

# The Quantum Symmetriser of a Yang–Baxter Solution Need Not Be a Quasi-Isomorphism

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## Abstract

For a set-theoretic solution  $(X, \sigma)$  of the Yang–Baxter equation, Farinati and García Galofre showed that a comparison map from braided (co)homology to the Hochschild (co)homology of the structure algebra  $A = kM(X, \sigma)$  can be chosen to be the quantum symmetriser of  $-\sigma$ , and that it factors through a complex  $A \otimes \mathfrak{B} \otimes A$ , where  $\mathfrak{B}$  is the Nichols algebra of  $-\sigma$ . They asked whether this complex is a resolution of  $A$ , and Lebed asked in a 2019 Oberwolfach report whether the quantum symmetriser is bijective for general solutions; the answer is known to be positive for involutive (characteristic zero) and idempotent solutions. We show that it is negative in general. For the permutation rack  $X = \mathbb{Z}/4$ ,  $x \triangleleft y = x + 1$ , a bijective non-degenerate solution, we give an explicit 2-cycle of weight 3 in  $A \otimes \mathfrak{B} \otimes A$  that is not a boundary; it comes from a cubic quantum Serre relation in  $\mathfrak{B}$ . With coefficients in the module on which  $X$  acts by 0, the image of the quantum symmetriser misses part of  $HH_3(A; k)$  for every quotient of the braided complex. We show that  $A \otimes \mathfrak{B} \otimes A$  is a resolution if and only if  $A$  is Koszul and  $\mathfrak{B}$  is the Koszul dual coalgebra of  $A$ , and that in characteristic zero every finite permutation rack whose permutation has a cycle of length divisible by 4 (resp. 3) fails in weight 3 (resp. 4). For finite non-degenerate solutions that are not involutive and have a finite-dimensional Nichols algebra, such as the dihedral quandle of order 3, a negative answer already follows from a remark of Farinati and García Galofre and a theorem of Jespers, Kubat and Van Antwerpen. By exact computation, 13 of the 29 isomorphism classes of bijective solutions on three points fail. This is an unrefereed note.

## 1 Introduction

Let  $k$  be a field. A *solution* is a pair  $(X, \sigma)$  of a set  $X$  and a bijection  $\sigma: X \times X \rightarrow X \times X$  satisfying  $(\sigma \times \text{id})(\text{id} \times \sigma)(\sigma \times \text{id}) = (\text{id} \times \sigma)(\sigma \times \text{id})(\text{id} \times \sigma)$ . Its *structure monoid*  $M$  is generated by  $X$  subject to  $xy = zt$  whenever  $\sigma(x, y) = (z, t)$ , and  $A = kM$ . Farinati and García Galofre [FG16] realised the braided (co)homology of solutions, which contains rack, quandle and biquandle (co)homology, by a differential graded bialgebra  $\mathcal{D} = \mathcal{D}(X, \sigma)$ : for an  $A$ -bimodule  $N$ , the complexes  $N \otimes_{A^e} \mathcal{D}$  and  $\text{Hom}_{A^e}(\mathcal{D}, N)$  are the braided (co)homology complexes with coefficients in  $N$ . As  $\mathcal{D}$  is degreewise a free  $A$ -bimodule, there are comparison maps to the Hochschild (co)homology of  $A$  [FG16, Cor. 27]. One of them is a morphism  $f: \mathcal{D} \rightarrow (A \otimes TA \otimes A, b')$  into the bar resolution given on generators by the quantum symmetriser QS of  $-\sigma$ , and  $\text{Im } f = A \otimes \mathfrak{B} \otimes A$ , where  $\mathfrak{B}$  is the Nichols algebra of  $(kX, -\sigma)$  [FG16, Thm. 32, Cors. 42–43]. They asked:

**Problem 1.1** ([FG16, Question 44]). *Is  $\text{Im } f = A \otimes \mathfrak{B} \otimes A$  a resolution of  $A$  as an  $A$ -bimodule, that is, is  $(A \otimes \mathfrak{B} \otimes A, b')$ , augmented by the multiplication  $A \otimes A \rightarrow A$ , acyclic?*

