

A Uniform Bound for Free Pairs of Bicyclic Units and for Free Points at Algebraic Integers

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Abstract

Gonçalves and del Río asked two questions in the problem session of Oberwolfach Report 55/2007 (Problem 28); the first is also the first half of Question 5.4 of their paper in J. Group Theory 11 (2008). A nonzero complex number z is a free point if the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ generate a free group, and $m(z)$ is the least positive integer k for which kz is a free point. For a finite non-Hamiltonian group G , $m(G)$ is the least positive integer m such that b_1 and b_2^m generate a free group whenever b_1, b_2 are bicyclic units of $\mathbb{Z}G$ that generate a group which is not nilpotent. Is $m(z)$ bounded over the nonzero algebraic integers z , and is $m(G)$ bounded over all such groups G ? We observe that both are bounded by 4. A nonzero algebraic integer has a Galois conjugate of modulus at least 1, freeness of a point is invariant under Galois conjugation, and every complex number of modulus at least 4 is a free point (Sanov, Brenner); hence kz is a free point for every integer k with $|k| \geq 4$, and $m(z) \leq 4$, with equality for $z = \pm 1$. For bicyclic units $1 + a$ and $1 + b$, the element ab acts on $\mathbb{Z}G$ by an integer matrix, so if it is not nilpotent it has an eigenvalue λ with $|\lambda| \geq 1$, and the two-dimensional reduction of Gonçalves and del Río turns $(1 + a, (1 + b)^m)$ into $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ m\lambda & 1 \end{pmatrix}$; hence $m(G) \leq 4$, also for the variant $M(G)$ that allows units of different types. The result is an elementary combination of known facts. Exact computations confirm $m(S_3) = 1$, $M(S_3) = 2$ and $m(S_4) = 2$ and give $M(S_4) = 2$; exhaustive computations with numerically located roots give $m(S_5) = M(S_5) = 2$. Whether $m(G) \leq 2$ for all G , the second part of Question 5.4, remains open. No answer was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and four later sources could not be read. This is an unrefereed note.

1 Introduction

1.1 The questions

Let G be a finite group and $\mathbb{Z}G$ its integral group ring. For $h \in G$ of order d put $\widehat{h} = 1 + h + \cdots + h^{d-1}$. For $x, h \in G$ the elements

$$\beta_{x,h} = 1 + (1 - h)x\widehat{h}, \quad \gamma_{x,h} = 1 + \widehat{h}x(1 - h)$$

are the *bicyclic units* of type β and of type γ [GdR08, §1]. Since $\widehat{h}(1 - h) = (1 - h)\widehat{h} = 0$, each of them has the form $1 + a$ with $a^2 = 0$; hence $(1 + a)^k = 1 + ka$ for every $k \in \mathbb{Z}$, and $1 + a$ is a unit with inverse $1 - a$. A pair (u, v) of elements of a group is a *free pair* if the homomorphism from the free group $F(x, y)$ that sends $x \mapsto u$, $y \mapsto v$ is injective, that is, if u and v form a basis of a free group of rank two.

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The source [GdR07], Problem 28 of the problem session of Oberwolfach Report 55/2007, posed by Gonçalves and del Río, works with the set $\mathcal{B}$ of the units $\beta_{x,h}$. It recalls the dichotomy of [GdR08]: for $b_1, b_2 \in \mathcal{B}$, either $\langle b_1, b_2 \rangle$ is nilpotent or $\langle b_1, b_2^m \rangle$ is free and non-cyclic for some positive integer m ; pairs of the second kind exist if and only if G is non-Hamiltonian (the source attributes this to Marciniak and Sehgal [MS97]). For a finite non-Hamiltonian group G it defines

$$m(G) = \min\{m \in \mathbb{N} : \langle b_1, b_2^m \rangle \text{ is free for all } b_1, b_2 \in \mathcal{B} \text{ with } \langle b_1, b_2 \rangle \text{ not nilpotent}\}, \quad (1)$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$, and following [GdR08, §5] we write $M(G)$ for the same minimum over all pairs of bicyclic units of arbitrary types. With

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_w = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \quad (w \in \mathbb{C}),$$

a nonzero complex number w is a *free point* if the group $\langle A, B_w \rangle$ is free, and $m(z) = \min\{k \in \mathbb{N} : kz \text{ is a free point}\}$ for $z \in \mathbb{C} \setminus \{0\}$. The source asks:

(Q1) Is the set $\{m(G) : G \text{ a finite non-Hamiltonian group}\}$ bounded?

(Q2) Is the set $\{m(z) : z \text{ a nonzero algebraic integer}\}$ bounded?

It adds that it is enough to consider symmetric groups, that $m(S_3) = 1$ and $m(S_4) = 2$, and that a positive answer to (Q2) gives a positive answer to (Q1). (Q1) is the first half of [GdR08, Question 5.4]; the second half asks whether $m(G) \leq 2$ for every finite non-Hamiltonian group G .

Conventions. (i) We call G Hamiltonian if all its subgroups are normal, abelian groups included. For such G every bicyclic unit equals 1, because $\beta_{x,h} = 1$, and likewise $\gamma_{x,h} = 1$, if and only if x normalises $\langle h \rangle$. These groups are excluded by the question; under the usage in which abelian groups are not Hamiltonian they would contribute the value 1, the condition in (1) being vacuous. (ii) For $w \neq 0$ the matrices A and B_w do not commute. A free group generated by two elements that do not commute has rank two, and the two elements are then a basis, since free groups of finite rank are Hopfian. So w is a free point if and only if $\langle A, B_w \rangle$ is a free pair; similarly “free”, “free non-cyclic” and “free pair” agree for $\langle b_1, b_2^m \rangle$ when $\langle b_1, b_2 \rangle$ is not nilpotent (Remark 3.4). (iii) In [GdR08, §5], $m(G)$ is defined with pairs of the same type, both β or both γ , and with “ ab not nilpotent” for $b_1 = 1 + a$, $b_2 = 1 + b$; Remark 3.4 shows that this gives the number (1). (iv) Some authors take $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ as the first matrix; z is free in that sense if and only if $2z$ is free in ours [GdR08, footnote 1].

The corpus record. The question is record OWR-1738-011 of the UnsolvedMath collection [ula26], where it is listed as open. The record prints the second matrix as $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. This matrix commutes with A , so, read literally, the group is abelian and never free of rank two; the source prints $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. The record also joins the two questions by “Equivalently”, while the source states only that a positive answer to the second question gives one to the first. Its page range 3209–3240 is that of the DOI metadata; the report is printed on pp. 3149–3180, and the problem is on p. 3176. We follow the source.

