

The Asymptotic Constant for Optimal Integration of C^k Functions on Bounded Lipschitz Domains: Existence and Dependence on the Volume Only

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Abstract

For a bounded open set $D \subset \mathbb{R}^d$ let $e_n(C^k(D), \text{INT})$ be the minimal worst-case error of algorithms that use n function values to integrate, over D , the functions all of whose partial derivatives up to order k are bounded by 1. In Oberwolfach Report 6/2020, Novak asked whether the asymptotic constant $\limsup_n n^{k/d} e_n(C^k(D), \text{INT})$ depends on D , for bounded Lipschitz domains of volume 1, and whether it is always a limit. We show that for all $k, d \geq 1$ there is a constant $c_{k,d} \in (0, \infty)$, defined on the unit cube by the derivatives of order k alone, such that $\lim_n n^{k/d} e_n(C^k(D), \text{INT}) = c_{k,d} \lambda(D)^{1+k/d}$ for every bounded open set D with the cone condition, in particular for every bounded Lipschitz domain. So the limit exists, at volume 1 it is the same number $c_{k,d}$ for all such D , and the bounds on the derivatives of order less than k do not change it. For every bounded open set, without any regularity, the lower limit is at least $c_{k,d} \lambda(D)^{1+k/d}$. For $d \geq 2$ a hypothesis on the boundary is needed: there are bounded, connected, Jordan measurable “comb” domains of volume 1 on which $n^{k/d} e_n(C^k(D), \text{INT}) \rightarrow \infty$. The proof adapts the scheme of Zador and of Graf and Luschgy from quantization theory: self-similarity on the cube, a cutoff argument that identifies the constants with and without boundary conditions, and a boundary layer that is controlled through the cone condition by local polynomial reproduction. For the Sobolev classes L_p^m with $1 < p < \infty$ the corresponding asymptotics were proved by Polovinkin (1975, 1989); the case $p = \infty$ with the full C^k norm treated here is not covered there. The value of $c_{k,d}$ and its dependence on d are not determined. This is an unrefereed note.

1 Introduction

1.1 The question

In his contribution to Oberwolfach Report 6/2020 [Nov20b], an extended abstract of joint work with Hinrichs, Krieg, Prochno, Ullrich and Woźniakowski, Novak considers integration $\text{INT}(f) = \int_{D_d} f d\lambda$ over a set $D_d \subset \mathbb{R}^d$, where $\lambda = \lambda_d$ is Lebesgue measure, by algorithms A_n that use n function values; the standing hypothesis is that D_d is a bounded Lipschitz domain with volume 1. The n -th minimal error of a class F is $e_n(F, \text{INT}) = \inf_{A_n} \sup_{f \in F} |\text{INT}(f) - A_n(f)|$, and the report recalls as a known fact that for the unit ball F of a norm

$$e_n(F, \text{INT}) = \inf_{x_1, \dots, x_n \in D_d} \sup \left\{ \int_{D_d} f d\lambda : f \in F, f(x_1) = \dots = f(x_n) = 0 \right\} \quad (1)$$

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(the theorem of Smolyak and Bakhvalov). The class $C^k(D_d)$ of the report consists of the C^k functions on D_d all of whose partial derivatives up to order k , the function itself included, are bounded by 1. Item (3) of the report asks whether the asymptotic constant $\limsup_{n \rightarrow \infty} e_n(C^k(D_d), \text{INT}) n^{k/d}$ depends on D_d , and continues: “Is it always a lim?” For $k = 1$ the report recalls the formula $\lim_n e_n(C^1(D_d), \text{INT}) n^{1/d} = \zeta_d \lambda_d(D_d) [d! \lambda_d(D_d)/2^d]^{1/d}$ with $\zeta_d \rightarrow 1$. In [Nov20a, Remark 7(2)] Novak guesses that the limit exists and does not depend on D_d when D_d is Jordan measurable with $\lambda_d(D_d) = 1$; in the survey [Nov24, Remark 7(2)] the same guess is stated with the addition “possibly under some assumptions on the boundary”. Items (1) and (2) of the report, on the dependence of the constants on the dimension, are different questions and are not treated here.

We use (1) as the definition of e_n , as the report does. For a nonempty bounded open set $D \subset \mathbb{R}^d$ we write $e_n(D)$ for $e_n(C^k(D), \text{INT})$, with nodes in D ; $k \geq 1$ and $d \geq 1$ are fixed throughout.

1.2 Results

Let $Q = (0, 1)^d$ be the open unit cube. For a finite set $P \subset Q$ let $r_S(Q, P)$ be the supremum of $\int_Q f$ over all $f \in C^k(Q)$ with $\sup_Q |D^\alpha f| \leq 1$ for $|\alpha| = k$ and $f = 0$ on P , and let $E_n = \inf\{r_S(Q, P) : P \subset Q, |P| \leq n\} \in [0, \infty]$. The precise definitions are in Section 2.

Theorem 1.1. *Let $k, d \geq 1$.*

(a) *The limit $c_{k,d} := \lim_{n \rightarrow \infty} n^{k/d} E_n$ exists, it equals $\inf_{n \geq 1} n^{k/d} E_n$, and $0 < c_{k,d} < \infty$. It is also the limit of $n^{k/d} E_n^0$, where E_n^0 is defined in the same way for the functions of class C^k on $\mathbb{R}^d$ that vanish outside Q (Proposition 3.3).*

(b) *For every nonempty bounded open set $D \subset \mathbb{R}^d$,*

$$\liminf_{n \rightarrow \infty} n^{k/d} e_n(C^k(D), \text{INT}) \geq c_{k,d} \lambda(D)^{1+k/d}.$$

(c) *If D is a nonempty bounded open set with the cone condition (Definition 2.1), in particular if D is a bounded Lipschitz domain, then*

$$\lim_{n \rightarrow \infty} n^{k/d} e_n(C^k(D), \text{INT}) = c_{k,d} \lambda(D)^{1+k/d}.$$

(d) *If $d \geq 2$, there is a bounded connected open set $D \subset \mathbb{R}^d$ with $\lambda(\partial D) = 0$ and $\lambda(D) = 1$ such that $n^{k/d} e_n(C^k(D), \text{INT}) \rightarrow \infty$.*

Under the standing hypothesis of the report (a bounded Lipschitz domain of volume 1) the answer to item (3) is therefore: the upper limit is a limit, and it does not depend on the domain; it equals $c_{k,d}$. For other volumes the constant depends on D only through the factor $\lambda(D)^{1+k/d}$, which is the form of the report’s formula for $k = 1$. The bounds $|D^\alpha f| \leq 1$ for $|\alpha| < k$ do not change the constant; it is the constant of the class defined by the derivatives of order k alone. Part (b) needs no regularity of D . Part (d), for $d \geq 2$, shows that some assumption on the boundary is needed for $C^k(D)$: Novak’s guess in [Nov20a, Remark 7(2)], stated for Jordan measurable sets, needs such an assumption, as the wording of [Nov24] anticipates. We do not know whether $\liminf$ and $\limsup$ can differ for a bounded open set without the cone condition, and the value of $c_{k,d}$ is not determined (Section 7).

Remark 1.2 (Readings of the class and of the nodes). The result does not depend on whether $C^k(D)$ is read as a class of functions in $C^k(D)$, in $C^k(\overline{D})$ or in $W^{k,\infty}(D)$ (with essential suprema), nor on whether the nodes may lie on ∂D (in the readings in which the functions are defined there). The lower bound (b) is obtained with restrictions to D of functions in $C^k(\mathbb{R}^d)$ all of whose derivatives up to order k are bounded by 1, and it ignores all nodes outside certain open grid cubes contained in

D ; the upper bound (c) uses only the bounds on the derivatives of order k and nodes in D , and it extends to $W^{k,\infty}$ by an approximation argument given at the end of Section 5. Every class between these two extremes, with nodes in D or in $\overline{D}$, therefore has the same asymptotic constant.