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A positive answer would give, for every bimodule  $N$ , a quotient  $N \otimes_{A^e} (\mathcal{D}/\ker f)$  of the braided complex on which QS induces an isomorphism onto  $HH_*(A; N)$ . The answer is positive for involutive  $\sigma$  if  $\mathbb{Q} \subseteq k$ , and for the flip and  $\sigma = \text{id}$  in every characteristic [FG16, Prop. 24 and the remark after Question 44]. For idempotent braidings, Lebed proved that QS induces quasi-isomorphisms from a quotient of the braided complex, the critical complex, to the Hochschild complex, with arbitrary coefficients [Leb17, Thm. 6]; she noted that it is natural to ask whether QS can be turned into a quasi-isomorphism in general [Leb17, §6] and later recorded that this is unknown [Leb20, §6]; see also [Leb18] and D. Yang’s related question [Yan16, Question 7.5]. In the report of a 2019 Oberwolfach mini-workshop she asks:

**Problem 1.2** ([Leb19, Problem 2, p. 3237]). *“Is the quantum symmetriser QS bijective for general solutions?”*

This is record OWR-17294-017 of the UnsolvedMath dataset [ula26]. Read literally it has trivial negative answers: for finite  $X$  the chain map  $QS_2 = 1 - \sigma$  kills the sums over  $\sigma$ -orbits, and the critical complex, designed for idempotent braidings, fails already for the flip on  $n \geq 2$  points (with trivial coefficients it has zero differential and  $n(n-1)$  generators in degree 2, while  $\dim HH_2 = n(n-1)/2$ ). Following [FG16, Leb17, Leb20] we read it as the question whether QS becomes a quasi-isomorphism on a suitable quotient of the braided complex; Problem 1.1 is the coefficient-free form. We show that the answer is negative. Below,  $K_n \subseteq (kX)^{\otimes n}$  is the Koszul dual coalgebra of  $A$  (Section 2).

**Theorem 1.3.** *Let  $\text{char } k \neq 2$ ,  $X = \mathbb{Z}/4$  and  $\sigma(x, y) = (y, x + 1)$ , the solution of the permutation rack  $x \triangleleft y = x + 1$ . It is bijective, left and right non-degenerate, and neither involutive nor idempotent.*

- (a) *The complex  $A \otimes \mathfrak{B} \otimes A$  is not acyclic: the 2-cycles  $z_{\text{re}}$  and  $z_{\text{im}}$  of weight 3 in (6), which have integer coefficients, are linearly independent modulo boundaries. So Problem 1.1 has a negative answer.*
- (b) *Let  $k_0$  be  $k$  with  $X$  acting by 0 on both sides, and let  $Y \rightarrow C_*(A; k_0)$  be a chain map into the Hochschild complex whose image in degree 3 lies in  $QS_3((kX)^{\otimes 3})$ , for instance the map induced by QS on a quotient or a subcomplex of the braided chain complex with coefficients in  $k_0$ . Then  $H_3(Y) \rightarrow HH_3(A; k_0)$  is not surjective. Dually,  $QS^*$  induces a non-injective map from  $HH^3(A; k_0)$  to the cohomology of every subcomplex of the braided cochain complex that contains the image of  $QS^*$ .*
- (c) *For  $k = \mathbb{Q}$ , exact computation gives Table 1,  $H_2(A \otimes \mathfrak{B} \otimes A) \cong \mathbb{Q}^2$  in weight 3,  $H_3 \cong \mathbb{Q}^3$  in weight 4, and no other homology in weights at most 4. The weight-3 part of  $HH_3(A; k_0)$  is  $K_3$ , of dimension 49, and the image of QS in it is  $\mathfrak{B}_3$ , of dimension 47.*

Part (b) covers the strongest reading of Problem 1.2: no choice of quotient helps. The failure is explained by a criterion (Proposition 2.3):  $A \otimes \mathfrak{B} \otimes A$  is a resolution if and only if  $A$  is Koszul and  $\mathfrak{B}_n = K_n$  for all  $n$ . For  $X = \mathbb{Z}/4$  the algebra  $A$  is Koszul, but  $\mathfrak{B}$  has a cubic quantum Serre relation, so  $\mathfrak{B}_3 \neq K_3$ . The same mechanism applies to many permutation racks.

**Theorem 1.4.** *Let  $\text{char } k = 0$ , let  $\varphi$  be a permutation of a finite set  $X$  and  $\sigma(x, y) = (y, \varphi(x))$ , the solution of the permutation rack  $x \triangleleft y = \varphi(x)$ . If  $\varphi$  has a cycle of length divisible by 4, then  $H_2(A \otimes \mathfrak{B} \otimes A)$  has a non-zero class of weight 3. If  $\varphi$  has a cycle of length divisible by 3, then  $H_3(A \otimes \mathfrak{B} \otimes A)$  has a non-zero class of weight 4.*

With exact computations for  $\mathbb{Z}/5$  and  $\mathbb{Z}/7$  (Table 2), the cyclic racks  $\mathbb{Z}/m$ ,  $x \triangleleft y = x + 1$ , fail for all  $3 \leq m \leq 8$ . When  $\mathfrak{B}$  is finite-dimensional, a negative answer follows from published results: Farinati and García Galofre remark after their Question 44 that a positive answer with  $\dim \mathfrak{B} < \infty$

forces  $A$  to have finite global dimension, and Jespers, Kubat and Van Antwerpen proved that for a finite bijective left non-degenerate solution this happens only for involutive solutions [JKV19, arXiv v2, Thm. 4.6].

