

Unbounded Gaps in the Tropical Secant Dimension Bound

Alper Ferudun*

22 October 2026

Abstract

The optimized affine winner-cell bound gives a combinatorial lower bound for the dimension of a secant cone of a monomial image. We show that it need not equal the actual dimension, answering negatively the unrestricted monomial formulation of a question in Draisma’s Oberwolfach report. For the six cubic monomials $x^3, y^3, z^3, x^2y, x^2z, yz^2$, the optimized two-cell bound is exactly five, whereas the two-point secant cone is all of $\mathbb{C}^6$. The upper bound follows from a planar separation obstruction; a Jacobian determinant of 216 proves the actual dimension. Disjoint coordinate blocks give, for every $r \geq 1$, a degree-three monomial cone with optimized bound $5r$ and secant dimension $6r$. The affine cones used here are not normal. No counterexample for normal toric embeddings or complete Segre–Veronese systems is claimed.

1 The bound and the question

Let $A \subset \mathbb{N}^m$ be a finite exponent set lying in an affine hyperplane not through the origin. Write

$$C_A = \overline{\{(x^\alpha)_{\alpha \in A} : x \in \mathbb{C}^m\}}, \quad kC_A = \overline{\{p_1 + \cdots + p_k : p_i \in C_A\}},$$

where closures are Zariski closures. We use the notation kC_A for the closed secant cone, not merely the set of sums.

For real affine forms $\ell_1, \dots, \ell_k$ on $\mathbb{R}^m$, define their strict winner cells

$$W_i(\ell) = \{\alpha \in A : \ell_i(\alpha) < \ell_j(\alpha) \text{ for every } j \neq i\}.$$

Set $\rho(W) = 1 + \dim \text{Aff } W$ for $W \neq \emptyset$ and $\rho(\emptyset) = 0$. The optimized affine partition bound is

$$T_k(A) = \max_{\ell_1, \dots, \ell_k} \sum_{i=1}^k \rho(W_i(\ell)). \quad (1)$$

Only finitely many cell patterns occur, so the maximum is well defined. Tied points contribute to no cell.

Draisma’s theorem gives $T_k(A) \leq \dim kC_A$. Question 1 following Theorem 7 in [1] asks whether optimization over the affine forms always gives equality. The theorem and question in that report allow arbitrary monomial maps of this form; normality, a complete linear system and an embedding of a homogeneous variety are not hypotheses. The corresponding general method and its applications are developed in [2, 3]. Our result concerns the fixed exponent set in (1): it does not optimize over changes of parametrization.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint, version 1.0. Licensed under CC BY 4.0.

Theorem 1. *Let*

$$A = \{(3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (2, 0, 1), (0, 1, 2)\}$$

and let

$$f(x, y, z) = (x^3, y^3, z^3, x^2y, x^2z, yz^2).$$

Then $C_A = \overline{f(\mathbb{C}^3)}$ is an irreducible cone of dimension three, and

$$T_2(A) = 5, \quad 2C_A = \mathbb{C}^6.$$

For every positive integer r , there is a homogeneous degree-three exponent set $A^{[r]} \subset \mathbb{N}^{3r}$, of cardinality $6r$, such that

$$\dim C_{A^{[r]}} = 3r, \quad T_2(A^{[r]}) = 5r, \quad 2C_{A^{[r]}} = \mathbb{C}^{6r}.$$

Thus the additive discrepancy between the optimized bound and the actual secant dimension is unbounded already for two-point sums.

2 A planar separation obstruction

Lemma 2. *Let a planar configuration B consist of the three vertices of a nondegenerate triangle and one point in the relative interior of each edge. For any two affine forms, the sum of the affine ranks of their strict winner cells is at most five.*

Proof. Every cell has affine rank at most three and at most its cardinality. If a point is tied, the two cells contain at most five points altogether, which proves the assertion.

Otherwise, a score of six would require each cell to be a noncollinear triple. Two of the three triangle vertices must belong to the same cell. The difference of the affine forms has the same strict sign at those two vertices and therefore at their edge point, a strict convex combination of the vertices. These three collinear points exhaust that cell's triple. This contradicts its required affine rank of three. $\square$

The set A in Theorem 1 lies in the plane $a_1 + a_2 + a_3 = 3$. Its three pure-power vectors are the vertices, and each of the other vectors lies strictly inside a different edge. Lemma 2 therefore gives $T_2(A) \leq 5$.