1.3 Relation to known results

The closest prior work we found is due to Polovinkin. In [Pol75] he considers the Sobolev-type spaces $L_p^m(\Omega)$ with $1 < p < \infty$ and $p > \max(n/m, 1)$, for bounded domains $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary, with the seminorm $(\int_{\Omega} (\sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^{\alpha} f|^2)^{p/2})^{1/p}$ and error functionals that vanish on polynomials of degree less than m ; he proves that the minimal norm $I(N, \Omega)$ of the error functional of an N -node cubature formula satisfies $I(N, \Omega) = N^{-m/n} (\text{mes } \Omega)^{1/q+m/n} B (1 + o(1))$ along some sequence $N \rightarrow \infty$, where $1/p + 1/q = 1$ and B is a constant defined on the periodic unit cube. In [Pol89, Theorem 2] he proves the asymptotics for all $N \rightarrow \infty$, with a weight $g \in L_q(\Omega)$, for $pm > n$ and a class of domains which contains every domain star-shaped with respect to a ball; the result is $N^{-m/n} T \|g\|_{L_r(\Omega)} (1 + o(1))$ with $r = qn/(qm + n)$, and the paper attributes the case of a constant weight to [Pol75] and to a note of 1978 [Pol78]. The exponent $1 + k/d$ of Theorem 1.1 is the value of $1/q + m/n$ at $p = \infty$, and Polovinkin's method (a constant of the cube and upper bounds for a boundary layer) is related to ours. His results do not settle the question of the report: (i) $p = \infty$ is excluded in both papers, and letting $p \rightarrow \infty$ in his asymptotics is not automatic (the limits in N and in p need not commute, and the norm changes); (ii) the class $C^k(D)$ is defined by the maximum over all multi-indices of the suprema of the partial derivatives, not by his Euclidean-type seminorm, and it includes the bounds of lower order, which we show do not change the constant; (iii) we obtain the limit for every bounded open set with the cone condition, hence for all bounded Lipschitz domains, and the lower bound for every bounded open set (the boundary conditions of [Pol89] are those of [Pol83], which we have not read); (iv) the comb shows that for $C^k(D)$ a hypothesis on the boundary cannot be dropped. Like [Pol89, Theorem 2] and unlike [Pol75], our limit is taken along all n . We regard Theorem 1.1(a)–(c) as the counterpart for $p = \infty$ and the full C^k norm, under the cone condition, of Polovinkin's theorem for L_p^m ; it is neither a new proof of a known theorem nor a new method.

For $k = 1$ the dependence on the volume is known: the formula of the report, and [Nov20a, Theorem 2], which rests on Chernaya [Che95b] and Gruber [Gru04]; a related paper of Chernaya [Che95a] gives weighted asymptotics for Hölder classes on Jordan measurable sets. These results concern the classes of functions with $|f(x) - f(y)| \leq \|x - y\|_B$ for all $x, y \in D$, for a norm $\|\cdot\|_B$ on $\mathbb{R}^d$, and they are stated for Jordan measurable sets. For the ℓ^1 norm and a convex D , the C^1 functions with $|\partial_i f| \leq 1$ belong to such a class, and every function of the class is a uniform limit of such C^1 functions (up to the bound $|f| \leq 1$); for a non-convex D the C^1 functions need not belong to the class. Weighted first-order results are due to Babenko, Babenko and Kovalenko [BBK21] and Kovalenko [Kov24]. For $C^1(D)$ our theorem recovers the dependence on the volume; it is new for $k \geq 2$ only in the sense of the list (i)–(iv) above. The lower bound $e_n(C^k(D)) \geq \min(1/2, c_k d n^{-k/d})$ of [HNUW17], uniform in d , and the weak order $n^{-k/d}$ on Lipschitz domains quoted in [Nov20a, Remark 7(2)] are consistent with Theorem 1.1, which says nothing uniform in d .

1.4 Method

The proof follows the scheme of Zador [Zad82] and of Graf and Luschgy [GL00] for the asymptotic quantization error, adapted to integration with higher smoothness. On the cube the class defined by the derivatives of order k is invariant under the scalings $f \mapsto h^{-k} f(a + h \cdot)$, which gives self-similarity and the existence of $c_{k,d}$ by a sandwich argument (Section 3). A cutoff lemma with a Leibniz estimate shows that the constants with and without boundary conditions coincide (Lemma 3.2); the functions with boundary conditions can be glued over disjoint cubes inside any open set, which gives the lower

bound (Section 4). For the upper bound the domain is filled with small cubes, and the layer near ∂D is controlled by a local estimate: a function of the class that vanishes on a principal lattice in a small cell is small on every convex set containing the cell, and the cone condition provides such convex sets (Section 5). The unisolvence of the principal lattice is classical [CY77, Nic72]. Section 6 contains the combs, Section 7 remarks on the constant, and Section 8 the checks of the finite ingredients.

2 Notation and three lemmas

For $x \in \mathbb{R}^d$ let $|x|$, $|x|_1$, $|x|_\infty$ be its Euclidean, ℓ^1 and ℓ^∞ norms. For a multi-index $\alpha \in \mathbb{Z}_{\geq 0}^d$ we write $|\alpha| = \alpha_1 + \dots + \alpha_d$, $\alpha! = \alpha_1! \dots \alpha_d!$ and $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$. $\mathcal{P}_{k-1}$ is the space of real polynomials on $\mathbb{R}^d$ of total degree at most $k-1$, of dimension $N_k = \binom{k-1+d}{d}$. For an open set $W \subset \mathbb{R}^d$ let

$$\begin{aligned}\mathcal{F}(W) &= \{f \in C^k(W) : \sup_W |D^\alpha f| \leq 1 \text{ for all } |\alpha| \leq k\}, \\ \mathcal{S}(W) &= \{f \in C^k(W) : \sup_W |D^\alpha f| \leq 1 \text{ for all } |\alpha| = k\}, \\ \mathcal{S}_0 &= \{g \in C^k(\mathbb{R}^d) : g = 0 \text{ on } \mathbb{R}^d \setminus Q, \sup |D^\alpha g| \leq 1 \text{ for all } |\alpha| = k\}.\end{aligned}$$

$\mathcal{F}(W)$ is the class $C^k(W)$ of the report, and $\mathcal{F}(W) \subset \mathcal{S}(W)$. For a finite set P we define, with values in $[0, \infty]$,

$$\begin{aligned}r_{\mathcal{F}}(W, P) &= \sup \left\{ \int_W f : f \in \mathcal{F}(W), f|_P = 0 \right\}, & r_{\mathcal{S}}(Q, P) &= \sup \left\{ \int_Q f : f \in \mathcal{S}(Q), f|_P = 0 \right\}, \\ r_0(P) &= \sup \left\{ \int_{\mathbb{R}^d} g : g \in \mathcal{S}_0, g|_P = 0 \right\},\end{aligned}$$

and

$$e_n(W) = \inf_{P \subset W, |P| \leq n} r_{\mathcal{F}}(W, P), \quad E_n = \inf_{P \subset Q, |P| \leq n} r_{\mathcal{S}}(Q, P), \quad E_n^0 = \inf_{P \subset Q, |P| \leq n} r_0(P).$$

By (1), $e_n(W) = e_n(C^k(W), \text{INT})$; a list of n nodes with repetitions is a set with at most n points. Functions in $\mathcal{S}(Q)$ are bounded on Q (Lemma 2.2 with x the centre of Q), so the integrals in $r_{\mathcal{S}}$ exist. Adding a point to P shrinks the set of admissible functions, so

$$E_{n+1} \leq E_n, \quad E_{n+1}^0 \leq E_n^0 \quad (n \geq 0). \quad (2)$$

The restriction to Q of a function in $\mathcal{S}_0$ lies in $\mathcal{S}(Q)$, so $r_0(P) \leq r_{\mathcal{S}}(Q, P)$ and $E_n^0 \leq E_n$.

Definition 2.1 (Cone condition). A nonempty open set $D \subset \mathbb{R}^d$ has the *cone condition* if there are $\rho > 0$ and $\theta \in (0, \pi/2]$ such that every $x \in D$ has a unit vector v_x with

$$C(x) := \{x\} \cup \{y \in \mathbb{R}^d : 0 < |y - x| < \rho, (y - x) \cdot v_x > |y - x| \cos \theta\} \subset D.$$

This is the cone condition of Adams and Fournier [AF03, 4.6], written out (a finite cone of height ρ and half-aperture θ with vertex x ; an aperture $\theta = \pi/2$ may be decreased). Every bounded Lipschitz domain has it: the strong local Lipschitz condition implies the uniform cone condition, which implies the cone condition; see [AF03, 4.6 and 4.11] and [Gri11, Theorem 1.2.2.2]. We use this standard fact without proof. The set $C(x)$ is convex, being the intersection of the ball $\{|y - x| < \rho\}$ with the convex set formed by x and the open circular cone with vertex x . For $d = 1$ every bounded open interval has the cone condition (a bounded open subset of $\mathbb{R}$ with infinitely many components does not).