**Corollary 1.5** ([FG16, JKV19]). *Let  $(X, \sigma)$  be a finite bijective left non-degenerate solution that is not involutive. If  $\dim \mathfrak{B} < \infty$ , then  $A \otimes \mathfrak{B} \otimes A$  is not a resolution of  $A$ . This applies to the dihedral quandle  $R_3$ , for which  $\mathfrak{B}$  is the 12-dimensional Fomin–Kirillov algebra.*

For  $R_3$  the failure is also equivalent, by Proposition 2.3, to the known fact that  $A = \text{FK}_3^!$  is not Koszul [Roo99, WZ19] (Section 5). Section 6 reports a census of the bijective solutions on two and three points.

## 2 The complex $A \otimes \mathfrak{B} \otimes A$

Let  $(X, \sigma)$  be a finite solution and  $V = kX$ ; we write  $\sigma$  also for its linear extension to  $V \otimes V$ . The subspace  $R = \text{Im}(1 - \sigma) \subseteq V \otimes V$  is spanned by the  $xy - zt$  with  $(z, t) = \sigma(x, y)$ , so  $A = T(V)/(R)$  is quadratic. It is graded by word length, called *weight*, with  $A_0 = k$ ,  $A_1 = V$  and  $\bar{A} = \bigoplus_{p \geq 1} A_p$ ; as  $A = kM$ , the monoid classes of words of length  $n$  form a basis of  $A_n$ .

Put  $c = -\sigma$  and  $c_i = \text{id}^{\otimes(i-1)} \otimes c \otimes \text{id}^{\otimes(n-i-1)}$ . For  $w \in S_n$  with a reduced expression  $w = s_{i_1} \cdots s_{i_l}$  let  $T_w = c_{i_1} \cdots c_{i_l}$ , which does not depend on the expression. The quantum symmetriser is  $\text{QS}_n = \sum_{w \in S_n} T_w$ , and  $\mathfrak{B}_n = \text{QS}_n(V^{\otimes n}) \subseteq V^{\otimes n}$ . The Nichols algebra of  $(V, c)$  is  $\bigoplus_n T^n(V)/\ker \text{QS}_n$  [AS02, §2.3], so  $\mathfrak{B}_n$  is isomorphic to its degree- $n$  part; see also [FG16, Cor. 43]. Let  $K_0 = k$ ,  $K_1 = V$  and  $K_n = \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes(n-2-i)}$  for  $n \geq 2$ ;  $K_n$  is dual to the degree- $n$  part of the quadratic dual  $A^!$  [PP05, Ch. 1].

**Lemma 2.1.** *For  $n \geq 2$ ,  $\mathfrak{B}_n \subseteq K_n$  and  $\mathfrak{B}_n \subseteq (V \otimes \mathfrak{B}_{n-1}) \cap (\mathfrak{B}_{n-1} \otimes V)$ .*

*Proof.* Fix  $1 \leq i \leq n-1$ . Every  $w \in S_n$  can be written uniquely as  $w = s_i^\epsilon w'$  with  $\epsilon \in \{0, 1\}$  and  $w'$  of minimal length in  $\{w', s_i w'\}$ , and then  $\ell(w) = \epsilon + \ell(w')$ . Hence  $\text{QS}_n = (1 + c_i) \sum_{w'} T_{w'}$ , and  $1 + c_i = 1 - \sigma_i$  has image  $V^{\otimes(i-1)} \otimes R \otimes V^{\otimes(n-i-1)}$ ; so  $\mathfrak{B}_n \subseteq K_n$ . Similarly,  $w = uv$  uniquely with  $u$  in the subgroup  $S_{n-1}$  permuting the first  $n-1$  positions,  $v$  of minimal length in  $S_{n-1}v$ , and  $\ell(w) = \ell(u) + \ell(v)$ ; hence  $\text{QS}_n = (\text{QS}_{n-1} \otimes \text{id}) \circ \sum_v T_v$  and  $\mathfrak{B}_n \subseteq \mathfrak{B}_{n-1} \otimes V$ . The other inclusion is symmetric.  $\square$