1.2 Results

Theorem A. *Let z be a nonzero algebraic integer. Then kz is a free point for every integer k with $|k| \geq 4$. Hence $m(z) \leq 4$. Moreover $m(1) = m(-1) = 4$, so that 4 is the maximum of $m(z)$ over all nonzero algebraic integers z .*

Theorem B. *Let G be a finite group and let $a, b \in \mathbb{Z}G$ satisfy $a^2 = b^2 = 0$, with ab not nilpotent. Then $((1+a)^m, (1+b)^n) = (1+ma, 1+nb)$ is a free pair for all integers m, n with $|mn| \geq 4$.*

Corollary C. *Let G be a finite non-Hamiltonian group, and let b_1, b_2 be bicyclic units of $\mathbb{Z}G$, of any types, such that $\langle b_1, b_2 \rangle$ is not nilpotent. Then (b_1, b_2^m) is a free pair for every integer $m \geq 4$, and so is (b_1^2, b_2^2) . Consequently $m(G) \leq M(G) \leq 4$; with Theorem A, both questions (Q1) and (Q2) have a positive answer.*

For Theorem A, a nonzero algebraic integer has a Galois conjugate z' with $|z'| \geq 1$, because the product of its conjugates is a nonzero rational integer; freeness of a point is invariant under Galois conjugation [CGS17, KK22]; and every $w \in \mathbb{C}$ with $|w| \geq 4$ is a free point (Sanov [San47], Brenner [Bre55]). For $z = \pm 1$ the multipliers $k = 1, 2, 3$ fail by explicit relations. Theorem B uses the same observation inside the regular representation: ab acts on $\mathbb{Z}G$ by an integer matrix, so it has an eigenvalue λ with $|\lambda| \geq 1$ when it is not nilpotent, and on a two-dimensional invariant subspace $1 + ma$ and $1 + nb$ act, after a conjugation, as A and $B_{mn\lambda}$, as in the proof of [GdR08, Theorem 1.1]. Remark 3.3 shows that the implication stated in the source holds as stated: if kz is a free point, so is jkz for every integer $j \neq 0$, and a bound $m(z) \leq K$ for all nonzero algebraic integers z gives $m(G) \leq M(G) \leq \text{lcm}(1, \dots, K)$.

Section 4 records computed values. By an exact computation, $m(S_3) = 1$, $M(S_3) = 2$ and $m(S_4) = M(S_4) = 2$ (Proposition 4.2); the first three values are those of [GdR07, GdR08]. Exhaustive computations with exact characteristic polynomials and numerically located roots give $m(S_5) = M(S_5) = 2$ and $M(G) \leq 2$ for twelve further small groups; in three of them the value at $m = 1$ depends on whether a specific algebraic integer of modulus less than 4 is a free point. In $\mathbb{Z}S_5$ there is a pair of bicyclic units for which every nonzero eigenvalue of ab has modulus less than 2 (Proposition 4.3). Since $m(S_4) = 2$, Corollary C gives $\sup_G m(G) \in \{2, 3, 4\}$; whether it equals 2 remains open.

1.3 What is new and what is not

The result is an elementary combination of known facts. The free-point theorem for $|w| \geq 4$ goes back to Sanov and Brenner (Lemma 2.1 and the remarks before it); the invariance of freeness under Galois conjugation is observed by Chorna, Geller and Shpilrain [CGS17, §2] and by Kim and Koberda [KK22, §1.1]; the reduction to 2×2 matrices is part of the proof of the criterion of Gonçalves and del Río [GdR08, Theorem 1.1]: if $a^2 = b^2 = 0$ in a ring of characteristic zero, then $(1 + a, 1 + b)$ is a free pair if and only if ab is transcendental or some eigenvalue of ab is a free point. That the freeness of $((1 + a)^m, (1 + b)^n)$ depends only on mn , as in Theorem B, is [GdR08, Corollary 3.1(1)] for positive m, n . An exponent for each pair separately was known: Salwa [Sal99] proved that $(1 + ma, 1 + mb)$ is a free pair for some $m \in \mathbb{N}$ when ab is not nilpotent, and [GdR08, Corollary 3.1(2)] takes m with $|m\lambda| \geq 4$ for one nonzero eigenvalue λ of ab ; this m depends on the pair, since λ may be small. For dihedral groups, Jiménez López [Jim08] showed $m(D_{2n}) \leq 2$, as reported in [GdR08, §5]. The survey of Jespers [Jes21] states the criterion as its Theorem 3.5; none of its open problems (Problems 2.3, 2.4, 8.1, 8.4 and 8.5) is the present question. What we add is the remark that integrality bounds the relevant eigenvalue, or one of its Galois conjugates, away from zero, which makes the exponent uniform. No answer was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and four later sources could not be read (see “Scope and priority” in Section 5).

2 Free points

The group $\text{SL}_2(\mathbb{C})$ acts on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot z = (pz + q)/(rz + s)$, and a matrix that moves some point is not the identity. For $n \in \mathbb{Z}$ we have $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $B_w^n = B_{nw}$, so $A^n \cdot z = z + n$ and $B_w^n \cdot z = z/(nwz + 1) = 1/(z^{-1} + nw)$, with the usual conventions at 0 and ∞ .

The next lemma is classical. Sanov [San47] proved that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a free group [CGS17, Theorem 1]; conjugation by $\text{diag}(2, 1)$ carries this pair to (A, B_4) . According to [KK22, §1], Sanov's result gives the real points with $|w| = 4$, and Brenner [Bre55] proved freeness for all real w with $|w| > 4$; [GdR08, §1] and [Jes21, Proposition 3.3(2)] state the lemma for all complex w with $|w| \geq 4$ and call it Sanov's theorem. We did not read [San47, Bre55] and include the standard ping-pong proof, which covers all complex w .

Lemma 2.1 (Sanov, Brenner). *Every $w \in \mathbb{C}$ with $|w| \geq 4$ is a free point.*

Proof. Let $X_1 = \{z \in \mathbb{C} : |z| > \frac{1}{2}\} \cup \{\infty\}$ and $X_2 = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$; they are disjoint and nonempty. Let $n \in \mathbb{Z} \setminus \{0\}$. If $z \in X_2$, then $|z + n| \geq |n| - |z| > \frac{1}{2}$, so $A^n(X_2) \subseteq X_1$. If $z \in X_1$, put $u = z^{-1}$ (and $u = 0$ for $z = \infty$), so that $|u| < 2$; then $B_w^n \cdot z = 1/(u + nw)$ and $|u + nw| \geq |n||w| - |u| > 4 - 2 = 2$, so $B_w^n(X_1) \subseteq X_2$.