Lemma 2.2 (Taylor's formula along a segment). *Let W be open, $f \in \mathcal{S}(W)$, and $x, y \in W$ such that the segment $[x, y]$ lies in W . Let T_x be the Taylor polynomial of f at x of degree $k-1$. Then for every multi-index β with $|\beta| = j \leq k-1$,*

$$|D^\beta f(y) - D^\beta T_x(y)| \leq \frac{|y-x|_1^{k-j}}{(k-j)!}.$$

Proof. $D^\beta T_x$ is the Taylor polynomial of $D^\beta f$ at x of degree $k-1-j$. Let $\gamma(s) = D^\beta f(x + s(y-x))$ for $s \in [0, 1]$. By the chain rule, with $m = k-j$, $\gamma^{(m)}(s) = \sum_{|c|=m} \frac{m!}{c!} (D^{\beta+c} f)(x + s(y-x)) (y-x)^c$, and since $|D^{\beta+c} f| \leq 1$ for $|\beta+c| = k$, the multinomial identity $\sum_{|c|=m} \frac{m!}{c!} |z_1|^{c_1} \cdots |z_d|^{c_d} = |z|_1^m$ gives $|\gamma^{(m)}| \leq |y-x|_1^m$. Taylor's formula for γ with the Lagrange remainder gives the claim. $\square$

Lemma 2.3 (The principal lattice). *Let $Z = \{i \in \mathbb{Z}_{\geq 0}^d : i_1 + \cdots + i_d \leq k-1\}$ and $U = \{(i + (1, \dots, 1))/(k+1) : i \in Z\}$, a set of N_k points in Q .*

- (a) *U is unisolvent for $\mathcal{P}_{k-1}$: a polynomial in $\mathcal{P}_{k-1}$ that vanishes on U is zero.*
- (b) *For $R \geq 1$ the number $\Lambda(R) = \max\{|D^\beta p(y)| : p \in \mathcal{P}_{k-1}, \max_U |p| \leq 1, |y|_\infty \leq R, |\beta| \leq k-1\}$ is finite.*
- (c) *Let $q = a + t[0, 1]^d$ with $t > 0$, $U_q = a + tU$, and $p \in \mathcal{P}_{k-1}$ with $|p| \leq \varepsilon$ on U_q . Then $|D^\beta p(y)| \leq \Lambda(R) \varepsilon t^{-|\beta|}$ whenever $|y-a|_\infty \leq Rt$ and $|\beta| \leq k-1$.*

Proof. (a) Since U is an affine image of Z , it suffices to show that $p \in \mathcal{P}_{k-1}$ vanishing on Z is zero (unisolvence of the principal lattice is classical [CY77, Nic72]; we include the short proof). We use induction on $d+k$. If $d=1$, p has degree at most $k-1$ and the k zeros $0, \dots, k-1$. If $k=1$, p is a constant vanishing at 0. Let $d \geq 2$, $k \geq 2$, $l(x) = x_1 + \cdots + x_d - (k-1)$ and $H = \{l=0\}$. The map $x \mapsto (x_1, \dots, x_{d-1})$ is an affine bijection of H onto $\mathbb{R}^{d-1}$; it carries the restrictions to H of polynomials in $\mathcal{P}_{k-1}$ to polynomials of degree at most $k-1$ in $d-1$ variables, and $Z \cap H$ onto the principal lattice $\{i' \in \mathbb{Z}_{\geq 0}^{d-1} : |i'| \leq k-1\}$ of the same order in dimension $d-1$. By induction $p=0$ on H , so $p = l p_1$ with $p_1 \in \mathcal{P}_{k-2}$. On $Z \setminus H = \{i : |i| \leq k-2\}$, which is the principal lattice for $\mathcal{P}_{k-2}$, we have $l \leq -1$, so p_1 vanishes there, and $p_1 = 0$ by induction.

(b) By (a), $\max_U |p|$ is a norm on the finite-dimensional space $\mathcal{P}_{k-1}$, so its unit ball is compact, and $(p, y) \mapsto D^\beta p(y)$ is continuous.

(c) If $\varepsilon = 0$ then $p = 0$ by (a). Otherwise $\tilde{p}(z) = p(a + tz)/\varepsilon$ satisfies $\max_U |\tilde{p}| \leq 1$ and $D^\beta \tilde{p}(z) = t^{|\beta|} (D^\beta p)(a + tz)/\varepsilon$; apply (b) at $z = (y-a)/t$. $\square$

Lemma 2.4 (Local estimate). *Let W be open, $R \geq 1$, $q = a + t[0, 1]^d$ with $t > 0$ and $U_q = a + tU$, and let $x \in W$ with $|x-a|_\infty \leq Rt$ be such that $[x, u] \subset W$ for every $u \in U_q$. If $f \in \mathcal{S}(W)$ vanishes on U_q , then*

$$|D^\beta f(x)| \leq C_1(R) t^{k-|\beta|} \quad (|\beta| \leq k-1), \quad C_1(R) := \Lambda(R) \frac{(d(R+1))^k}{k!}.$$

Proof. Let $T = T_x$ be the Taylor polynomial of f at x of degree $k-1$. For $u \in U_q$ we have $|u-a|_\infty < t$, hence $|u-x|_\infty \leq (R+1)t$ and $|u-x|_1 \leq d(R+1)t$, and Lemma 2.2 with $\beta = 0$ gives $|T(u)| = |f(u) - T(u)| \leq (d(R+1)t)^k/k!$. By Lemma 2.3(c) at $y = x$, $|D^\beta T(x)| \leq \Lambda(R)(d(R+1)t)^k t^{-|\beta|}/k!$, and $D^\beta f(x) = D^\beta T(x)$ for $|\beta| \leq k-1$. $\square$

3 The constant of the cube

For an open cube $Q' = a + hQ$ ($a \in \mathbb{R}^d$, $h > 0$) we define $\mathcal{S}(Q')$ and $r_{\mathcal{S}}(Q', \cdot)$ as for Q .

Proposition 3.1. (a) $r_{\mathcal{S}}(Q, U) \leq C_1(1)$; hence $E_n \leq C_1(1) < \infty$ for $n \geq N_k$.

(b) For $Q' = a + hQ$ and finite $P \subset Q$, $r_{\mathcal{S}}(Q', a + hP) = h^{k+d} r_{\mathcal{S}}(Q, P)$. If $g \in \mathcal{S}_0$, then $x \mapsto h^k g((x-a)/h)$ is a function of class C^k on $\mathbb{R}^d$ that vanishes outside Q' , has derivatives of order k bounded by 1, and has integral $h^{k+d} \int g$.

(c) $E_{m^d n} \leq m^{-k} E_n$ for all $m, n \geq 1$.

(d) The limit $c_{k,d} = \lim_n n^{k/d} E_n$ exists and equals $\inf_{n \geq 1} n^{k/d} E_n < \infty$.

(e) There is $c_* > 0$ with $E_n^0 \geq c_* n^{-k/d}$ for all $n \geq 1$. Consequently $c_{k,d} \geq c_* > 0$.

Proof. (a) Apply Lemma 2.4 with $W = Q$ (convex), $q = [0, 1]^d$, $a = 0$, $t = 1$, $R = 1$: every $f \in \mathcal{S}(Q)$ vanishing on U satisfies $|f| \leq C_1(1)$ on Q .

(b) For $f \in \mathcal{S}(Q')$ put $g(y) = h^{-k} f(a + hy)$, $y \in Q$. Then $D^\alpha g(y) = h^{|\alpha|-k} (D^\alpha f)(a + hy)$, which for $|\alpha| = k$ is $(D^\alpha f)(a + hy)$; so $f \mapsto g$ is a bijection of $\mathcal{S}(Q')$ onto $\mathcal{S}(Q)$, f vanishes on $a + hP$ if and only if g vanishes on P , and $\int_{Q'} f = h^d \int_Q f(a + hy) dy = h^{k+d} \int_Q g$. The statement on $\mathcal{S}_0$ is proved in the same way.

(c) If $E_n = \infty$ there is nothing to prove. Let $\varepsilon > 0$ and $P \subset Q$ with $|P| \leq n$ and $r_{\mathcal{S}}(Q, P) \leq E_n + \varepsilon$. Let $Q_j = j/m + Q/m$ for $j \in \{0, \dots, m-1\}^d$ and $P' = \bigcup_j (j/m + P/m)$, so $|P'| \leq m^d n$. If $f \in \mathcal{S}(Q)$ vanishes on P' , its restriction to Q_j lies in $\mathcal{S}(Q_j)$ and vanishes on $j/m + P/m$, so $\int_{Q_j} f \leq m^{-k-d} (E_n + \varepsilon)$ by (b). The faces of the Q_j are null sets, so $\int_Q f \leq m^{-k} (E_n + \varepsilon)$. Hence $E_{m^d n} \leq r_{\mathcal{S}}(Q, P') \leq m^{-k} (E_n + \varepsilon)$.

(d) Let $V_n = n^{k/d} E_n \in [0, \infty]$ and let n_0 be such that $V_{n_0} < \infty$ (by (a), $n_0 = N_k$ will do). For $N \geq n_0$ put $m = \lfloor (N/n_0)^{1/d} \rfloor \geq 1$, so that $m^d n_0 \leq N < (m+1)^d n_0$. By (2) and (c), $E_N \leq E_{m^d n_0} \leq m^{-k} E_{n_0}$, hence

$$V_N \leq N^{k/d} m^{-k} E_{n_0} \leq \left(\frac{m+1}{m} \right)^k V_{n_0}.$$

As $N \rightarrow \infty$, $m \rightarrow \infty$, so $\limsup_N V_N \leq V_{n_0}$. Since n_0 was any index with $V_{n_0} < \infty$, $\limsup_N V_N \leq \inf_n V_n \leq \liminf_N V_N$.