Let  $(A \otimes \bar{A}^{\otimes n} \otimes A, b')$  be the normalised bar resolution of  $A$ , and consider  $C_n = A \otimes \mathfrak{B}_n \otimes A$  inside it, using  $V^{\otimes n} \subseteq \bar{A}^{\otimes n}$ . By Lemma 2.1 the inner terms of  $b'$  vanish on  $C_n$  (they multiply two adjacent factors, and  $R$  maps to 0 in  $A_2$ ), so

$$d(a \otimes v_1 \cdots v_n \otimes a') = av_1 \otimes v_2 \cdots v_n \otimes a' + (-1)^n a \otimes v_1 \cdots v_{n-1} \otimes v_n a', \quad (1)$$

extended linearly in the middle factor, and  $d(C_n) \subseteq C_{n-1}$ . Thus  $C$  is a subcomplex of the bar resolution, augmented by the multiplication  $C_0 = A \otimes A \rightarrow A$ , and it is graded by weight ( $a \otimes t \otimes a'$  with  $a \in A_p$ ,  $t \in \mathfrak{B}_n$ ,  $a' \in A_q$  has weight  $p + n + q$ ). In [FG16] the unnormalised bar resolution is used; the image of  $f$  lies in  $\bigoplus_n A \otimes V^{\otimes n} \otimes A$ , on which the projection to the normalised resolution is injective, so  $(\text{Im } f, b')$  is isomorphic to  $C$  [FG16, Thm. 32, Cors. 42–43]. Let  $H_A(t) = \sum_n \dim A_n t^n$  and  $H_{\mathfrak{B}}(t) = \sum_n \dim \mathfrak{B}_n t^n$ .

**Lemma 2.2** (Euler characteristic). *Let  $\chi_w$  be the Euler characteristic of the weight- $w$  part of the augmented complex  $C \rightarrow A$ . Then  $H_A(t)^2 H_{\mathfrak{B}}(-t) - H_A(t) = \sum_w \chi_w t^w$ . So if  $C \rightarrow A$  is exact, then  $H_A(t) H_{\mathfrak{B}}(-t) = 1$ ; and if  $H_A(t) H_{\mathfrak{B}}(-t) = 1 + ct^w + O(t^{w+1})$  with  $c \neq 0$ , then it is not exact in weight  $w$ .*

*Proof.* The weight- $w$  part  $0 \rightarrow C_{w,w} \rightarrow \cdots \rightarrow C_{0,w} \rightarrow A_w \rightarrow 0$  is a finite complex of finite-dimensional spaces, and  $\dim C_{n,w}$  is the coefficient of  $t^w$  in  $t^n H_A(t)^2 \dim \mathfrak{B}_n$ . As  $H_A(0) = 1$ , the lowest non-zero coefficient of  $H_A(t)H_{\mathfrak{B}}(-t) - 1$  equals that of  $\sum_w \chi_w t^w$ .  $\square$

We call  $A$  Koszul if  $\mathrm{Tor}_n^A(k, k)$  is concentrated in weight  $n$  for all  $n$ , where  $k = A/\bar{A}$ ; equivalently, the Koszul complex  $A \otimes K_{\bullet} \rightarrow k$ , with differential  $a \otimes v_1 \cdots v_n \mapsto av_1 \otimes v_2 \cdots v_n$ , is exact [Pri70], [PP05, Ch. 2].

**Proposition 2.3.** *The augmented complex  $C = A \otimes \mathfrak{B} \otimes A \rightarrow A$  is a resolution if and only if  $A$  is Koszul and  $\mathfrak{B}_n = K_n$  for all  $n$ .*

*Proof.* (F1) Computing Tor with the normalised bar resolution,  $k \otimes_A (-) \otimes_A k$  gives  $(\bar{A}^{\otimes \bullet}, b)$  with  $b = \sum_{i=1}^{n-1} (-1)^i \mu_i$ , where  $\mu_i$  multiplies the factors  $i$  and  $i+1$ . In weight  $n$  this is  $V^{\otimes n}$  in degree  $n$  and zero in degree  $n+1$ , and the kernel of  $b$  on  $V^{\otimes n}$  is  $K_n$ , as the terms for different  $i$  lie in different summands and  $\ker(V \otimes V \rightarrow A_2) = R$ . So  $\mathrm{Tor}_n^A(k, k)_n = K_n$ , represented by  $K_n \subseteq V^{\otimes n}$ . (F2) Let  $P \rightarrow A$  be an augmented complex of right  $A$ -modules  $P_n = U_n \otimes A$ , with weight-graded  $U_n$  and a right  $A$ -linear weight-preserving differential. If  $P \otimes_A k$  is exact, so is  $P$ : the subcomplexes  $F_j = \bigoplus_n U_n \otimes A_{\geq j}$  have subquotients  $F_j/F_{j+1} = (P \otimes_A k) \otimes A_j$ , and in each weight the filtration is finite.