The pair of affine forms

$$\ell_1(a) = 0, \quad \ell_2(a) = a_1 - 1$$

has no ties. Its first winner cell is

$$\{(3, 0, 0), (2, 1, 0), (2, 0, 1)\},$$

a noncollinear triple of affine rank three. The other cell is $\{(0, 3, 0), (0, 0, 3), (0, 1, 2)\}$, a collinear triple of affine rank two. Hence $T_2(A) = 5$. This proves the optimized value over all real affine forms, not only over a finite list of sampled cuts.

3 The secant dimension

All six monomials have degree three. Since every complex scalar has a cube root, their image and its closure are cones. The first three coordinate functions x^3, y^3, z^3 are algebraically independent, so the image closure has dimension at least three. The domain gives the reverse inequality. Irreducibility follows from irreducibility of the domain.

Consider the addition map

$$F : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^6, \quad F(p, q) = f(p) + f(q).$$

At $p = (1, 1, 1)$, $q = (1, 2, 3)$, its Jacobian is

$$J = \begin{pmatrix} 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 12 & 0 \\ 0 & 0 & 3 & 0 & 0 & 27 \\ 2 & 1 & 0 & 4 & 1 & 0 \\ 2 & 0 & 1 & 6 & 0 & 1 \\ 0 & 1 & 2 & 0 & 9 & 12 \end{pmatrix}.$$

The columns are ordered as x, y, z, x', y', z' . Eliminating the lower-left block against the upper-left block $3I_3$ gives the Schur complement

$$S = \begin{pmatrix} 2 & -3 & 0 \\ 4 & 0 & -8 \\ 0 & 5 & -6 \end{pmatrix}, \quad \det S = 8, \quad \det J = 27 \cdot 8 = 216.$$

The complex inverse function theorem implies that the image of F contains a nonempty Euclidean open subset of $\mathbb{C}^6$. No nonzero complex polynomial vanishes on such a subset. Consequently the image is Zariski dense. Since it is contained in $C_A + C_A$, the closed secant cone $2C_A$ is all of $\mathbb{C}^6$. This establishes the first part of Theorem 1.

4 Amplifying the discrepancy

For $1 \leq b \leq r$, let $\iota_b : \mathbb{R}^3 \longrightarrow \mathbb{R}^{3r}$ insert a vector into the b -th three-coordinate block, with zeros elsewhere. Define

$$A^{[r]} = \bigcup_{b=1}^r \iota_b(A).$$

The corresponding monomial map is the product of r copies of f . Thus $C_{A^{[r]}} = C_A^r$, an irreducible cone of dimension $3r$. The two-point addition map has, after a simultaneous reordering of rows and columns, a block diagonal Jacobian with r copies of J . Its determinant has absolute value 216^r . The preceding argument gives $2C_{A^{[r]}} = \mathbb{C}^{6r}$.

It remains to compute the optimized bound, because the two global winner cells need not be confined to individual blocks.

Lemma 3. *For every subset $W \subseteq A^{[r]}$,*

$$\rho(W) = \dim \operatorname{span} W = \sum_{b=1}^r \dim \operatorname{span}(W \cap \iota_b(A)).$$

For an empty set its linear span has dimension zero.

Proof. Every exponent has total coordinate sum three. If $w_0 \in W \neq \emptyset$, the coordinate-sum functional vanishes on all differences $w - w_0$, whereas it is nonzero on w_0 . Therefore $\dim \text{span } W = 1 + \dim \text{Aff } W$. The second equality holds because the coordinate-block subspaces form a direct sum. The empty case follows from the conventions. $\square$

Restrict any two global affine forms to each block. By Lemma 3, their total score on $A^{[r]}$ is the sum of their scores on the block configurations, including any ties or empty cells. Each summand is at most five by Lemma 2, so $T_2(A^{[r]}) \leq 5r$. The global pair

$$\ell_1(a) = 0, \quad \ell_2(a) = \sum_{b=1}^r a_{3b-2} - 1$$

restricts in each block to the attaining pair used above. Its score is $5r$. This completes the proof of Theorem 1.

