y] \rangle\rangle_F$ is free abelian of rank two. Therefore

$$\langle\langle [x, y] \rangle\rangle_F = F'.$$

The subgroup F' is characteristic in F and F is normal in G , so F' is normal in G . It contains the longitude; thus $N_0 = \langle\langle [x, y] \rangle\rangle_G \leq F'$. Conversely, the normal closure in G contains the normal closure in F , proving $N_0 = F' = G''$.

Quotienting by F' abelianizes the fiber in (3). The induced action of t on the column-coordinate basis (x, y) is A , which gives the stated semidirect product. $\square$

In particular, the zero-filling group is metabelian: it has an abelian normal subgroup $\mathbb{Z}^2$ and cyclic quotient. Geometrically, zero surgery caps the puncture of each fiber, producing the corresponding torus bundle. The calculation above also identifies the kernel exactly.

3 A word killed in every free subgroup

Lemma 4. *In the free group on u, v , the word*

$$w(u, v) = [[u, v], [u, v^2]]$$

is nonidentity and belongs to the second derived subgroup. It evaluates to the identity under every substitution in any metabelian group.

Proof. Both inner commutators belong to the derived subgroup, so their commutator belongs to the second derived subgroup. For nontriviality, write $X = x^{-1}$ and $Y = y^{-1}$. Free reduction gives

$$w(x, y) = \mathbf{xyXYxyYxYxYxyxYYX}.$$

This word has eighteen letters and no adjacent inverse pair, so it is nonempty in the free group. Replacing x, y by a free basis u, v preserves nontriviality. In a metabelian group the derived subgroup is abelian, so the two inner commutators commute and w evaluates to 1. $\square$

Proof of Theorem 1. The bundle description gives hyperbolicity, and Lemma 2 gives the no-finite-surgery hypothesis. Proposition 3 identifies the zero-filling kernel. For every $H \leq G$, inclusion respects derived subgroups, so $H'' \leq G'' = N_0$. If H is free of rank two, Lemma 4 supplies a nonidentity element of H'' . Since $0 \in \mathbb{Q}$, this contradicts (1). $\square$

More generally, if one quotient of a group satisfies a nontrivial group law, its quotient map cannot be injective on a free subgroup on the variables of that law. The single exceptional quotient at slope zero is therefore enough here. No classification of all free subgroups of G is required.

This distinction is compatible with the existence of individual persistent elements and of cyclic persistent subgroups. The word obstruction uses noncommuting elements in a free subgroup's derived subgroup; it gives no corresponding contradiction for a cyclic group.

4 Reproducibility and attribution

The supplementary standard-library Python script checks the displayed free word with two reduction procedures and a separate exact matrix substitution. It also checks the inverse monodromy on 1,457 reduced words, compares the reducers on all 5,461 words of length at most six, and tests the metabelian identity on 2,025 pairs in $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$. Normal and optimized Python runs agree. These finite diagnostics supplement the general proof; they do not establish the topological inputs or constitute proof-assistant verification.

The knot presentation, longitude and surgery facts used here are classical and credited above. This short note records their elementary consequence for the universal question and makes no absolute priority claim. AI tools assisted the author with source research, algebraic checks and drafting. The manuscript is self-audited and unrefereed; no independent expert review is claimed.

References

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