(e) Fix $\varphi \in C^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$, $\varphi \not\equiv 0$, $\text{supp } \varphi$ a compact subset of Q , and $\sup |D^\alpha \varphi| \leq 1$ for $|\alpha| = k$; let $I_\varphi = \int \varphi > 0$. Let $P \subset Q$ with $|P| \leq n$, let $M = \lceil (2n)^{1/d} \rceil$ and split Q into the M^d open cubes $j/M + Q/M$. At most n of them contain a point of P , so at least $M^d - n \geq M^d/2$ contain none. Let $g(x) = \sum M^{-k} \varphi(M(x - j/M))$, the sum over the empty cubes. The terms have disjoint compact supports in the empty cubes, so $g \in C^\infty(\mathbb{R}^d)$, $|D^\alpha g| \leq 1$ for $|\alpha| = k$, $g = 0$ outside Q and $g = 0$ on P ; thus $g \in \mathcal{S}_0$ and $r_0(P) \geq \int g \geq \frac{M^d}{2} M^{-k-d} I_\varphi = \frac{I_\varphi}{2} M^{-k}$. Since $M \leq (2n)^{1/d} + 1 \leq 2(2n)^{1/d}$, we get $E_n^0 \geq c_* n^{-k/d}$ with $c_* = I_\varphi 2^{-k-1-k/d}$. Finally $c_{k,d} = \lim_n n^{k/d} E_n \geq \liminf_n n^{k/d} E_n^0 \geq c_*$, because $E_n \geq E_n^0$. $\square$

The next lemma is the key to the comparison of the free and the Dirichlet constants (a “firewall” in the language of quantization). Fix $\vartheta \in C^\infty(\mathbb{R})$ with $0 \leq \vartheta \leq 1$, $\vartheta = 0$ on $(-\infty, 1/4]$ and $\vartheta = 1$ on $[3/4, \infty)$, and let $T_k = \max(1, \max_{m \leq k} \sup |\vartheta^{(m)}|)$. For $0 < \delta \leq 1/4$ let

$$\eta(s) = \vartheta\left(\frac{s-\delta}{\delta}\right) \vartheta\left(\frac{1-\delta-s}{\delta}\right), \quad \chi(x) = \prod_{i=1}^d \eta(x_i).$$

The first factor of η is 0 for $s \leq 5\delta/4$ and 1 for $s \geq 7\delta/4$; the second is 1 for $s \leq 1 - 7\delta/4$ and 0 for $s \geq 1 - 5\delta/4$. Since $7\delta/4 < 1 - 7\delta/4$, on each of the two half-lines $s \leq 1 - 7\delta/4$ and $s \geq 7\delta/4$ one factor

is identically 1; hence $|\eta^{(m)}| \leq T_k \delta^{-m}$ for $m \leq k$. So $\chi \in C^\infty$, $0 \leq \chi \leq 1$, $\text{supp } \chi \subset [5\delta/4, 1 - 5\delta/4]^d$ is a compact subset of Q , $|D^c \chi| \leq T_k^d \delta^{-|c|}$ for $|c| \leq k$, and $\chi = 1$ on $(7\delta/4, 1 - 7\delta/4)^d$. In particular,

$$\text{if } x \in Q \text{ and } \chi(x) \neq 1 \text{ or } D^c \chi(x) \neq 0 \text{ for some } c \neq 0, \text{ then } x_i \notin (2\delta, 1 - 2\delta) \text{ for some } i. \quad (3)$$

Lemma 3.2 (Firewall). *Let $M \geq 4$ and $1 \leq L \leq M/4$ be integers and $\delta = L/M$. Let B be the set of $j \in \{0, \dots, M-1\}^d$ with $j_i < 2L$ or $j_i \geq M - 2L$ for some i , and $c_j = j/M + [0, 1/M]^d$. Then $\bigcup_{j \in B} c_j = \{x \in [0, 1]^d : x_i \notin (2\delta, 1 - 2\delta) \text{ for some } i\}$, $\#B = M^d - (M - 4L)^d \leq 4dLM^{d-1}$, and for every $n \geq 0$*

$$E_n^0 \geq \left(1 + \frac{A}{L}\right)^{-1} \left[E_{n+N_k \# B} - 4d\delta C_1(1)M^{-k}\right], \quad A := C_1(1)T_k^d 2^k.$$

Proof. A cell c_j with $j_i < 2L$ lies in $\{x_i \leq 2\delta\}$, one with $j_i \geq M - 2L$ in $\{x_i \geq 1 - 2\delta\}$, and these cells cover $\{x_i \leq 2\delta\}$ and $\{x_i \geq 1 - 2\delta\}$; this gives the union. The complement of B consists of the j with $2L \leq j_i \leq M - 2L - 1$ for all i , so $\#B = M^d - (M - 4L)^d \leq d \cdot 4L \cdot M^{d-1}$.

Let $P \subset Q$ with $|P| \leq n$, and let $P' = P \cup \bigcup_{j \in B} (j/M + U/M)$; these points lie in Q and $|P'| \leq n + N_k \# B$. Let $f \in \mathcal{S}(Q)$ vanish on P' , and let $g = f\chi$ on Q and $g = 0$ outside Q . Since $\text{supp } \chi$ is a compact subset of Q , $g \in C^k(\mathbb{R}^d)$; moreover $g = 0$ outside Q and on P .

Derivatives. Let $|\alpha| = k$. By Leibniz' rule $D^\alpha g = \sum_{b \leq \alpha} \binom{\alpha}{b} D^b f D^{\alpha-b} \chi$ on Q . At a point where all derivatives of χ of positive order vanish, $D^\alpha g = \chi D^\alpha f$ and $|D^\alpha g| \leq 1$. Every other point $x \in Q$ lies, by (3), in a cell c_j with $j \in B$, and Lemma 2.4 (with $W = Q$, $q = c_j$, $t = 1/M$, $R = 1$) gives $|D^b f(x)| \leq C_1(1)M^{-(k-|b|)}$ for $|b| < k$. Since $M\delta = L \geq 1$ and $k - |b| \geq 1$,

$$|D^\alpha g(x)| \leq 1 + \sum_{b < \alpha} \binom{\alpha}{b} C_1(1)M^{-(k-|b|)} T_k^d \delta^{-(k-|b|)} \leq 1 + C_1(1)T_k^d 2^k / L = 1 + A/L.$$

Hence $g/(1 + A/L) \in \mathcal{S}_0$ vanishes on P , and $r_0(P) \geq (1 + A/L)^{-1} \int_Q f\chi$.

Integral. $1 - \chi$ vanishes outside the union of the cells c_j , $j \in B$, a set of measure $1 - (1 - 4\delta)^d \leq 4d\delta$, and on these cells $|f| \leq C_1(1)M^{-k}$ by Lemma 2.4 with $\beta = 0$. Hence $\int_Q f\chi \geq \int_Q f - 4d\delta C_1(1)M^{-k}$.

Taking the supremum over f gives $r_0(P) \geq (1 + A/L)^{-1} [r_{\mathcal{S}}(Q, P') - 4d\delta C_1(1)M^{-k}]$, and $r_{\mathcal{S}}(Q, P') \geq E_{|P'|} \geq E_{n+N_k \# B}$ by (2). Taking the infimum over P proves the lemma. $\square$

Proposition 3.3. $\lim_{n \rightarrow \infty} n^{k/d} E_n^0 = c_{k,d}$.

Proof. Since $E_n^0 \leq E_n$, $\limsup_n n^{k/d} E_n^0 \leq c_{k,d}$. Fix an integer $L \geq 1$. For $n \geq (4L)^d$ let $M = \lceil n^{1/d} \rceil \geq 4L$ and $\delta = L/M$, and let $n' = n + N_k \# B \leq n + 4dN_k L(n^{1/d} + 1)^{d-1}$; thus $n'/n \rightarrow 1$. Moreover $n^{k/d} \cdot 4d\delta C_1(1)M^{-k} = 4dLC_1(1)n^{k/d}M^{-k-1} \leq 4dLC_1(1)n^{-1/d} \rightarrow 0$, and $n^{k/d} E_{n'} = (n/n')^{k/d} n'^{k/d} E_{n'} \rightarrow c_{k,d}$ by Proposition 3.1(d). By Lemma 3.2, $\liminf_n n^{k/d} E_n^0 \geq c_{k,d}/(1 + A/L)$. Let $L \rightarrow \infty$; A depends only on k and d . $\square$

Together, Propositions 3.1 and 3.3 prove Theorem 1.1(a).