Let W be a nontrivial reduced word in $x^{\pm 1}, y^{\pm 1}$, written as an alternating product of nonzero powers of x and of y . If W begins and ends with a power of x , the two inclusions, applied from the right, show that $W(A, B_w)$ maps X_2 into X_1 ; hence $W(A, B_w) \neq I$. If W begins and ends with a power of y , then $W(A, B_w)$ maps X_1 into X_2 , and again $W(A, B_w) \neq I$. If $W = x^{n_1} \cdots y^m$, choose $c \in \mathbb{Z} \setminus \{0, -n_1\}$: the word $x^c W x^{-c} = x^{c+n_1} \cdots y^m x^{-c}$ is reduced and begins and ends with a power of x , so $W(A, B_w)$ is conjugate to a matrix different from I . The case $W = y^m \cdots x^{n_k}$ is the same with $c \notin \{0, n_k\}$. Hence the homomorphism $F(x, y) \rightarrow \text{SL}_2(\mathbb{C})$, $x \mapsto A$, $y \mapsto B_w$, is injective. $\square$

Lemma 2.2. *Let $w \in \mathbb{C} \setminus \{0\}$.*

- (a) *If w is algebraic and $\sigma: \mathbb{Q}(w) \rightarrow \mathbb{C}$ is a field embedding, then w is a free point if and only if $\sigma(w)$ is.*
- (b) *w is a free point if and only if $-w$ is, if and only if $\bar{w}$ is.*
- (c) *If w is a free point, then so is jw for every $j \in \mathbb{Z} \setminus \{0\}$.*

Proof. (a) Applying σ to the entries is an injective ring homomorphism $M_2(\mathbb{Q}(w)) \rightarrow M_2(\mathbb{C})$; it restricts to an injective group homomorphism $\Sigma: \text{SL}_2(\mathbb{Q}(w)) \rightarrow \text{SL}_2(\mathbb{C})$ with $\Sigma(A) = A$ and $\Sigma(B_w) = B_{\sigma(w)}$. For every word W we have $\Sigma(W(A, B_w)) = W(A, B_{\sigma(w)})$, so $W(A, B_w) = I$ if and only if $W(A, B_{\sigma(w)}) = I$.

(b) With $D = \text{diag}(1, -1)$ we have $DAD^{-1} = A^{-1}$ and $DB_w D^{-1} = B_{-w}$, so $W(A, B_w) = I$ if and only if $W(A^{-1}, B_{-w}) = I$; since $x \mapsto x^{-1}$, $y \mapsto y$ is an automorphism of $F(x, y)$, (A^{-1}, B_{-w}) is a free pair if and only if (A, B_{-w}) is. Complex conjugation of the entries is an injective homomorphism of $\text{SL}_2(\mathbb{C})$ that fixes A and maps B_w to $B_{\bar{w}}$.

(c) $B_{jw} = B_w^j$. The endomorphism φ of $F(x, y)$ with $\varphi(x) = x$ and $\varphi(y) = y^j$ is injective, because it maps a nontrivial reduced word $x^{n_1} y^{m_1} x^{n_2} \cdots$ to the word $x^{n_1} y^{jm_1} x^{n_2} \cdots$, which is again reduced and nontrivial. The homomorphism $x \mapsto A$, $y \mapsto B_{jw}$ is the composite of φ with $x \mapsto A$, $y \mapsto B_w$, hence injective if the latter is. $\square$

Part (a) is observed by Chorna, Geller and Shpilrain [CGS17, §2] for the pairs $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ and by Kim and Koberda [KK22, §1.1], who credit [CGS17]; for example $4 - \sqrt{2}$ is a free point because $4 + \sqrt{2}$ is.

Lemma 2.3. *A nonzero algebraic integer z has a Galois conjugate z' , that is, a root of the minimal polynomial of z over $\mathbb{Q}$, with $|z'| \geq 1$.*

Proof. The minimal polynomial f of z is monic with integer coefficients, because z is an algebraic integer, and $f(0) \neq 0$, because f is irreducible and $z \neq 0$. If $z_1, \dots, z_d$ are the roots of f , then $|z_1 \cdots z_d| = |f(0)| \geq 1$, so $|z_i| \geq 1$ for some i . $\square$

Proposition 2.4. *For $k \in \{1, 2, 3\}$ the matrix $X_k = AB_k^{-1} = AB_{-k} = \begin{pmatrix} 1-k & 1 \\ -k & 1 \end{pmatrix}$ has determinant 1, trace $2 - k$, and order exactly 6, 4 and 3, respectively. Hence the nontrivial reduced words $(xy^{-1})^6$, $(xy^{-1})^4$, $(xy^{-1})^3$ vanish at (A, B_k) for $k = 1, 2, 3$, and $(xy)^6$, $(xy)^4$, $(xy)^3$ vanish at (A, B_{-k}) . In particular $\pm 1, \pm 2, \pm 3$ are not free points.*

Proof. By the Cayley–Hamilton theorem $X^2 = (\text{tr } X)X - I$ for $X \in \text{SL}_2(\mathbb{C})$. For trace 1 this gives $X^3 = X^2 - X = -I$ and $X^6 = I$; for trace 0, $X^2 = -I$ and $X^4 = I$; for trace -1 , $X^3 = -X^2 - X = I$. The orders are exact, since $X_k \neq I$ for all k , $X_1^3 = X_2^2 = -I$ and $X_1^2 = X_1 - I \neq I$. Finally AB_{-k} is the value of xy^{-1} at (A, B_k) and of xy at (A, B_{-k}) . $\square$

Proof of Theorem A. Let $z \neq 0$ be an algebraic integer and $k \in \mathbb{Z}$ with $|k| \geq 4$. By Lemma 2.3 the minimal polynomial f of z has a root z' with $|z'| \geq 1$, and since f is irreducible there is a field embedding $\sigma: \mathbb{Q}(z) \rightarrow \mathbb{C}$ with $\sigma(z) = z'$. Then $|kz'| \geq 4$, so $\sigma(kz) = kz'$ is a free point by Lemma 2.1, and kz is a free point by Lemma 2.2(a), as $\mathbb{Q}(kz) = \mathbb{Q}(z)$. Hence $m(z) \leq 4$. By Proposition 2.4, the points $\pm 1, \pm 2, \pm 3$ are not free, so $m(1) = m(-1) = 4$. $\square$

Remark 2.5. In the same way $m(\pm 2) = m(\pm 3) = 2$ and $m(\pm 4) = 1$, so the values 1, 2 and 4 occur. The hypothesis that z is an algebraic integer enters only through Lemma 2.3. In the normalisation with first matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (convention (iv)) the bound becomes 2, attained at $z = 1$.