If  $C \rightarrow A$  is exact, it is an exact complex of free right  $A$ -modules, bounded on the right, hence split, so  $C \otimes_A k \rightarrow k$  is a free resolution of the left module  $k$ . By (1),  $k \otimes_A C \otimes_A k = \mathfrak{B}_{\bullet}$  has zero differential, so  $\mathrm{Tor}_n^A(k, k) \cong \mathfrak{B}_n$  lies in weight  $n$  and  $A$  is Koszul. The inclusion of  $C$  into the bar resolution lifts  $\mathrm{id}_A$ , so it induces an isomorphism on  $\mathrm{Tor}^A(k, k)$ ; after  $k \otimes_A (-) \otimes_A k$  it is the inclusion  $\mathfrak{B}_n \subseteq V^{\otimes n}$ , so by (F1)  $\mathfrak{B}_n = K_n$ . Conversely, if  $\mathfrak{B} = K$ , then  $C \otimes_A k$  is the Koszul complex, which is exact when  $A$  is Koszul; by (F2) with  $U_n = A \otimes \mathfrak{B}_n$ ,  $C \rightarrow A$  is exact.  $\square$

The criterion is elementary and presumably known to experts; we state it because it explains all examples below. Let  $k_0 = A/\bar{A}$ , that is,  $k$  with  $X$  acting by 0 on both sides.

**Lemma 2.4.** *The normalised Hochschild complex  $C_*(A; k_0)$  is  $(\bar{A}^{\otimes \bullet}, b)$ , so  $HH_n(A; k_0) = \mathrm{Tor}_n^A(k, k)$ , with weight- $n$  part  $K_n$ . The braided chain complex with coefficients in  $k_0$  is  $T(V)$  with zero differential, and QS maps  $v \in V^{\otimes n}$  to  $\mathrm{QS}_n(v) \in \mathfrak{B}_n$ .*

*Proof.* With coefficients  $k_0$  the outer terms of the Hochschild differential vanish; now use (F1). In [FG16, §§2,4] and [Leb17, Thm. 2] the braided differential is a signed sum of terms in each of which one letter, braided to an end of the word, acts on the coefficients; for  $k_0$  all of them vanish. The description of QS is [FG16, Cor. 43] and [Leb17, Thm. 5].  $\square$

### 3 The rack $\mathbb{Z}/4$

Let  $X = \mathbb{Z}/4$ ,  $\sigma(x, y) = (y, x+1)$  and  $\mathrm{char} k \neq 2$ . Both sides of the braid relation send  $(x, y, z)$  to  $(z, y+1, x+2)$ ;  $\sigma$  is bijective, left and right non-degenerate, has no fixed points, and  $\sigma^2(x, y) = (x+1, y+1)$ , so it is neither involutive nor idempotent. Let  $L = k(i)$  with  $i^2 = -1$  (so  $L = k$  if  $k$  contains a square root of  $-1$ ),  $V_L = V \otimes_k L$ , and  $u_j = \sum_{x \in \mathbb{Z}/4} i^{-jx} x \in V_L$  for  $j \in \mathbb{Z}/4$ ; as 4 is invertible, the  $u_j$  form a basis. The objects of Section 2 for  $V_L$  arise from those for  $V$  by extension of scalars, since  $R$ ,  $\mathrm{QS}_n$  and  $d$  are defined over  $\mathbb{Z}$ :  $A_L = A \otimes_k L$ ,  $\mathfrak{B}_{n,L} = \mathfrak{B}_n \otimes_k L$ ,  $K_{n,L} = K_n \otimes_k L$  and  $C_L = C \otimes_k L$ , so  $H_*(C_L) = H_*(C) \otimes_k L$ . With  $F(x) = x+1$  we have  $Fu_j = i^j u_j$  and  $\sigma(v \otimes w) = w \otimes F(v)$ , so

$$\sigma(u_a \otimes u_b) = i^a u_b \otimes u_a, \quad (1 - \sigma)(u_a \otimes u_b) = u_a u_b - i^a u_b u_a. \quad (2)$$

So  $c = -\sigma$  is of diagonal type, and  $A_L$ ,  $\mathfrak{B}_L$ ,  $K_{n,L}$  and  $C_L$  are graded by the number of occurrences of each  $u_j$  (the *multidegree*); let  $\alpha_j$  be the multidegree of  $u_j$ . Up to (6) we work over  $L$  and omit the index  $L$ .

