

Independent Sets in Two-Regular Uniform Hypergraphs

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Abstract

For every integer $r \geq 2$, we prove the Balogh–Bollobás–Narayanan independent-set bound for finite simple r -uniform, 2-regular hypergraphs. If such a hypergraph has n vertices, its number of weak independent sets is at most $(2^{2(r-1)} + 3(2^{r-1} - 1)^2)^{n/(2r)}$. The proof uses the edge-cover dual and regional entropy inequalities of Sarantis, Tetali and Zheng. We choose a fractional-matching factor with induced odd-cycle components by a chord-reduction argument. The remaining closed cycles satisfy a uniform transfer-matrix bound. This extends the previously stated degree-two cases, odd r and $r = 4$, to all uniformities without a finite rooted-region enumeration. The conjecture for general degree $d \geq 3$ is not resolved here.

1 Statement and relation to the conjecture

All hypergraphs are finite and simple: their edges are distinct sets. A hypergraph is r -uniform if every edge has r vertices, and d -regular if every vertex belongs to d edges. An independent set means a *weak* independent set, namely a vertex set containing no edge in its entirety. Write $i(H)$ for their number, including the empty set.

The canonical hypergraph $H_{r,d}$ has d marked vertices and d disjoint blocks of $r-1$ other vertices. Its edges are all unions of one marked vertex and one block. Balogh, Bollobás and Narayanan [1] conjectured that every d -regular r -uniform hypergraph on n vertices satisfies

$$i(H) \leq i(H_{r,d})^{n/(rd)}, \quad i(H_{r,d}) = 2^{(r-1)d} + (2^d - 1)(2^{r-1} - 1)^d.$$

This is also Conjecture 1 in Balogh’s contribution to [3, p. 80].

Theorem 1. *Let $r \geq 2$ be an integer, and let H be a finite simple r -uniform, 2-regular hypergraph on n vertices. Set*

$$T_r = 2^{2(r-1)} + 3(2^{r-1} - 1)^2.$$

Then $i(H) \leq T_r^{n/(2r)}$. Disjoint unions of copies of $H_{r,2}$ attain the bound when $2r$ divides n .

Sarantis, Tetali and Zheng [2, Theorems 1.8 and 1.9] prove the degree-two bound for odd r and for $r = 4$. Their edge-cover duality, regional entropy inequality, pair estimate and doubling argument are the inputs to our proof. We give those steps explicitly for completeness. The additional observation is that an edge-and-odd-cycle factor may be chosen with every odd cycle induced. A short estimate then treats closed cycles for every even r . In particular, the rooted-region computation of [2, §5 and Appendix B] is not needed for this extension.

Theorem 1 concerns only $d = 2$. It does not establish the general-degree conjecture, classify all equality cases, or give the exact finite- n maximum when $2r$ does not divide n .

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2 Edge covers and regional bounds

An edge cover of a loopless multigraph is a subset of its edge *copies* incident to every vertex. Denote their number by $\text{ec}(F)$. The equivalent form of our result is the following.

Theorem 2. *If F is a finite loopless r -regular multigraph on N vertices with every edge multiplicity at most $r - 1$, then*

$$\text{ec}(F) \leq T_r^{N/4}.$$

To see the equivalence, make one vertex of F for every hyperedge of H . Each vertex of H belongs to two distinct hyperedges and gives an edge of F between them. Regularity of F follows from uniformity of H . Multiplicity r would give two identical hyperedges, so it is excluded. Conversely the stars of a multigraph as in Theorem 2, viewed as sets of edge copies, form a simple r -uniform, 2-regular hypergraph. In either direction, complements of independent sets correspond to edge covers, and $n = rN/2$.

Let $S = \text{supp}(F)$ be the simple support graph. Give a support edge of multiplicity m_e weight $w_e = 2^{m_e} - 1$, the number of nonempty choices from its parallel copies. Thus

$$\text{ec}(F) = \sum_{\substack{J \subseteq E(S) \\ J \text{ covers } V(S)}} w(J), \quad w(J) = \prod_{e \in J} w_e.$$

Partition $V(S)$ into nonempty regions B . Edges joining distinct regions are boundary edges; let ∂B be those incident to B . For $U \subseteq B$, put

$$Z_B(U) = \sum_{\substack{J \subseteq E(S[B]) \\ J \text{ covers } U}} w(J).$$

Covering vertices outside U is allowed. A boundary state $\eta \subseteq \partial B$ hits a subset $V_B(\eta)$ of B . Define

$$A_B(\eta) = w(\eta)^{1/2} Z_B(B \setminus V_B(\eta)), \quad \Xi_B = \sum_{\eta \subseteq \partial B} A_B(\eta)^2.$$