3 Bicyclic units

The first lemma is part of [GdR08, Corollary 3.1(3)]; we include the short proof.

Lemma 3.1. *Let R be a ring and $a, b \in R$ with $a^2 = b^2 = 0$. If ab is nilpotent, then the group $\langle 1 + a, 1 + b \rangle$ is nilpotent.*

Proof. Let $(ab)^r = 0$. A product of factors from $\{a, b\}$ with two equal neighbours vanishes, and an alternating product of $2r + 1$ factors is $(ab)^r a$ or $b(ab)^r$; hence every product of at least $2r + 1$ factors is zero. For $i \geq 1$ let S_i be the additive subgroup of R generated by the products of at least i factors from $\{a, b\}$. Then $S_i S_j \subseteq S_{i+j}$ and $S_{2r+1} = 0$, so every $1 + s$ with $s \in S_i$ is a unit with inverse $\sum_{k \geq 0} (-s)^k \in 1 + S_i$, and $U_i = 1 + S_i$ is a group. For $u = 1 + s \in U_i$ and $v = 1 + t \in U_j$ we have $uv - vu = st - ts$, hence $u^{-1}v^{-1}uv = 1 + u^{-1}v^{-1}(st - ts) \in U_{i+j}$, because $(1 + S_1)S_{i+j} \subseteq S_{i+j}$. Thus $U_1 \supseteq U_2 \supseteq \dots \supseteq U_{2r+1} = 1$ is a central series of U_1 , and U_1 contains $\langle 1 + a, 1 + b \rangle$. $\square$

The next proposition is the two-dimensional reduction in the proof of [GdR08, Theorem 1.1], in the form needed here.

Proposition 3.2. *Let $X, Y \in M_N(\mathbb{C})$ with $X^2 = Y^2 = 0$, and let $\lambda \neq 0$ be an eigenvalue of XY . If $m, n \in \mathbb{Z}$, $m \neq 0$, and $mn\lambda$ is a free point, then $(I + mX, I + nY)$ is a free pair.*

Proof. Choose $v \neq 0$ with $XYv = \lambda v$ and put $u = Yv$. Then $Xu = \lambda v \neq 0$, so $u \neq 0$; moreover $Xv = \lambda^{-1}X^2Yv = 0$ and $Yu = Y^2v = 0$. If $u = cv$ for some $c \in \mathbb{C}$, then $\lambda v = Xu = cXv = 0$, which is impossible; so v and u are linearly independent. The plane W spanned by $e_1 = \lambda v$ and $e_2 = u$ is invariant under X and Y : $Xe_1 = 0$, $Xe_2 = e_1$, $Ye_1 = \lambda e_2$ and $Ye_2 = 0$. In the basis (e_1, e_2) ,

$$X|_W = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y|_W = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

As $X^2 = Y^2 = 0$, we have $(I + X)^m = I + mX$ and $(I + Y)^n = I + nY$. The group $H = \langle I + mX, I + nY \rangle$ preserves W , and restriction to W is a homomorphism $\psi: H \rightarrow \text{GL}_2(\mathbb{C})$ with $\psi(I + mX) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

and $\psi(I + nY) = \begin{pmatrix} 1 & 0 \\ n\lambda & 1 \end{pmatrix}$. With $D = \text{diag}(m, 1)$,

$$D^{-1} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} D = A, \quad D^{-1} \begin{pmatrix} 1 & 0 \\ n\lambda & 1 \end{pmatrix} D = B_{mn\lambda}.$$

Since $mn\lambda$ is a free point, the homomorphism $F(x, y) \rightarrow \text{SL}_2(\mathbb{C})$, $x \mapsto A$, $y \mapsto B_{mn\lambda}$, is injective. It is the composite of $x \mapsto I + mX$, $y \mapsto I + nY$ with ψ and with conjugation by D ; hence the first map is injective, that is, $(I + mX, I + nY)$ is a free pair. $\square$

Proof of Theorem B. Let $N = |G|$. For $c \in \mathbb{Z}G$ let $L_c \in M_N(\mathbb{Z})$ be the matrix of the map $y \mapsto cy$ of $\mathbb{Z}G$ in the basis G . The map $c \mapsto L_c$ is an injective ring homomorphism, because L_c applied to the basis vector 1 returns the coordinates of c ; so it restricts to an injective group homomorphism from the unit group of $\mathbb{Z}G$ to $\text{GL}_N(\mathbb{Z})$. The eigenvalues of ab in the sense of [GdR08] are those of L_{ab} .

Put $X = L_a$ and $Y = L_b$. These are integer matrices with $X^2 = Y^2 = 0$, and $XY = L_{ab}$ is not nilpotent. Write $\det(tI - XY) = t^e q(t)$ with $q(0) \neq 0$. Then q is a monic polynomial with integer coefficients; its roots are the nonzero eigenvalues of XY , counted with multiplicity; and $\deg q \geq 1$, since otherwise $(XY)^N = 0$ by the Cayley–Hamilton theorem. The product of these eigenvalues has modulus $|q(0)|$, a positive integer, so some nonzero eigenvalue λ of XY satisfies $|\lambda| \geq 1$.