5 Scope, related work and exact checks

The example does not contradict the lower-bound theorem. It contradicts universal equality with the actual secant dimension in the unrestricted monomial setting of [1]. In particular the comparison is with an explicitly computed actual dimension, not merely the expected dimension.

The affine cone is not normal. Indeed, in its monomial coordinate ring

$$R_A = \mathbb{C}[x^3, y^3, z^3, x^2y, x^2z, yz^2],$$

the element

$$xyz = \frac{(x^2y)(x^2z)}{x^3}$$

belongs to the fraction field and is integral over R_A , since its cube lies in R_A . It is not in R_A : every generator has degree three, and the degree-three monomial xyz is absent from their linear span. No conclusion is drawn here about exactness restricted to normal toric embeddings or complete Segre–Veronese systems.

The dependence of tropical lower bounds on a parametrization is already visible in [2, Example 3.3], where an example attributed to Sturmfels uses dense linear coordinate functions. That example is not monomial. Develin’s discussion of nonpure tropical secant complexes [4, Section 4] also cautions against equating tropical and expected dimensions. Neither caution substitutes for comparing the optimized value with the actual dimension of the specific monomial secant. Homogeneous varieties and Segre–Veronese systems have their own nondefectivity results; see [3, 5]. The present sparse cubic system is not a complete Segre–Veronese system. A bounded review of these primary sources and targeted searches did not locate this explicit six-monomial discrepancy or its block amplification. This is not an assertion of absolute priority.

The accompanying pure-Python checker uses integer and rational arithmetic. It enumerates all $3^6 = 729$ strict-positive, strict-negative and zero sign patterns of the difference of the affine forms, using exact Fourier–Motzkin elimination for feasibility. There are 87 feasible patterns, and the largest score is five. It also computes $\det J$ independently by elimination and the Leibniz formula and checks several block instances. These are reproducibility checks; the all-parameter proofs above do not depend on finite sampling. No proof-assistant verification or independent review is claimed.

AI tools assisted with mathematical exploration, source retrieval, drafting and exact-checker development. The author is responsible for the statements and the unrefereed manuscript. The

archived problem identifier is OWR-1740-002 in `ulamai/UnsolvedMath`, version 1.6.0. The publisher copy of [1] places Theorem 7 and Question 1 on pages 3293–3294; the MFO repository copy numbers the same pages 3353–3354.

References

- [1] J. Draisma, Some tropical geometry of algebraic groups, minimal orbits, and secant varieties, in *Tropical Geometry*, Oberwolfach Report 57/2007, Theorem 7 and Question 1. [doi:10.4171/OWR/2007/57](https://doi.org/10.4171/OWR/2007/57).
- [2] J. Draisma, A tropical approach to secant dimensions, *J. Pure Appl. Algebra* **212** (2008), 349–363. [doi:10.1016/j.jpaa.2007.05.022](https://doi.org/10.1016/j.jpaa.2007.05.022). Preprint [arXiv:math/0605345v3](https://arxiv.org/abs/math/0605345v3).
- [3] K. Baur and J. Draisma, Secant dimensions of low-dimensional homogeneous varieties, *Adv. Geom.* **10** (2010), 1–29. [doi:10.1515/advgeom.2010.001](https://doi.org/10.1515/advgeom.2010.001). Preprint [arXiv:0707.1605](https://arxiv.org/abs/0707.1605).
- [4] M. Develin, Tropical secant varieties of linear spaces, *Discrete Comput. Geom.* **35** (2006), 117–129. [doi:10.1007/s00454-005-1182-2](https://doi.org/10.1007/s00454-005-1182-2). Preprint [arXiv:math/0405115](https://arxiv.org/abs/math/0405115).
- [5] A. Laface, A. Massarenti and R. Rischter, On secant defectiveness and identifiability of Segre–Veronese varieties, *Rev. Mat. Iberoam.* **38** (2022), 1605–1635. [doi:10.4171/RMI/1336](https://doi.org/10.4171/RMI/1336).