**Lemma 3.1.**  $R_L$  is spanned by the  $u_a u_b$  with  $(a, b) \notin \{(0, 0), (1, 3), (3, 1)\}$  and by  $u_1 u_3 - i u_3 u_1$ . In particular  $u_0 u_1 = u_1 u_0 = u_1 u_1 = 0$  in  $A$ , and  $A_2$  has the basis  $u_0^2, u_3 u_1$ .

*Proof.* If  $a = b$ , (2) gives  $(1 - i^a)u_a^2$ , and  $1 - i^a$  is invertible for  $a \neq 0$  as  $(1 - i)(1 + i) = 2$ . If  $a \neq b$ , the elements for  $(a, b)$  and  $(b, a)$  span  $\langle u_a u_b, u_b u_a \rangle$  if  $1 - i^{a+b} \neq 0$ , that is, if  $a + b \neq 0$ , and the line of  $u_1 u_3 - i u_3 u_1$  if  $\{a, b\} = \{1, 3\}$ .  $\square$

Let  $w_1 = u_1 u_1 u_0$ ,  $w_2 = u_1 u_0 u_1$ ,  $w_3 = u_0 u_1 u_1$ , which span the  $(\alpha_0 + 2\alpha_1)$ -part  $W$  of  $V_L^{\otimes 3}$ . By (2),  $c \otimes \text{id}$  sends  $w_1, w_2, w_3$  to  $-i w_1, -i w_3, -w_2$ , and  $\text{id} \otimes c$  sends them to  $-i w_2, -w_1, -i w_3$ . Summing the six products gives the matrix of  $\text{QS}_3$  on  $W$ , with  $\text{QS}_3(w_j)$  in column  $j$ :

$$\text{QS}_3|_W = \begin{pmatrix} 1 - i & -1 + i & 1 - i \\ -1 - i & 2 & -1 + i \\ -1 + i & -1 - i & 1 - i \end{pmatrix}. \quad (3)$$

The form  $\ell(a_1 w_1 + a_2 w_2 + a_3 w_3) = i a_1 + (1 + i) a_2 + a_3$  vanishes on the three columns. Since  $\text{QS}_3$  preserves multidegrees,  $\mathfrak{B}_3 \cap W = \text{QS}_3(W) \subseteq \ker \ell$ , and as  $\ell(w_3) = 1$ ,

$$w_3 = u_0 u_1 u_1 \notin \mathfrak{B}_3, \quad \text{while} \quad W \subseteq K_3, \quad (4)$$

because every adjacent pair in  $w_1, w_2, w_3$  lies in  $R$  by Lemma 3.1. The kernel of (3) is spanned by  $\kappa = w_1 + (1 + i)w_2 + i w_3 = (\text{ad}_c u_1)^2(u_0)$ , a quantum Serre element, whose vanishing in  $\mathfrak{B}$  is predicted by [AS02, Lemma 3.7(b)] since  $q_{11}q_{10}q_{01} = 1$  for  $q_{ab} = -i^a$ . Consider

$$z = u_0 \otimes u_1 u_1 \otimes 1 - 1 \otimes u_0 u_1 \otimes u_1 \in A \otimes \mathfrak{B}_2 \otimes A, \quad \mathfrak{B}_2 = R. \quad (5)$$

*Proof of Theorem 1.3.* (a) By (1) and Lemma 3.1,  $dz = u_0 u_1 \otimes u_1 \otimes 1 + u_0 \otimes u_1 \otimes u_1 - u_0 \otimes u_1 \otimes u_1 - 1 \otimes u_0 \otimes u_1 u_1 = 0$ . The weight-3 part of  $C_3$  is  $1 \otimes \mathfrak{B}_3 \otimes 1$ , and  $d(1 \otimes t \otimes 1) = \theta(t) - \rho(t)$  with  $\theta(v_1 v_2 v_3) = v_1 \otimes v_2 v_3 \otimes 1 \in A_1 \otimes V^{\otimes 2} \otimes A_0$  and  $\rho(v_1 v_2 v_3) = 1 \otimes v_1 v_2 \otimes v_3 \in A_0 \otimes V^{\otimes 2} \otimes A_1$ . These lie in different summands and  $\theta$  is injective. As  $z = \theta(w_3) - \rho(w_3)$ , an equation  $z = d(1 \otimes t \otimes 1)$  would force  $t = w_3 \in \mathfrak{B}_3$ , contradicting (4). So  $[z] \neq 0$  in  $H_2(C_L)$ . These computations are polynomial identities in  $i$ , valid for both square roots of  $-1$ ; with  $-i$ , that is, with  $u_3$  in place of  $u_1$ , they show that  $\bar{z} = u_0 \otimes u_3 u_3 \otimes 1 - 1 \otimes u_0 u_3 \otimes u_3$  is a non-trivial class of multidegree  $\alpha_0 + 2\alpha_3$ . As  $C$  is multigraded,  $[z]$  and  $[\bar{z}]$  are linearly independent. Put  $a = x_0 - x_2$  and  $b = x_1 - x_3$ , so that  $u_1 = a - ib$ ,  $u_3 = a + ib$  and  $u_0 = x_0 + x_1 + x_2 + x_3$ . Then  $z = z_{\text{re}} + i z_{\text{im}}$  and  $\bar{z} = z_{\text{re}} - i z_{\text{im}}$ , where