Let $m, n \in \mathbb{Z}$ with $|mn| \geq 4$. Then $|mn\lambda| \geq 4$, so $mn\lambda$ is a free point by Lemma 2.1, and $(I + mX, I + nY)$ is a free pair by Proposition 3.2. Since $L_{1+ma} = I + mX$ and $L_{1+nb} = I + nY$ and $c \mapsto L_c$ is injective, $(1 + ma, 1 + nb)$ is a free pair. $\square$

Proof of Corollary C. Write $b_1 = 1 + a$ and $b_2 = 1 + b$, with $a^2 = b^2 = 0$ (Section 1.1). If $\langle b_1, b_2 \rangle$ is not nilpotent, then ab is not nilpotent by Lemma 3.1. Theorem B with (m, n) replaced by $(1, m)$ and by $(2, 2)$ shows that $(b_1, b_2^m) = (1 + a, 1 + mb)$ for $m \geq 4$ and $(b_1^2, b_2^2) = (1 + 2a, 1 + 2b)$ are free pairs. So $m = 4$ satisfies the condition in (1) and the condition defining $M(G)$; the pairs in (1) are among those defining $M(G)$, so $m(G) \leq M(G) \leq 4$. Together with Theorem A this answers (Q1) and (Q2). $\square$

Remark 3.3 (The implication stated in the source). Suppose that $m(z) \leq K$ for every nonzero algebraic integer z , and let $L = \text{lcm}(1, \dots, K)$. Let $b_1 = 1 + a$ and $b_2 = 1 + b$ be bicyclic units of $\mathbb{Z}G$ with ab not nilpotent. Every nonzero eigenvalue λ of ab is a nonzero algebraic integer, being a root of the polynomial q in the proof of Theorem B. Some $k \leq K$ makes $k\lambda$ a free point; then $L\lambda = (L/k)k\lambda$ is a free point by Lemma 2.2(c), and Proposition 3.2 with $(m, n) = (1, L)$ shows that (b_1, b_2^L) is a free pair. Hence $m(G) \leq M(G) \leq L$, and the implication from (Q2) to (Q1) holds as stated in the source. The strong form of Theorem A, that kz is free for every $|k| \geq 4$, gives the bound 4 directly, since 4λ is then a free point; the proof of Theorem B avoids Galois conjugation by choosing an eigenvalue of modulus at least 1 from the start.

Remark 3.4 (Readings of the definitions). Let $b_1 = 1 + a$ and $b_2 = 1 + b$ be bicyclic units of $\mathbb{Z}G$.

(a) $\langle b_1, b_2 \rangle$ is nilpotent if and only if ab is nilpotent: one direction is Lemma 3.1; if ab is not nilpotent, then $\langle b_1, b_2 \rangle$ contains the free pair (b_1, b_2^4) and is not nilpotent. This is [GdR08, Corollary 3.1(3)].

(b) If $\langle b_1, b_2 \rangle$ is not nilpotent and $m \neq 0$, then b_1 and b_2^m do not commute: otherwise $m(ab - ba) = b_1 b_2^m - b_2^m b_1 = 0$, so $ab = ba$, as $\mathbb{Z}G$ is torsion-free, and $(ab)^2 = a^2 b^2 = 0$, against (a). By convention (ii), $\langle b_1, b_2^m \rangle$ is then free if and only if it is free and non-cyclic, if and only if (b_1, b_2^m) is a free pair.

(c) The $\mathbb{Z}$ -linear map $c \mapsto c^*$ with $g^* = g^{-1}$ for $g \in G$ is an anti-automorphism of $\mathbb{Z}G$, and $\hat{h}^* = \hat{h}$, since $\langle h^{-1} \rangle = \langle h \rangle$. It maps $\beta_{x,h}$ to $\gamma_{x^{-1}, h^{-1}}$, and so induces a bijection from the units of type β onto those of type γ . The map $c \mapsto (c^*)^{-1}$ is an automorphism of the unit group of $\mathbb{Z}G$; for units u, v it maps $\langle u, v \rangle$ onto $\langle u^*, v^* \rangle$, sending $u \mapsto (u^*)^{-1}$ and $v \mapsto (v^*)^{-1}$. Hence (u, v^m) is a free pair if and only if $(u^*, (v^*)^m)$ is, and $\langle u, v \rangle$ is nilpotent if and only if $\langle u^*, v^* \rangle$ is. So the condition of (1) for pairs

of type (β, β) holds for a given m if and only if it holds for pairs of type (γ, γ) , and with (a) and (b) the definition of $m(G)$ in [GdR08, §5] gives the number (1).

(d) Theorem B applies to all square-zero elements of $\mathbb{Z}G$, for example to the units $1 + (1 - h)x\hat{H}$ and $1 + \hat{H}x(1 - h)$, where H is a subgroup of G , $h \in H$, $x \in G$ and $\hat{H}$ is the sum of the elements of H [GdR08, §4]. Integrality enters only through the lower bound $|\lambda| \geq 1$.

4 Computed values

This section is not needed for Theorems A and B. Permutations act on $\{1, \dots, n\}$, and $(pq)(i) = p(q(i))$. The following variant of Lemma 2.1 decides some non-real eigenvalues; we did not trace its first appearance in the literature and include the proof.

Lemma 4.1. *Every $w \in \mathbb{C}$ with $|\operatorname{Im} w| > 2$ is a free point.*

Proof. Let $J(z) = z^{-1}$ on $\hat{\mathbb{C}}$; then $B_w^n = J T_{nw} J$ with $T_c(u) = u + c$. Let $X_1 = \{z \in \mathbb{C} : |\operatorname{Re} z| \geq \frac{1}{2}\} \cup \{\infty\}$ and $X_2 = \{z \in \mathbb{C} : |\operatorname{Re} z| < \frac{1}{2}, z \neq 0\}$. For $z \neq 0$ we have $\operatorname{Re} z \geq \frac{1}{2}$ if and only if $|z - 1| \leq |z|$, that is, $|1 - z^{-1}| \leq 1$, and $\operatorname{Re} z \leq -\frac{1}{2}$ if and only if $|1 + z^{-1}| \leq 1$. Hence $J(X_1)$ lies in $U = \{u : |u - 1| \leq 1\} \cup \{u : |u + 1| \leq 1\}$, a set of points with $|\operatorname{Im} u| \leq 1$ that contains $J(\infty) = 0$. Let $n \in \mathbb{Z} \setminus \{0\}$. If $z \in X_2$, then $|\operatorname{Re}(z + n)| > \frac{1}{2}$, so $A^n(X_2) \subseteq X_1$. If $z \in X_1$ and $u = J(z)$, then $|\operatorname{Im}(u + nw)| \geq |n| |\operatorname{Im} w| - |\operatorname{Im} u| > 1$, so $u + nw$ is a nonzero complex number outside U , and $B_w^n \cdot z = J(u + nw)$ is finite and nonzero; it does not lie in X_1 , since otherwise $u + nw = J(B_w^n \cdot z) \in J(X_1) \subseteq U$. Hence $B_w^n(X_1) \subseteq X_2$, and the ping-pong argument in the proof of Lemma 2.1 applies. $\square$

Proposition 4.2. $m(S_3) = 1$, $M(S_3) = 2$ and $m(S_4) = M(S_4) = 2$.