Lemma 3 (Regional inequality, [2, §4.2]). *For every such partition,*

$$\log \text{ec}(F) \leq \frac{1}{2} \sum_B \log \Xi_B.$$

Proof. Let ω range over subsets of all boundary edges and write $\omega_B = \omega \cap \partial B$. Conditional on ω , internal edge choices are independent across regions. The half weights count each boundary edge once, so

$$\text{ec}(F) = \sum_{\omega} \prod_B A_B(\omega_B).$$

Normalize these summands to a probability law μ . Its entropy satisfies

$$\log \text{ec}(F) = H_{\mu}(\omega) + \sum_B \mathbb{E}_{\mu} \log A_B(\omega_B).$$

Every boundary coordinate occurs in exactly two projections ω_B , hence Shearer's entropy inequality gives $H_{\mu}(\omega) \leq \frac{1}{2} \sum_B H_{\mu}(\omega_B)$. Here the inequality follows directly by ordering the coordinates,

expanding each entropy by the chain rule, and noting that conditioning on fewer preceding coordinates increases entropy. For a regional marginal q , nonnegativity of relative entropy against $A_B(\eta)^2/\Xi_B$ gives

$$\mathbb{E}_q \log A_B + \frac{1}{2} H(q) \leq \frac{1}{2} \log \Xi_B.$$

The actual marginals give zero mass to zero-weight states. Summing these inequalities proves the claim. $\square$

For evaluating a single Ξ_B , boundary edges incident to one vertex may be aggregated: if their total multiplicity is b , their nonempty selections have total weight $2^b - 1$. The original global boundary coordinates remain distinct.

Lemma 4 (Pair estimate, [2, Lemma 4.5]). *For a two-vertex region whose internal edge has multiplicity $1 \leq m \leq r - 1$, one has $\Xi_B \leq T_r$. Consequently Theorem 2 holds whenever S has a perfect matching.*

Proof. Put $p = 2^m - 1$ and $h = 2^{r-m} - 1$. Neither, either, or both endpoints may be hit by the boundary. The conditional internal weights are respectively $p, p, 1 + p$, so

$$\Xi_B = p^2 + 2hp^2 + h^2(1 + p)^2.$$

Direct substitution gives

$$T_r - \Xi_B = 4(2^{m-1} - 1)(2^{r-m-1} - 1) \geq 0.$$

For a perfect matching, use its pairs as regions and apply Lemma 3. $\square$

3 Induced cycle factors and doubling

Lemma 5. *A finite simple graph with a fractional perfect matching has a spanning subgraph whose components are single edges and induced odd cycles.*

Proof. Take an extreme point of the nonempty compact polytope

$$\{x \in \mathbb{R}_{\geq 0}^{E(S)} : \sum_{e \ni v} x_e = 1 \text{ for every } v\}.$$

The incidence columns of its positive support are linearly independent; a dependence would permit a small perturbation in both directions. A support vertex of degree one forces its incident weight to equal one, so that component is a single edge. Every other component has minimum degree at least two. Column independence gives at most as many edges as vertices, and the degree condition gives the reverse inequality. Such a component is a cycle. An even cycle has the alternating incidence dependence, so these cycles are odd. This is the usual half-integral factor argument, also used in [2, Lemma 4.6].

Choose such a factor minimizing the total number of vertices in its cycle components. Suppose one cycle C has a chord in S . The two paths in C between the chord endpoints have opposite parities. The even-length path, together with the chord, is a smaller odd cycle. The internal vertices of the other path form an even path and can be paired by consecutive edges. Replacing C by this smaller odd cycle and those pairs gives another spanning edge-and-odd-cycle factor. Since the chord endpoints were nonadjacent, at least two vertices leave cycle components. This contradicts minimality. $\square$

Lemma 6 (Boundary cycle, [2, Lemma 4.7]). *Let B be the vertex set of an odd cycle in S , and suppose some edge leaves B . With all induced edges internal to the region, $\Xi_B \leq T_r^{|B|/2}$.*

Proof. Write b_v for the total outgoing multiplicity at $v \in B$. Its two cycle edges have positive multiplicity, so $0 \leq b_v \leq r - 2$. Make two copies of $F[B]$, and join the two copies of each v by b_v parallel edges if $b_v > 0$. Call the result $\widehat{F}$. It is loopless and r -regular, with maximum multiplicity at most $r - 1$. Local aggregation gives exactly

$$\Xi_B = \sum_{D \subseteq B} \left(\prod_{v \in D} (2^{b_v} - 1) \right) Z_B(B \setminus D)^2 = \text{ec}(\widehat{F}).$$