$$\begin{aligned} z_{\text{re}} &= u_0 \otimes (aa - bb) \otimes 1 - 1 \otimes u_0 a \otimes a + 1 \otimes u_0 b \otimes b, \\ z_{\text{im}} &= -u_0 \otimes (ab + ba) \otimes 1 + 1 \otimes u_0 a \otimes b + 1 \otimes u_0 b \otimes a \end{aligned} \quad (6)$$

have integer coefficients. They are linearly independent in  $H_2(C)$ : a linear relation between their classes with coefficients in  $k$  would hold in  $H_2(C_L)$ , where  $[z_{\text{re}}] = ([z] + [\bar{z}])/2$  and  $[z_{\text{im}}] = ([z] - [\bar{z}])/(2i)$  are linearly independent. The identities used can also be checked in the monoid:  $A_2$  has the basis  $P, Q$ , where  $P$  is the class of the pairs  $(x, y)$  with  $y - x \in \{0, 1\}$  and  $Q$  that of the pairs with  $y - x \in \{2, 3\}$ , and  $u_0 a = u_0 b = 0$ ,  $aa = bb = 2P - 2Q$ ,  $ab + ba = 0$  in  $A_2$ .

(b) By Lemma 2.4, the weight-3 part of  $HH_3(A; k_0)$  is  $K_3$ , represented in  $V^{\otimes 3}$ , and the image of  $H_3(Y)$  in it lies in  $\mathfrak{B}_3$ ; by (4),  $\mathfrak{B}_{3,L} \neq K_{3,L}$ , so  $\mathfrak{B}_3 \neq K_3$ , as dimensions do not change under extension of scalars. (The same holds for the unnormalised Hochschild complex, whose projection to  $C_*(A; k_0)$  is a quasi-isomorphism preserving weights.) In cohomology, the braided cochain complex with coefficients in  $k_0$  has zero differential, and so has each subcomplex  $E$ . Choose a linear form  $g$  on  $V^{\otimes 3}$  that vanishes on  $\mathfrak{B}_3$  but not on  $K_3$ , and extend it by zero to a Hochschild 3-cochain. It is a cocycle, as  $\bar{A}^{\otimes 4}$  has no weight-3 part, and not a coboundary, as coboundaries vanish on  $K_3$ ; but its image  $g \circ \text{QS}_3$  in  $E$  is zero.

| $n$                   | 0 | 1 | 2  | 3  | 4   | 5   |
|-----------------------|---|---|----|----|-----|-----|
| $\dim A_n$            | 1 | 4 | 2  | 1  | 1   | 1   |
| $\dim \mathfrak{B}_n$ | 1 | 4 | 14 | 47 | 152 | 489 |
| $\dim K_n$            | 1 | 4 | 14 | 49 | 171 | 597 |

Table 1: The rack  $\mathbb{Z}/4$  over  $\mathbb{Q}$  (exact computation;  $\dim A_n = 1$  for all  $n \geq 3$ ). So  $H_A(t)H_{\mathfrak{B}}(-t) = 1 + 2t^3 - 11t^4 + 36t^5 + \dots$ , while  $H_A(t)H_K(-t) = 1 + O(t^6)$ .