The values $m(S_3) = 1$ and $m(S_4) = 2$ are stated in the source; $m(S_3) = 1$ is the case $n = 3$ of the result for dihedral groups D_{2n} with n prime quoted in [GdR08, §5], $M(S_3) = 2$ is [GdR08, Example 5.2], and $m(S_4) = 2$ is [GdR08, Example 5.3]. The value $M(S_4) = 2$ is not stated there.

Proof. The exact program `check_note.py` (Section 5) lists the bicyclic units of both types: $6 + 6$ nontrivial ones in $\mathbb{Z}S_3$ and $156 + 156$ in $\mathbb{Z}S_4$, so that there are 157 of one type counting 1, as in [GdR08, Example 5.3]. Simultaneous conjugation by an element of G preserves the types, the nilpotency of ab and freeness, so the first unit of a pair runs over representatives of the conjugation orbits of its type ($1 + 1$ in S_3 , $8 + 8$ in S_4) and the second over all units of its type, for all four orders of types. In S_3 and S_4 every irreducible character χ has degree at most 3, so $\rho(a)^2 = 0$ forces $\rho(a)$ to have rank at most 1, and the only possible nonzero eigenvalue of $\rho(ab)$ is $\chi(ab)$ [GdR08, Corollary 3.3]. The program takes the set Λ of the nonzero values $\chi(ab)$ and then certifies it without using characters: with $p(t) = t^2 \prod_{\lambda \in \Lambda} (t - \lambda)$ it checks $p(ab) = 0$ in $\mathbb{Z}G$ and $(p/(t - \lambda))(ab) \neq 0$ for each $\lambda \in \Lambda$. Since $c \mapsto L_c$ is injective, this proves that Λ is exactly the set of nonzero eigenvalues of ab . All of them are nonzero integers, and ab is nilpotent if and only if $\Lambda = \emptyset$. For a pair with ab not nilpotent and $m \geq 1$, the pair $(1 + a, (1 + b)^m)$ is free if $|m\lambda| \geq 4$ for some $\lambda \in \Lambda$, by Lemma 2.1 and Proposition 3.2. Otherwise every eigenvalue of mab lies in $\{0, \pm 1, \pm 2, \pm 3\}$; none of these is a free point (Proposition 2.4; $B_0 = I$), and the pair is not free by [GdR08, Theorem 1.1]. So every pair is decided for every m ; the counts are in Table 1. Every pair is free at $m = 2$, the same-type pairs of $\mathbb{Z}S_3$ already at $m = 1$; non-free pairs at $m = 1$ are, for example, $\beta_{(1,2),(2,3)}$ and $\gamma_{(2,3),(1,2)}$ in $\mathbb{Z}S_3$, for which ab has the single nonzero eigenvalue -3 , and $\beta_{(2,3),(3,4)}$ and $\beta_{(1,2)(3,4),(1,3)}$ in $\mathbb{Z}S_4$, for which the nonzero eigenvalues are ± 2 . This gives the four values; by Remark 3.4 it does not matter whether $m(G)$ is computed over pairs of type (β, β) only or over all pairs of the same type, and the program computes both. $\square$

group	types	pairs	ab not nilp.	$\min \max \lambda $	not free at $m = 1$	free at $m = 2$
S_3	same	12	8	9	0	8
S_3	mixed	12	12	3	8	12
S_4	same	2,496	1,676	2	24	1,676
S_4	mixed	2,496	1,948	2	96	1,948

Table 1: Exact counts of `check_note.py`. “Same” means the type orders (β, β) and (γ, γ) together, “mixed” means (β, γ) and (γ, β) ; “pairs” counts the pairs examined, the first unit running over orbit representatives and the second over all units; $\min \max |\lambda|$ is the minimum, over the pairs with ab not nilpotent, of the largest modulus of an eigenvalue of ab . The two orders in each row give equal counts.

Exhaustive computations. The following statements were obtained by computer, with exact characteristic polynomials and numerically located roots; we label them as tested rather than proved. For the pairs of bicyclic units of $\mathbb{Z}S_5$ (3,780 nontrivial units of each type, 36 conjugation orbits per type; first unit over orbit representatives, second over all units), the characteristic polynomial of $\rho(ab)$ for every irreducible representation ρ was computed exactly by the programs with which the results were first obtained, for the type orders (β, β) and (β, γ) , and by the first two verification runs, for all four type orders, by three different exact methods (Section 5). The roots were located numerically, in double precision by the original programs and with 50 to 60 digits by these runs, and the cases with $|m\lambda| = 4$ were decided exactly. The third verification run, with a fourth exact method, found that every irreducible factor of these polynomials has degree 1 or 2 and decided every case by exact comparisons of integers; its counts agree with those that follow. Among the 230,500 same-type and 250,820 mixed pairs with ab not nilpotent (both type orders counted), the minimum of the largest modulus of an eigenvalue of ab is $\sqrt{3}$; at $m = 1$, 1,768 same-type and 2,672 mixed pairs are not free and 2,424 and 4,512 are undecided; at $m = 2$ every pair is free, by an integer eigenvalue with $|2\lambda| \geq 4$ or by Lemma 2.1 or Lemma 4.1 with a margin of at least 0.47. Since bicyclic units of $\mathbb{Z}S_4$ are bicyclic units of $\mathbb{Z}S_5$ of the same type, $m(S_5) \geq m(S_4)$. Hence $m(S_5) = M(S_5) = 2$, and $M(H) \leq 2$ for every non-Hamiltonian subgroup H of S_5 (tested).

A natural route to $m(G) \leq 2$ would be to show that ab always has an eigenvalue of modulus at least 2; the proof of Theorem B would then give $M(G) \leq 2$. The value $\sqrt{3}$ above shows that this fails in $\mathbb{Z}S_5$; here is an explicit pair, checked exactly.

Proposition 4.3. *In $\mathbb{Z}S_5$ let $1 + a = \beta_{x,h}$ with $x = (1, 2)$, $h = (2, 3)(4, 5)$, and $1 + b = \beta_{y,k}$ with $y = (1, 3, 4, 2, 5)$, $k = (1, 2)(3, 4)$. The minimal polynomial of ab is $t(t-1)(t+1)(t^2-2t+3)(t^2+2t+3)$. Hence ab is not nilpotent, its nonzero eigenvalues ± 1 and $\pm 1 \pm i\sqrt{2}$ have modulus 1 or $\sqrt{3}$, and nevertheless $(1 + a, (1 + b)^2)$ is a free pair.*

Proof. With $p(t) = t(t-1)(t+1)(t^2-2t+3)(t^2+2t+3) = t^7 + t^5 + 7t^3 - 9t$, the program `check_note.py` checks $p(ab) = 0$ in $\mathbb{Z}S_5$ and $(p/f)(ab) \neq 0$ for each of the five irreducible factors f of p ; so p is the minimal polynomial of ab . Its root $\lambda = 1 + i\sqrt{2}$ is an eigenvalue of ab with $\text{Im}(2\lambda) = 2\sqrt{2} > 2$, so 2λ is a free point by Lemma 4.1, and $(1 + a, 1 + 2b)$ is a free pair by Proposition 3.2. $\square$

For this pair the status at $m = 1$ depends on whether $1 + i\sqrt{2}$ is a free point, which we did not decide.