Choose a vertex v with $b_v > 0$. Match its two copies to each other; after deleting them, each copy of the odd cycle is an even path, which has a perfect matching. Thus the support of $\widehat{F}$ has a perfect matching. Lemma 4, proved without using the general theorem, yields $\text{ec}(\widehat{F}) \leq T_r^{2|B|/4}$. $\square$

4 Closed cycles and completion of the proof

Lemma 7. *If a connected component of F has support an odd cycle of length k , then r is even and its edge-cover count is strictly less than $T_r^{k/4}$.*

Proof. The cyclic regularity equations $m_{i-1} + m_i = r$ force $m_i = r/2$ for every i , because k is odd. Let $w = 2^{r/2} - 1 \geq 1$. Recording whether each support edge is selected gives the transfer matrix

$$K = \begin{pmatrix} 0 & w \\ 1 & w \end{pmatrix}.$$

The edge-cover sum is $c_k = \text{tr } K^k$; the zero entry disallows two consecutive absent edges. The eigenvalues satisfy $\lambda_+ > 0 > \lambda_-$ and $\lambda_{\pm}^2 - w\lambda_{\pm} - w = 0$. Since k is odd, $c_k = \lambda_+^k + \lambda_-^k < \lambda_+^k$.

It remains to show $\lambda_+^4 < T_r$. Put

$$a = w^3 + 2w^2, \quad b = w^3 + w^2, \quad T = (w + 1)^4 - 3(w + 1)^2 + 3 = T_r.$$

The quadratic relation gives $\lambda_+^4 = a\lambda_+ + b$. Define

$$u = \frac{2w - 1}{w^2(w + 2)}, \quad R = w + 1 - u = \frac{T - b}{a}.$$

For $w \geq 1$, $0 < u < 1$ and $R > w$; indeed the denominator of u minus its numerator is $w^3 + 2w(w - 1) + 1 > 0$. Also

$$R^2 - wR - w = 1 - (w + 2)u + u^2 = \frac{(w - 1)^2}{w^2} + u^2 > 0.$$

The quadratic is strictly increasing on $[w, \infty)$, is negative at w , and vanishes at $\lambda_+ > w$. Hence $R > \lambda_+$ and $\lambda_+^4 < aR + b = T$. This proves the assertion. $\square$

Proof of Theorem 2. The empty graph is immediate. Otherwise S has the fractional perfect matching $x_e = m_e/r$. Choose the factor of Lemma 5 and use its components as regions. For an edge component, Lemma 4 gives $\Xi_B \leq T_r$. For an odd-cycle component with boundary, Lemma 6 gives $\Xi_B \leq T_r^{|B|/2}$. A cycle component without boundary is induced by Lemma 5, so it has no chords or other edges. It is therefore covered by Lemma 7. Its only boundary state is empty, giving $\Xi_B = \text{ec}(F[B])^2 < T_r^{|B|/2}$. All regions consequently satisfy $\Xi_B \leq T_r^{|B|/2}$, and Lemma 3 proves $\log \text{ec}(F) \leq N \log T_r/4$. $\square$

The dual of $H_{r,2}$ is a four-cycle with opposite multiplicities $r-1, r-1$ and the other opposite multiplicities $1, 1$. Its edge-cover count is T_r . The duality and multiplication over components now prove Theorem 1, including the asserted sharpness.

5 Checks and scope

The proof is analytic and does not require enumerating rooted regions. The supplied pure-Python checker independently counts edge covers by support-edge dynamic programming and by inclusion-exclusion over uncovered vertices. It tests all 4 742 labelled multigraphs in its stated small domains and 400 seeded regular examples, checks 84 525 chord reductions, 2 016 pair identities and 800 closed cycle instances, and verifies the transfer-matrix threshold identity coefficient by coefficient. The exact domains and outputs accompany the source package. These are regression checks, not a formal verification of the general theorem.

The current prior-art comparison is bounded, not an exhaustive priority determination. The contribution is the all-uniformity degree-two extension using the regional method of [2]; the general-degree conjecture remains outside the result. This unrefereed preprint was developed with AI assistance for mathematical exploration, literature retrieval, drafting and exact computation. The argument was self-audited; no independent expert review or proof-assistant certification is asserted.

References

- [1] J. Balogh, B. Bollobás and B. Narayanan, Counting independent sets in regular hypergraphs, *Journal of Combinatorial Theory, Series A* **180** (2021), 105405. doi:10.1016/j.jcta.2021.105405.
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