4 The lower bound on every bounded open set

Lemma 4.1. *Let $g \in \mathcal{S}_0$. Then $D^\beta g = 0$ on $\mathbb{R}^d \setminus Q$ and $\sup |D^\beta g| \leq 2^{-(k-|\beta|)} \leq 1$ for every $|\beta| \leq k$. In particular $\mathcal{S}_0$ lies in the full class $\mathcal{F}(\mathbb{R}^d)$, and $r_0(P) \leq 2^{-k}$ for every finite P .*

Proof. Each $D^\beta g$ with $|\beta| \leq k$ is continuous and vanishes on the open set $\mathbb{R}^d \setminus \overline{Q}$, hence on $\mathbb{R}^d \setminus Q$. The bound holds for $|\beta| = k$. Let $|\beta| < k$ and assume the bound for $\beta + e_1$. For $x \in Q$ let z be the point obtained from x by replacing x_1 with the nearer of 0 and 1, so $|x_1 - z_1| \leq 1/2$ and $z \in \partial Q$. Since $D^\beta g \in C^1$ and $D^\beta g(z) = 0$, $|D^\beta g(x)| = \left| \int_{z_1}^{x_1} D^{\beta+e_1} g(s, x_2, \dots, x_d) ds \right| \leq \frac{1}{2} 2^{-(k-|\beta|-1)}$. $\square$

Proposition 4.2. *For every nonempty bounded open set $D \subset \mathbb{R}^d$, $\liminf_{n \rightarrow \infty} n^{k/d} e_n(D) \geq c_{k,d} \lambda(D)^{1+k/d}$.*

Proof. Gluing. Let $h = 2^{-l}$ with $l \geq 0$, and let $Q_1, \dots, Q_J$ ($J = J_h$) be the open cubes $a_i + hQ$, $a_i \in h\mathbb{Z}^d$, whose closures lie in D ; J is finite because D is bounded. Let $P \subset D$ with $|P| \leq n$ and $n_i = |P \cap Q_i|$, so $\sum_i n_i \leq n$. For each i choose $g_i \in \mathcal{S}_0$ vanishing on $(P \cap Q_i - a_i)/h \subset Q$, and let

$$G(x) = \sum_{i=1}^J h^k g_i((x - a_i)/h).$$

By Lemma 4.1 each term is of class C^k on $\mathbb{R}^d$ and vanishes, with all its derivatives up to order k , outside Q_i . The Q_i are disjoint, so $D^\beta G(x) = h^{k-|\beta|} (D^\beta g_i)((x - a_i)/h)$ for $x \in Q_i$ and $D^\beta G = 0$ outside $\bigcup_i Q_i$; by Lemma 4.1 and $h \leq 1$, $|D^\beta G| \leq 1$ for all $|\beta| \leq k$. So $G \in C^k(\mathbb{R}^d)$ has all derivatives up to order k bounded by 1, and $G|_D \in \mathcal{F}(D)$. G vanishes on P : on the points in Q_i by the choice of g_i , and on the other points because $G = 0$ outside $\bigcup Q_i$. Since $\int_D G = h^{k+d} \sum_i \int g_i$ and the g_i can be chosen independently,

$$r_{\mathcal{F}}(D, P) \geq h^{k+d} \sum_{i=1}^J r_0((P \cap Q_i - a_i)/h) \geq h^{k+d} \sum_{i=1}^J E_{n_i}^0, \quad (4)$$

where $E_0^0 = r_0(\emptyset) \leq 2^{-k}$.

Convexity. Let $0 < \varepsilon < c_{k,d}$ and assume $J \geq 1$. By Proposition 3.3 there is $N_\varepsilon \geq 1$ with $E_m^0 \geq (c_{k,d} - \varepsilon) m^{-k/d}$ for $m \geq N_\varepsilon$, and by (2) $E_m^0 \geq (c_{k,d} - \varepsilon) \max(m, N_\varepsilon)^{-k/d}$ for all $m \geq 0$. The function $s \mapsto s^{-k/d}$ is convex and decreasing on $(0, \infty)$, and $\sum_i \max(n_i, N_\varepsilon) \leq n + JN_\varepsilon$, so by Jensen's inequality

$$\sum_{i=1}^J \max(n_i, N_\varepsilon)^{-k/d} \geq J \left(\frac{n + JN_\varepsilon}{J} \right)^{-k/d}.$$

With (4) and $h^{k+d} J^{1+k/d} = (Jh^d)^{1+k/d}$ this gives, for every $P \subset D$ with $|P| \leq n$,

$$r_{\mathcal{F}}(D, P) \geq (c_{k,d} - \varepsilon) (Jh^d)^{1+k/d} (n + JhN_\varepsilon)^{-k/d},$$

so the same lower bound holds for $e_n(D)$, and $\liminf_n n^{k/d} e_n(D) \geq (c_{k,d} - \varepsilon) (Jh^d)^{1+k/d}$.

Exhaustion. The union of the closed dyadic cubes of side 2^{-l} contained in D increases with l , and every point of the open set D lies in such a cube for l large; by monotone convergence $J_h h^d \rightarrow \lambda(D)$ as $h = 2^{-l} \rightarrow 0$. Letting $\varepsilon \rightarrow 0$ and then $h \rightarrow 0$ proves the claim. $\square$

No regularity of D was used, and only nodes in the open cubes Q_i played a role; this proves Theorem 1.1(b) and the part of Remark 1.2 that concerns the lower bound.

5 The upper bound under the cone condition

In this section D is a nonempty bounded open set with the cone condition, with constants ρ and θ as in Definition 2.1, and

$$\beta(r) = \lambda(\{y \in \mathbb{R}^d : \text{dist}(y, \partial D) \leq r\}) \quad (r > 0).$$

Lemma 5.1. $\lambda(\partial D) = 0$; $\beta(r) > 0$ for $r > 0$, and $\beta(r) \rightarrow 0$ as $r \rightarrow 0$.

Proof. Let $z \in \partial D$ and $0 < r \leq \rho$, and pick $x \in D$ with $|x - z| < r/2$. The truncated cone $C(x) \cap \{|y - x| < r/2\}$ lies in $D \cap \{|y - z| < r\}$ and has measure $\kappa(r/2)^d$, where $\kappa > 0$ is the measure of the part of the unit ball in an open circular cone of half-aperture θ . So the density of D in the balls around z of radius $r \leq \rho$ is at least $\kappa 2^{-d}/\omega_d > 0$ (ω_d the volume of the unit ball), and z is not a point of density 1 of the measurable set $\mathbb{R}^d \setminus D \supset \partial D$. By Lebesgue's density theorem $\lambda(\partial D) = 0$. The sets $\{\text{dist}(\cdot, \partial D) \leq r\}$ are compact and decrease to the closed set ∂D as $r \downarrow 0$, so $\beta(r) \rightarrow \lambda(\partial D) = 0$. Finally $\partial D \neq \emptyset$ because D is bounded and nonempty, so $\beta(r) > 0$. $\square$

Lemma 5.2 (Boundary layer). *Let $R = 4\sqrt{d}/\sin \theta$, $0 < s \leq \rho$, $t = s/R$, and $L \subset D$. There is a finite set $P_L \subset D$ with*

$$|P_L| \leq N_k t^{-d} \lambda(\{y \in D : \text{dist}(y, L) < s\})$$

such that every $f \in \mathcal{S}(D)$ vanishing on P_L satisfies $|f(x)| \leq C_1(R) t^k$ for all $x \in L$.

Proof. For $x \in D$ let $c_x = x + \frac{s}{2}v_x$ and let B_x be the open ball of radius $\frac{s}{2}\sin \theta$ around c_x . We claim $B_x \subset C(x)$. Let $y \in B_x$, $u = y - x$, $\sigma = u \cdot v_x$ and $w = u - \sigma v_x$, and write $\ell = s/2$. Then $(\sigma - \ell)^2 + |w|^2 < \ell^2 \sin^2 \theta$, so $|\sigma - \ell| < \ell$ and $\sigma > 0$; moreover

$$\sigma^2 \sin^2 \theta - \cos^2 \theta (\ell^2 \sin^2 \theta - (\sigma - \ell)^2) = (\sigma - \ell \cos^2 \theta)^2 \geq 0,$$

and $|w|^2 < \ell^2 \sin^2 \theta - (\sigma - \ell)^2$. If $\cos \theta > 0$ this gives $\sigma^2 \sin^2 \theta > |w|^2 \cos^2 \theta$, that is $\sigma^2 > |u|^2 \cos^2 \theta$; if $\cos \theta = 0$, $\sigma > 0 = |u| \cos \theta$ directly. In both cases $u \cdot v_x > |u| \cos \theta$, and $|u| < \ell \sin \theta + \ell \leq s \leq \rho$, so $y \in C(x)$. Also $B_x \subset \{|y - x| < s\}$.

Consider the grid of closed cubes $q = a + t[0, 1]^d$, $a \in t\mathbb{Z}^d$, and let q_x be one that contains c_x . Every point of q_x is within $\sqrt{d}t = \frac{s}{4}\sin \theta$ of c_x , so $q_x \subset B_x \subset C(x)$. Since $C(x)$ is convex and contains x and U_{q_x} , the segments $[x, u]$, $u \in U_{q_x}$, lie in D . If $q_x = a + t[0, 1]^d$, then $|x - a|_\infty \leq |x - c_x|_\infty + |c_x - a|_\infty \leq \frac{s}{2} + t \leq s = Rt$, and $R \geq 1$. By Lemma 2.4 with $\beta = 0$, every $f \in \mathcal{S}(D)$ that vanishes on U_{q_x} satisfies $|f(x)| \leq C_1(R) t^k$.