The computation for S_5 was repeated for twelve groups that are not isomorphic to subgroups of S_5 (Table 2): in floating point for the first eleven by the original programs; for all twelve by the second verification run, with exact characteristic polynomials of the regular representation and numerically located roots; and for all twelve by the third, with exact minimal polynomials of ab , which again have only irreducible factors of degree 1 or 2, so that every case was decided exactly. Every pair with ab

not nilpotent is free at $m = 2$, so $M(G) \leq 2$. In three of them the value at $m = 1$ is undecided. In D_{24} the undecided same-type pairs have nonzero eigenvalues exactly $\pm 2\sqrt{3}$, so $m(D_{24}) = 1$ if and only if $2\sqrt{3}$ is a free point; this agrees with the result of Jiménez López [Jim08] as reported in [GdR08, §5]. In $\text{GL}(2, 3)$ the undecided same-type pairs have nonzero eigenvalues ± 2 and $2 \pm i\sqrt{2}$, or ± 2 and $-2 \pm i\sqrt{2}$, so $m(\text{GL}(2, 3)) = 1$ if and only if $2 + i\sqrt{2}$ is a free point. In $C_7 \rtimes C_3$ the undecided pairs are of mixed type, with nonzero eigenvalues the roots of $t^2 - 7t + 14$ or of $t^2 + 7t + 14$, so $M(C_7 \rtimes C_3) = 1$ if and only if $\frac{1}{2}(7 + i\sqrt{7})$ is a free point. By Lemma 2.2(a), (b) the choice of sign and of conjugate does not matter. We did not decide whether these three points are free.

G	$ G $	$m(G)$	$M(G)$	G	$ G $	$m(G)$	$M(G)$
$C_3 \rtimes C_4$	12	1	1	D_{24}	24	1* or 2	2
Q_{16}	16	1	1	Heisenberg 3^{1+2}	27	1	1
$C_5 \rtimes C_4$, C_4 via C_2	20	1	1	$(C_3 \times C_3) \rtimes C_4$	36	1	2
$C_7 \rtimes C_3$	21	1	1* or 2	$C_5 \rtimes C_8$, C_8 via C_4	40	1	1
$C_3 \rtimes C_8$, C_8 via C_2	24	1	1	$C_5 \rtimes C_8$, C_8 via C_2	40	1	1
$\text{SL}(2, 3)$	24	1	1	$\text{GL}(2, 3)$	48	1* or 2	2

Table 2: Tested values for twelve groups. 1* or 2: the value is 1 if and only if $2\sqrt{3}$ (D_{24}), $2 + i\sqrt{2}$ ($\text{GL}(2, 3)$) or $\frac{1}{2}(7 + i\sqrt{7})$ ($C_7 \rtimes C_3$) is a free point. $C_3 \rtimes C_4$ and $C_5 \rtimes C_4$ are the dicyclic groups of orders 12 and 20; “via C_k ” means that the acting group operates through its quotient of order k , faithfully on the normal subgroup. In $(C_3 \times C_3) \rtimes C_4$ a generator of C_4 acts by $(x, y) \mapsto (-y, x)$, that is, faithfully.

So $\sup_G m(G) \in \{2, 3, 4\}$, and none of the tested groups needs an exponent larger than 2. Whether $m(G) \leq 2$ for every finite non-Hamiltonian group, the second half of [GdR08, Question 5.4], remains open; by Proposition 4.3 a proof cannot rest on the moduli of the eigenvalues alone.

5 Computations and reproducibility

Theorems A and B, Corollary C and all lemmas are proved without computation. The computations check the finite statements: Propositions 2.4, 4.2 and 4.3 exactly, and the statements on S_5 and on the groups of Table 2 at the level described in Section 4. The reproducibility package has five parts; its `README.md` gives the command lines and the running times.

The exact program of this note, `check_note.py` (standard library only, integer arithmetic, below one second). Part 1 checks Proposition 2.4 and the formulas for A^n and B_w^n ; Part 2 lists the bicyclic units of $\mathbb{Z}S_3$ and $\mathbb{Z}S_4$, certifies the eigenvalue sets as in the proof of Proposition 4.2, decides every pair for $m = 1, 2, 3, 4$, prints Table 1 and asserts the four values of Proposition 4.2; Part 3 checks Proposition 4.3. It reports no failure.

The programs with which the results were first obtained (`original/`): a library for permutation groups, group rings, bicyclic units and the characters of symmetric groups (Murnaghan–Nakayama rule); the relations of Proposition 2.4; all ordered pairs in $\mathbb{Z}S_3$ and $\mathbb{Z}S_4$ with exact spectra; the pairs in $\mathbb{Z}S_4$ and $\mathbb{Z}S_5$ with exact characteristic polynomials of $\rho(ab)$, obtained from exact power sums by Newton’s identities with an integrality check; the minimal polynomial of Proposition 4.3 by exact elimination; and the eleven groups of Table 2 other than the second $C_5 \rtimes C_8$, in floating point.

Two independent verification runs, each with its own programs and their recorded outputs, in `verification_run_A/` and `verification_run_B/`. The first computes the spectra in $\mathbb{Z}S_3$, $\mathbb{Z}S_4$ and $\mathbb{Z}S_5$ from six integer representations whose constituents are all irreducible representations, and cross-checks them with the regular representation. The second computes exact minimal polynomials by

Krylov sequences in $\mathbb{Z}S_3$ and $\mathbb{Z}S_4$, characteristic polynomials on four integral representations of S_5 by the Faddeev–LeVerrier method modulo several primes with Chinese remaindering, and characteristic polynomials of the regular representation for the groups of Table 2, given by multiplication tables that are checked to be groups; it also tests the free points $\pm 1, \pm 2, \pm 3$ by a search for short relations, the explicit pairs of [GdR08, Examples 5.2 and 5.3], the units of three Hamiltonian groups, and general square-zero elements of $\mathbb{Z}S_3$.

A *third independent verification run* (`independent_run_2/`), with its programs and their recorded outputs. It builds Young’s natural representations of S_3 , S_4 and S_5 over $\mathbb{Z}$ from standard polytabloids and computes, for every pair, the coefficients of the characteristic polynomial of $\rho(ab)$ as sums of principal minors (in $\mathbb{Z}S_4$ also over all ordered pairs, which checks the reduction to orbit representatives); for the twelve groups of Table 2, built as permutation, matrix or affine groups and compared with all subgroups of S_5 , it computes the minimal polynomial of ab exactly. It also checks Proposition 2.4 and Proposition 4.3, the latter in both product conventions, and contains two detector tests.

On 14 October 2026 every program of the package was run again from the layout of the package, and the programs of the first four parts once more by the third verification run from the extracted archive `source.zip` (Python 3.13.5, with NumPy, SymPy and mpmath for all programs except `check_note.py`; at most 72 seconds per program). All outputs agree with the recorded ones except for lines that report running times.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs.

The first run re-derived every step of the proofs of Theorems A and B, of Proposition 2.4, Lemma 4.1 and Proposition 4.3, read the source and Sections 1 to 5 of [GdR08], and recomputed with its own exact programs the published values, the value $M(S_4) = 2$, the data for S_5 and the minimal polynomial of Proposition 4.3; its two methods for the spectra agree. The second run re-derived the proofs, recomputed every finite statement, including the counts of Table 1, with three exact methods of its own, replaced the floating-point computation for the further groups by one with exact characteristic polynomials and added a twelfth group, found that the value at $m = 1$ is undecided in three groups rather than only in D_{24} , ran the original programs again and reproduced their outputs, searched the literature and checked the bibliography. Neither run found a mathematical error. The corrections they required concern the description of the corpus record (the matrix, “Equivalently”, the pages), the remark on the implication stated in the source, which is now proved (Remark 3.3), the equality of the two definitions of $m(G)$ (Remark 3.4), the citation of $m(S_3) = 1$, the numbering of the problems of [Jes21], the list of undecided groups, the restriction to non-Hamiltonian subgroups of S_5 , the journal version of [KK22], the list of sources that could not be read, and two details in the description of the computations; all were made. When this note was written, the program `check_note.py` was added; it certifies the eigenvalue sets of Proposition 4.2 without using characters, and its counts agree with those of the earlier programs.

A third independent run, also AI-assisted, examined the final text. It read Problem 28 of [GdR07], Sections 1 to 5 of [GdR08], the cited passages of [CGS17, KK22, Jes21] and the abstracts of [Sal99, MS97, Jim08] again, re-derived every proof of the note line by line, and checked every citation against these texts and every DOI with Crossref. With programs of its own it recomputed every finite statement of Sections 2 and 4: Propositions 2.4, 4.2 and 4.3, Table 1 (also over all ordered pairs of $\mathbb{Z}S_4$), the statements on S_5 , and Table 2 with the three undecided spectra. Every irreducible factor of the polynomials that occurred has degree 1 or 2, so this run decided every case by exact comparisons of integers; all its numbers agree with those of the note. It ran every program of the package again from the archive `source.zip` and reproduced the recorded outputs, and it repeated the literature search. It found no mathematical error. It required seven corrections of the text, which were made:

the attribution of Lemma 2.1 (Section 2), the statement in the abstract that only the first question is part of [GdR08, Question 5.4], two logical connectives (the end of Section 1.2 and Corollary C), the description of which programs computed which type orders in $\mathbb{Z}S_5$ and of which runs located roots numerically (Section 4), and the action in $(C_3 \times C_3) \rtimes C_4$ (Table 2). Its programs and outputs are in `independent_run_2/`. No proof depends on a computation.

Scope and priority. The note answers both questions of [GdR07] as posed there, with the bound 4, which is sharp for (Q2). It does not answer the second half of [GdR08, Question 5.4], whether $m(G) \leq 2$ for every finite non-Hamiltonian group; it only shows $\sup_G m(G) \in \{2, 3, 4\}$. The questions, the definitions and the values $m(S_3) = 1$, $M(S_3) = 2$ and $m(S_4) = 2$ are due to Gonçalves and del Río [GdR07, GdR08]. The result is an elementary combination of known facts: the theorem of Sanov and Brenner [San47, Bre55], the invariance of freeness under Galois conjugation [CGS17, KK22], and the two-dimensional reduction in the proof of [GdR08, Theorem 1.1]; exponents for each pair separately are due to Salwa [Sal99] and to [GdR08, Corollary 3.1(2)], and dihedral groups were treated by Jiménez López [Jim08]. No priority is claimed.

What was read: Problem 28 of [GdR07], on p. 3176 of the open report on the publisher’s site; [GdR08] in arXiv version 2, Sections 1 to 5 and the references; [KK22] in the arXiv version, Section 1 up to the statements of the main results, and the references; [CGS17] in arXiv version 4, the beginning of Section 1 and the passage of Section 2 on Galois conjugation; [Jes21] in the arXiv version, Proposition 3.3, Theorem 3.5 and the statements of all its problems; and the abstracts of [Sal99], [MS97] and [Jim08], the last of which states the results on dihedral groups quoted in [GdR08, §5]. We did not read [San47, Bre55] nor the full texts of [MS97, Jim08]; what we say about these four works beyond their abstracts is taken from [CGS17, KK22, GdR07, GdR08]. Four later sources that could contain the observation were not read, because their texts are not openly accessible: the survey [GdR13], Chapter 14 (“Free Groups”) of [JdR16], [Gon17] and [Kim13]. If one of them contains it, priority belongs there.

The searches were made on 14 October 2026, when the results were first obtained and by the three verification runs: queries to arXiv (bicyclic units, free pairs, free points, parabolic generators, and the authors of the problem), to zbMATH, to Crossref, and to OpenAlex and Semantic Scholar for the works citing [GdR08], [GdR07], [Sal99], [GdR13], [Kim13] and [Jim08] and for later works of the two proposers, and eight web searches. None of the works found states a uniform bound for $m(G)$, $M(G)$ or $m(z)$; for the 54 works that cite one of these six, the titles and the 35 abstracts available in OpenAlex were read, and the full texts only for the open sources listed above. No answer was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. All DOIs were checked with Crossref, and the pages of [GdR08] with zbMATH.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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