Let $\mathcal{Q}$ be the set of grid cubes q contained in $\{y \in D : \text{dist}(y, L) < s\}$ and $P_L = \bigcup_{q \in \mathcal{Q}} U_q \subset D$. For $x \in L$ we have $q_x \subset B_x \subset D \cap \{|y - x| < s\}$, so $q_x \in \mathcal{Q}$, and the estimate holds for every $f \in \mathcal{S}(D)$ vanishing on P_L . The cubes in $\mathcal{Q}$ have disjoint interiors and volume t^d , which gives the bound on $\#\mathcal{Q}$ and $|P_L| \leq N_k \#\mathcal{Q}$. $\square$

Proposition 5.3. $\limsup_{n \rightarrow \infty} n^{k/d} e_n(D) \leq c_{k,d} \lambda(D)^{1+k/d}$. *The same bound holds when $\mathcal{F}(D)$ is replaced by the larger class $\mathcal{S}(D)$.*

Proof. Cubes and layer. Let $h = 2^{-l}$ be so small that $J = J_h \geq 1$, with $Q_1, \dots, Q_J$ as in the proof of Proposition 4.2, and let $L_h = D \setminus \bigcup_i \overline{Q_i}$. If $x \in L_h$, a closed grid cube of side h that contains x is not one of the $\overline{Q_i}$, so it is not contained in D ; being connected and meeting D , it meets ∂D . Hence $\text{dist}(x, \partial D) \leq \sqrt{d}h$ on L_h , $\lambda(L_h) \leq \beta(\sqrt{d}h)$, and for $s > 0$ the set $\{y \in D : \text{dist}(y, L_h) < s\}$ lies in $\{\text{dist}(\cdot, \partial D) < \sqrt{d}h + s\}$.

Nodes. Fix $\eta \in (0, 1)$ and let $\beta_h = \beta(2\sqrt{d}h) > 0$. For $n \geq 1$ put

$$m = \lfloor (1 - \eta)n/J \rfloor, \quad t = (N_k \beta_h / (\eta n))^{1/d}, \quad s = Rt,$$

with $R = 4\sqrt{d}/\sin \theta$ as in Lemma 5.2. For n large we have $m \geq N_k$ and $s \leq \min(\rho, \sqrt{d}h)$; we consider only such n . Since $0 < E_m < \infty$ (Proposition 3.1(a),(e)), we can choose $P_m \subset Q$ with $|P_m| \leq m$ and $r_S(Q, P_m) \leq (1 + 1/m)E_m$. Lemma 5.2 with $L = L_h$ gives $P_{L_h} \subset D$ with $|P_{L_h}| \leq N_k t^{-d} \beta(2\sqrt{d}h) = \eta n$. Let

$$P = \bigcup_{i=1}^J (a_i + hP_m) \cup P_{L_h} \subset D, \quad |P| \leq Jm + \eta n \leq n.$$

Estimate. Let $f \in \mathcal{S}(D)$ vanish on P (for instance $f \in \mathcal{F}(D)$). D is the union of the cubes $\overline{Q_i}$ and of L_h , and the faces are null sets. The restriction of f to Q_i lies in $\mathcal{S}(Q_i)$ and vanishes on $a_i + hP_m$, so $\int_{Q_i} f \leq h^{k+d}(1 + 1/m)E_m$ by Proposition 3.1(b); and $|f| \leq C_1(R)t^k$ on L_h . Hence

$$e_n(D) \leq r_{\mathcal{F}}(D, P) \leq Jh^{k+d}\left(1 + \frac{1}{m}\right)E_m + C_1(R)\beta(\sqrt{d}h)\left(\frac{N_k\beta_h}{\eta n}\right)^{k/d}.$$

As $n \rightarrow \infty$, $n/m \rightarrow J/(1 - \eta)$ and $m^{k/d}E_m \rightarrow c_{k,d}$, so $n^{k/d}Jh^{k+d}(1 + 1/m)E_m \rightarrow c_{k,d}(Jh^d)^{1+k/d}(1 - \eta)^{-k/d}$, and $Jh^d \leq \lambda(D)$. Therefore

$$\limsup_{n \rightarrow \infty} n^{k/d}e_n(D) \leq c_{k,d}\lambda(D)^{1+k/d}(1 - \eta)^{-k/d} + C_1(R)\left(\frac{N_k}{\eta}\right)^{k/d}\beta(\sqrt{d}h)\beta(2\sqrt{d}h)^{k/d}.$$

R depends only on d and θ . Let $h \rightarrow 0$ (the last term tends to 0 by Lemma 5.1) and then $\eta \rightarrow 0$. $\square$

Propositions 4.2 and 5.3 prove Theorem 1.1(c).

The reading $W^{k,\infty}$. Let $\mathcal{S}^W(W)$ be defined as $\mathcal{S}(W)$ with $W^{k,\infty}(W)$ and essential suprema in place of $C^k(W)$ and suprema. Such functions have representatives of class C^{k-1} , and Lemma 2.2 holds for them along every segment contained in W : it holds for the mollifications $f * \varphi_\varepsilon$, which are smooth near the segment and satisfy the same bounds, and the derivatives of order less than k of $f * \varphi_\varepsilon$ converge locally uniformly. Hence Lemmas 2.4 and 5.2 hold for $\mathcal{S}^W$. Moreover, for every finite $P \subset Q$ the supremum $r_{\mathcal{S}}(Q, P)$ does not change when $\mathcal{S}(Q)$ is replaced by $\mathcal{S}^W(Q)$. Indeed, let $f \in \mathcal{S}^W(Q)$ vanish on P ; its derivatives of order less than k are Lipschitz on the convex set Q , so f extends continuously to $\overline{Q}$. For $\mu > 1$ and the centre c of Q , $f_\mu(x) = f(c + (x - c)/\mu)$ is defined on a neighbourhood of $\overline{Q}$ and has derivatives of order k bounded by $\mu^{-k} \leq 1$; for small ε the mollification $f_{\mu,\varepsilon}$ is smooth near $\overline{Q}$ with the same bounds. Fix $\xi \in C^\infty$ with support in the unit ball and $\xi(0) = 1$, and $r > 0$ smaller than half the distances between the points of P and than their distances to ∂Q , and put $g = f_{\mu,\varepsilon} - \sum_{p \in P} f_{\mu,\varepsilon}(p)\xi((x - p)/r)$. Then g vanishes on P , its derivatives of order k are bounded by $1 + \omega r^{-k} \max_{p \in P} |f_{\mu,\varepsilon}(p)|$ with $\omega = \max_{|\alpha|=k} \sup |D^\alpha \xi|$, and as $\mu \rightarrow 1$ and $\varepsilon \rightarrow 0$ we have $f_{\mu,\varepsilon} \rightarrow f$ uniformly on $\overline{Q}$, hence $\max_P |f_{\mu,\varepsilon}| \rightarrow 0$ and $\int_Q g \rightarrow \int_Q f$. Dividing g by the bound gives functions in $\mathcal{S}(Q)$ that vanish on P with integrals tending to $\int_Q f$. With these two facts the proof of Proposition 5.3 applies to the class $\mathcal{S}^W(D)$, which contains the $W^{k,\infty}$ reading of $C^k(D)$. This is the part of Remark 1.2 that concerns the upper bound.

6 Comb domains

Lemma 6.1 (Scaling of the full class). *Let $W \subset \mathbb{R}^d$ be a bounded open set and $0 < \tau \leq 1$. Then $e_n(\tau W) \geq \tau^{k+d}e_n(W)$ for all n .*

Proof. Let $P \subset \tau W$ with $|P| \leq n$, and let $g \in \mathcal{F}(W)$ vanish on P/τ . Then $f(x) = \tau^k g(x/\tau)$ satisfies $D^\alpha f(x) = \tau^{k-|\alpha|}(D^\alpha g)(x/\tau)$, so $f \in \mathcal{F}(\tau W)$ because $\tau \leq 1$; f vanishes on P , and $\int_{\tau W} f = \tau^{k+d} \int_W g$. Hence $r_{\mathcal{F}}(\tau W, P) \geq \tau^{k+d}r_{\mathcal{F}}(W, P/\tau) \geq \tau^{k+d}e_n(W)$. $\square$

Proposition 6.2. *Let $d \geq 2$, $k \geq 1$ and $0 < \delta < k/d$. Put $c_0 = \delta/(2(1 + \delta))$, $w_j = c_0 j^{-1-\delta}$ ($j \geq 1$), $a_1 = 0$, $a_{j+1} = a_j + 2w_j$, $I_j = (a_j, a_j + w_j)$, and*

$$D_2 = ((0, 1) \times (-1, 0)) \cup \bigcup_{j \geq 1} (I_j \times (-1, 1)) \subset \mathbb{R}^2, \quad D' = D_2 \times (0, 1)^{d-2}.$$

Then D' is a bounded connected open set with $\lambda(\partial D') = 0$ and $\lambda(D') = 1 + \sum_j w_j$, and $e_n(D') \geq (c_0 I_\psi / \delta)(n+1)^{-\delta}$ for all $n \geq 0$, with $I_\psi > 0$ defined below. For $D = \lambda(D')^{-1/d} D'$, which has the same properties and volume 1, $n^{k/d} e_n(D) \rightarrow \infty$.

Proof. Since $\sum_{j \geq 1} j^{-1-\delta} \leq 1 + \int_1^\infty s^{-1-\delta} ds = 1 + 1/\delta$, we have $\sum_j 2w_j \leq 1$, so the I_j are disjoint open subintervals of $(0, 1)$, separated by gaps of length w_j , and a_j increases to a limit $a^* \leq 1$. D_2 is open and bounded, and connected because every tooth $I_j \times (-1, 1)$ meets the base $(0, 1) \times (-1, 0)$. A point of ∂D_2 is a limit of points of D_2 , either of points in the base or in a single tooth (then it lies on the boundary of the base or of that tooth), or of points in teeth I_j with $j \rightarrow \infty$ (then its first coordinate is a^*). So ∂D_2 lies in a countable union of segments, and $\partial D' \subset (\partial D_2 \times [0, 1]^{d-2}) \cup (\overline{D_2} \times \partial(0, 1)^{d-2})$ is a null set. The measure of D' is $1 + \sum_j w_j$ (the base and the parts of the teeth above it).

Fix $\psi \in C^\infty(\mathbb{R})$ with $\psi \geq 0$, $\psi \not\equiv 0$, $\text{supp } \psi \subset [1/4, 3/4]$ and $\sup |\psi^{(m)}| \leq 1$ for $m \leq k$, and let $I_\psi = \int \psi > 0$. Let $P \subset D'$ with $|P| \leq n$, and let J_P be the set of j such that some point of P has its first two coordinates in $I_j \times (0, 1)$; then $|J_P| \leq n$. Define f on D' by $f(x) = \psi(x_2)$ if $(x_1, x_2) \in I_j \times (0, 1)$ for some $j \notin J_P$, and $f(x) = 0$ otherwise. The open sets $D' \cap \{x_2 < 1/4\}$, on which $f = 0$, and $D' \cap \{x_2 > 0\}$, which is the disjoint union of the open boxes $(I_j \times (0, 1)) \times (0, 1)^{d-2}$ on each of which f is $\psi(x_2)$ or 0, cover D' . Hence $f \in C^\infty(D')$, its only nonzero derivatives are $\partial_2^m f = \psi^{(m)}(x_2)$ on some boxes, and $f \in \mathcal{F}(D')$. It vanishes on P by the choice of J_P . Since w_j decreases and $|J_P| \leq n$,

$$r_{\mathcal{F}}(D', P) \geq \int_{D'} f = I_\psi \sum_{j \notin J_P} w_j \geq I_\psi \sum_{j > n} w_j \geq c_0 I_\psi \int_{n+1}^\infty s^{-1-\delta} ds = \frac{c_0 I_\psi}{\delta} (n+1)^{-\delta}.$$

This bounds $e_n(D')$ from below. With $\tau = \lambda(D')^{-1/d} < 1$, Lemma 6.1 gives $e_n(D) \geq \tau^{k+d} e_n(D')$, and $n^{k/d} (n+1)^{-\delta} \rightarrow \infty$ because $\delta < k/d$. $\square$

This proves Theorem 1.1(d). The comb has no cone condition: points in the middle of a thin tooth admit no cone of fixed size inside D . The point of the example is that $C^k(D)$ constrains the derivatives only inside D , so the values on neighbouring teeth are independent. A class defined by a Lipschitz condition $|f(x) - f(y)| \leq \|x - y\|$ for all $x, y \in D$, as in the first-order theorems of Chernaya and Gruber [Che95b, Che95a, Gru04] quoted in [Nov20a, Theorem 2], which are stated for Jordan measurable sets, does not allow the jumps of f across the narrow gaps; so the example does not contradict those theorems. It shows that Novak's guess [Nov20a, Remark 7(2)] for the class $C^k(D)$ needs an assumption on the boundary, as the wording of [Nov24, Remark 7(2)] anticipates.

7 Remarks and open questions

Remark 7.1 (Explicit bounds). Since $c_{k,d} = \inf_n n^{k/d} E_n$ (Proposition 3.1(d)), every finite $P \subset Q$ gives the upper bound $c_{k,d} \leq |P|^{k/d} r_{\mathcal{S}}(Q, P)$, and Proposition 3.1(e) gives $c_{k,d} \geq I_\varphi 2^{-k-1-k/d}$ for every admissible bump φ . For example, for $k = d = 1$ and the nodes $(2i-1)/(2n)$, every f with $|f'| \leq 1$ vanishing at the nodes satisfies $f(x) \leq \text{dist}(x, P)$, whose integral is $1/(4n)$; so $c_{1,1} \leq 1/4$.

Remark 7.2 (The case $k = 1$; not used). For $k = 1$ and $f \in \mathcal{S}(Q)$ vanishing on P , Lemma 2.2 gives $f(x) \leq \min_{p \in P} |x - p|_1$, so $c_{1,d}$ is at most the lower limit of $n^{1/d} \inf_{|P| \leq n} \int_Q \min_{p \in P} |x - p|_1 dx$, the ℓ^1 quantization coefficient of order 1 of the uniform distribution on the cube [Zad82, GL00]. We expect equality, by approximating the ℓ^1 distance function to P by C^1 functions that vanish on P and have $|\partial_i f| \leq 1$; this would identify $c_{1,d} \lambda(D)^{1+1/d}$ with the report's formula, in which $[d! \lambda(D)/2^d]^{1/d} = (\lambda(D)/\lambda(B_1))^{1/d}$ for the ℓ^1 unit ball B_1 . We have not written out this approximation, and the identification is neither proved nor used here.

Remark 7.3 (The case $d = 1$). For $d = 1$, Theorem 1.1(c) gives $\lim_n n^k e_n(C^k(a, b), \text{INT}) = c_{k,1}(b - a)^{k+1}$. If $g \in \mathcal{S}_0$, the derivatives of g up to order k vanish at 0 and 1 (Lemma 4.1), so $g|_{[0,1]}$ extends to a 1-periodic function of class C^k with $|g^{(k)}| \leq 1$; and the restriction of such a periodic function to $(0, 1)$ lies in $\mathcal{S}((0, 1))$. So $E_n^0 \leq E_n^{\text{per}} \leq E_n$ for the corresponding minimal radius E_n^{per} of the periodic class, and by Theorem 1.1(a) $c_{k,1} = \lim_n n^k E_n^{\text{per}}$. The classical theory of optimal quadrature for periodic functions with bounded k -th derivative would then give $c_{k,1} = K_k(2\pi)^{-k}$ with the Favard constants K_k (so $1/4, 1/32, 1/192$ for $k = 1, 2, 3$); we have not compared this with the sources, and we state it only as an expectation. The linear programs of the second and the third independent verification runs (Section 8) are consistent with these values.

Remark 7.4 (Other seminorms; not proved). By Theorem 1.1, the constant is determined by the seminorm $\max_{|\alpha|=k} \sup |D^\alpha f|$ alone. For another translation-invariant top-order seminorm, for instance the supremum of $|D_v^k f|$ over unit vectors v (the class with a tilde in [Nov20b]), we expect that the proof goes through with three changes: the norm used in Lemma 2.2, the normalisation of the bump in Proposition 3.1(e), and the constant of the Leibniz estimate in Lemma 3.2. We state this only as a remark.

Open questions. (1) The value of $c_{k,d}$ for $k \geq 2$, and for $k = 1$ in the sense of Remark 7.2; its behaviour in d is the subject of items (1) and (2) of [Nov20b], which are not addressed here. (2) Whether $\liminf_n n^{k/d} e_n(D) < \limsup_n n^{k/d} e_n(D)$ is possible for a bounded open set without the cone condition; by Theorem 1.1(b) the lower limit is always at least $c_{k,d} \lambda(D)^{1+k/d}$. (3) Weighted integrals $\int_D f g$ with a positive continuous weight g , for which Polovinkin's and Chernaya's results suggest the constant $c_{k,d} (\int_D g^{d/(d+k)})^{(d+k)/d}$; we have not treated them.

8 Computations and reproducibility

No proof in this note depends on a computation. The exact program of this note, `check_note.py` (standard library of Python, exact integers and rationals, about two seconds), checks the finite ingredients of the proofs: (A) Lemma 2.3(a): the collocation matrix of the monomials of degree at most $k - 1$ at U has full rank N_k for $d = 1, 2$ and $k \leq 10$, $d = 3$ and $k \leq 7$, $d = 4$ and $k \leq 6$, $d = 5$ and $k \leq 5$, $d = 6$ and $k \leq 4$ (up to $N_k = 126$), and $U \subset Q$; as negative controls, collinear sets and sets with a repeated point are detected as not unisolvent; (B) the multinomial identity of Lemma 2.2 for $d \leq 4$, $m \leq 6$, on grids large enough to prove it; (C) Lemma 3.2: $\#B = M^d - (M - 4L)^d$ by direct count ($d \leq 3$, $M \leq 24$), $\#B \leq 4dLM^{d-1}$ and $1 - (1 - 4L/M)^d \leq 4dL/M$ ($d \leq 6$, $4 \leq M \leq 60$), the description of the union of the cells c_j , $j \in B$, on 67,964 rational points ($d = 2$, $M \leq 16$), and the position of the transition intervals of η ; (D) the inequalities $2n \leq M^d \leq 2^{d+1}n$ for $M = \lceil (2n)^{1/d} \rceil$ used in Proposition 3.1(e) ($d \leq 6$, $n \leq 20,000$); (E) the identity in the proof of Lemma 5.2; (F) the telescoping inequalities behind the tail bound of Proposition 6.2 ($\delta = 1, 2, 3$ exactly, $\delta = 1/2$ with rational enclosures of square roots) and the normalisation of c_0 ; (G) the value $1/(4n)$ in Remark 7.1 for $n \leq 50$. All checks pass ("TOTAL failures: 0").

The second independent verification run (next paragraph) also made numerical computations with programs of its own (Python with `numpy` and `scipy`), which are in the package with their recorded outputs; they were run again at the writing and by the third verification run with identical outputs. They are consistency checks, not part of any proof. For $d = 1$ and $k \leq 4$, linear programs on fine grids gave, for equidistant nodes, values of $n^k r_0(P)$ that increase and values of $n^k r_S(Q, P)$ that decrease towards $K_k(2\pi)^{-k}$ (for $k = 2$, $n = 64$: $0.031055 \leq 1/32 \leq 0.031470$), and for the full class on intervals of four lengths ℓ the same values of $n^k r / \ell^{k+1}$ as soon as the bounds of lower order are inactive. For $d = 2$ and $k = 1$, the integrals of the ℓ^1 geodesic distance to diamond lattices (upper bounds for r_S)

in the unit square and in an L-shaped domain of area $3/4$, multiplied by $n^{1/2}\lambda^{-3/2}$, decreased from 0.488 and 0.494 ($n = 32$ and 24) to 0.4705 and 0.4708 ($n = 8192$ and 6144), close to $\sqrt{2}/3 \approx 0.4714$, the normalised mean ℓ^1 distance in a tiling of the plane by ℓ^1 balls (the last values lie slightly below it because of the discretisation). For $d = 2$ and $k = 2$, a finite-difference model with square lattices of $m \times m$ cells gave $nr/\lambda^2 = 0.0713$ and 0.0755 for the square and the L-shape at $m = 4$, and 0.0664 and 0.0669 at $m = 16$; the difference decays like $1/m$, as the boundary layer of Section 5 suggests.

The third verification run wrote its own programs as well; they are in the package with their outputs. An exact program (Python with sympy and mpmath) checks the unisolvence of U for all $k, d \leq 5$ by fraction-free integer elimination, with negative controls, and the two lattices of the induction step; $\#B$ by a coordinate-wise count and by enumeration, with the two bounds of Lemma 3.2, for $d \leq 6$ and $4 \leq M \leq 60$; the multinomial identity and the identity in the proof of Lemma 5.2 symbolically, together with random tests of the inclusions $q_x \subset B_x \subset C(x)$; the tail bound of Proposition 6.2 through the Hurwitz zeta function for ten values of $\delta \in [1/20, 3]$ and $n \leq 1000$; and the sandwich inequality of Proposition 3.1(d) in exact integer form. Linear programs for $d = 1$ (scipy) reproduce the values $1/4$ (midpoints, $k = 1$) and $1/24$ ($k = 2$, one node at $1/2$) and, for equidistant nodes, $k \leq 4$ and $n \leq 64$, place $K_k(2\pi)^{-k}$ between the Dirichlet and the free values. They also illustrate that E_n may be finite for some $n < N_k$, which the proofs do not exclude (for $k = 2$ and $d = 1$, $E_1 = 1/24$, attained only with the node $1/2$, since $x - x_0$ has a nonzero integral for $x_0 \neq 1/2$). These are consistency checks as well.

Verification. The results were first obtained with complete proofs and without computation. Two independent verification runs, both AI-assisted, followed; each read the source and wrote its own programs. The first run checked the statement against the report, re-derived every lemma, case distinction and limit of Sections 2 to 6, and checked that the answer does not depend on the readings of Remark 1.2; its own computations confirmed the unisolvence of the principal lattice for $k, d \leq 5$ (in exact arithmetic), the counting inequalities of Lemma 3.2, the comb tail bound and the value $c_{1,1} \leq 1/4$. It required six corrections of the text, which were made: the cone condition for $d = 1$ is stated for bounded open intervals; Remark 1.2 was added; Remark 7.2 is labelled as neither proved nor used; the restriction $d \geq 2$ is stated wherever the comb is mentioned; the words “for n large” are explicit in the proof of Proposition 5.3; and more information on Polovinkin’s papers was needed, which the second run supplied. The second run downloaded the report and the proposer’s later texts, re-derived the proof line by line, read [Pol75, Pol89] in the Russian originals, checked the bibliographic data with Crossref and made the computations described above. It required the description of [Pol75, Pol89] given in the introduction and corrections of the bibliography, which were made, and no change to the theorem or the proof. It also pointed out that the classes of [Nov20a, Theorem 2] are defined by Lipschitz conditions between all pairs of points of D .

A third independent verification run, also AI-assisted, then examined the final text of the note. It read the report and the two later texts of the proposer again, and [Pol75, Pol89] in the copies fetched from Math-Net.ru at the writing (Math-Net.ru did not respond at the time of this run; the copies are identical to those fetched by the second run). It re-derived every proof of Sections 2 to 6 and the argument for the reading $W^{k,\infty}$, checked all DOIs with Crossref and the papers of Chernaya with zbMATH, made the computations described in Section 8 with programs of its own, and re-ran all programs of the package from the source archive with identical outputs. It found no mathematical error and required corrections, which were made: [Nov20a, Theorem 2] rests on the paper [Che95b] of Chernaya, which is now cited; the existence of the limit along all n was removed from the list of differences with Polovinkin’s results, since [Pol89, Theorem 2] has it; the relation between the Lipschitz classes of [Nov20a, Theorem 2] and the classes of C^1 functions is stated precisely; and two statements in the verification report and the description of the programs were made precise.

Scope and priority. The note answers item (3) of [Nov20b] under the standing hypothesis of the report: for bounded Lipschitz domains, and more generally for bounded open sets with the cone condition, the limit exists and depends on the domain only through its volume. The lower bound holds for every bounded open set, and for $d \geq 2$ the combs show that some assumption on the boundary is needed. The value of $c_{k,d}$, its dependence on d (items (1) and (2) of the report), and the existence of the limit without the cone condition are not determined. The question is Novak’s [Nov20b], from joint work with Hinrichs, Krieg, Prochno, Ullrich and Woźniakowski; he states the guess in [Nov20a, Remark 7(2)] and again, with the qualification on the boundary, in [Nov24, Remark 7(2)]. The representation (1) is the report’s known fact and is used as the definition. The method is the quantization scheme of Zador and of Graf and Luschgy [Zad82, GL00]; the unisolvence of the principal lattice is classical [CY77, Nic72]; that bounded Lipschitz domains have the cone condition is standard [AF03, Gri11]. The closest prior work is Polovinkin’s [Pol75, Pol89], who proved the corresponding asymptotics for L_p^m , $1 < p < \infty$, with a Sobolev-type seminorm; Theorem 1.1(a)–(c) is the counterpart for $p = \infty$ and the full C^k norm, under the cone condition. For $k = 1$ the dependence on the volume is known [Nov20b, Nov20a, Che95b, Che95a, Gru04], and weighted first-order results are due to Babenko, Babenko and Kovalenko [BBK21] and Kovalenko [Kov24]; the note recovers the case $k = 1$ for $C^1(D)$. No priority is claimed.

What was read: the abstract of [Nov20b] on pp. 386–388 of the open PDF of the report; [Nov20a] in the arXiv version (Sections 1–3, Section 4 in part) and the part of [Nov24] (arXiv version) that contains its Remark 7; [Pol75, Pol89] in the Russian originals on Math-Net.ru (the definitions, the statements and the structure of the proofs; the proofs were not checked); the introduction of [BBK21]; the abstract and the introduction of [Kov24] (by the second verification run); the zbMATH reviews of the two papers [Che95a, Che95b] of Chernaya and of papers of Derets’, Boikov, Polovinkin (1988), Ramazanov, Reztsov, Osipov, Yumova and Noskov. [Gru04] was cited through [Nov20a], and the papers of Chernaya were not read beyond these reviews. We did not read [Pol78, Pol83, Ram74, SV97] or the papers of Babenko of 1976–77, and [AF03, Gri11] were not re-read for this note. The searches, made on 19 and 20 October 2026, were queries to the arXiv API (asymptotic constants for integration, cubature and sampling; asymptotically optimal cubature; Zador’s method and integration; exact asymptotics; optimal quadrature on domains; cubature on Lipschitz and general domains; the authors Kovalenko and Babenko), the lists of works citing [Nov20a] and [Nov24] in Semantic Scholar and OpenAlex, queries to zbMATH and Crossref, and a few web searches. No printed answer to item (3) of [Nov20b] for $k \geq 2$ was found in the sources searched; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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