(c) The dimensions and ranks were computed with exact arithmetic over  $\mathbb{Q}$  in the  $X$ -basis and over  $\mathbb{Q}(i)$  in the basis  $u_j$ . The classes of (a) span  $H_2$  in weight 3; the classes in weight 4 have multidegrees  $4\alpha_1$ ,  $4\alpha_3$  and  $2\alpha_0 + \alpha_1 + \alpha_3$ .  $\square$

*Remark 3.2.* In the basis  $u_j$  the relations of  $A$  are a quadratic Gröbner basis for the deglex order with  $u_1 > u_3$ : the rewriting rules are  $u_a u_b \mapsto 0$  and  $u_1 u_3 \mapsto i u_3 u_1$ , and the ambiguities  $u_a u_1 u_3$  and  $u_1 u_3 u_b$  resolve. The normal words of length  $\geq 2$  are the  $u_0^n$  and  $u_3 u_1$ , so  $A$  is a PBW algebra, hence Koszul [Pri70], and the failure in Theorem 1.3 is entirely due to  $\mathfrak{B}_3 \neq K_3$ .

*Remark 3.3* (the constant character). Let  $k_1$  be  $k$  with every  $x \in X$  acting by 1, the coefficients of rack homology. The image of the braided chain complex with coefficients  $k_1$  under QS, that is, its quotient by  $\ker \text{QS}$ , is  $(\mathfrak{B}_\bullet, \partial)$  with  $\partial(v_1 \cdots v_n) = \epsilon(v_1)v_2 \cdots v_n + (-1)^n v_1 \cdots v_{n-1}\epsilon(v_n)$ ,  $\epsilon(x) = 1$ . For  $X = \mathbb{Z}/4$  and  $k = \mathbb{Q}$ , exact computation gives  $H_n(\mathfrak{B}_\bullet, \partial) = \mathbb{Q}, \mathbb{Q}, 0, \mathbb{Q}$  for  $n = 0, 1, 2, 3$ , whereas  $(K_\bullet, \partial)$ , which computes  $HH_*(A; k_1) = H_*(M; \mathbb{Q})$  because  $A$  is Koszul, gives  $\mathbb{Q}, \mathbb{Q}, 0, 0$ . So QS is not injective on  $H_3$  of this quotient. We did not study other quotients for these coefficients.

*Remark 3.4* (the quadratic quotient). The sums  $\omega = \sum_j e_{x_j} e_{y_j}$  over the  $\sigma$ -orbits  $\{(x_j, y_j)\}$  in  $X \times X$  generate a differential Hopf ideal  $\langle \omega \rangle$  of  $\mathcal{D}$  [FG16, §3.6]; for involutive  $\sigma$  and  $\mathbb{Q} \subseteq k$  it is the ideal of [FG16, §3.5], and  $\mathcal{D}/\langle \omega \rangle$  is then acyclic in positive degrees [FG16, Prop. 24]. Since  $f$  is multiplicative [FG16, Cor. 42] and  $f(\omega) = 1 \otimes (1 - \sigma)(\sum_j x_j y_j) \otimes 1 = 0$ ,  $f$  vanishes on  $\langle \omega \rangle$  and factors through  $\mathcal{D}/\langle \omega \rangle$ . By the braid relation (compare [FG16, Remark 26]), moving a letter across an orbit sum gives another orbit sum, so  $(\mathcal{D}/\langle \omega \rangle)_2 \cong A \otimes (V \otimes V / \ker(1 - \sigma)) \otimes A \cong C_2$ . Hence  $\mathcal{D}/\langle \omega \rangle \rightarrow C$  is bijective in degrees  $\leq 2$  and surjective in degree 3, and  $H_2(\mathcal{D}/\langle \omega \rangle) \cong H_2(C) \neq 0$  for  $X = \mathbb{Z}/4$ .

## 4 Permutation racks

Let  $\varphi$  be a permutation of a finite set  $X$ ,  $\sigma(x, y) = (y, \varphi(x))$ , and  $\text{char } k = 0$ . Over a field  $L \supseteq k$  containing the necessary roots of unity, the linearisation  $F$  of  $\varphi$  is diagonalisable: for a cycle  $Z$  of  $\varphi$  with a point  $x_Z$  and  $\lambda^{|Z|} = 1$ , the vectors  $u = \sum_{j=0}^{|Z|-1} \lambda^{-j} \varphi^j(x_Z)$  satisfy  $Fu = \lambda u$  and form a basis  $\mathcal{U}$  of  $V_L$ . As  $\sigma(v \otimes w) = w \otimes F(v)$ ,

$$\sigma(u \otimes u') = \lambda_u u' \otimes u \quad (u, u' \in \mathcal{U}), \quad (7)$$

where  $\lambda_u$  is the eigenvalue of  $u$ . So  $R_L$  is spanned by the  $uu' - \lambda_u u' u$ , and the objects of Section 2 are graded by multidegrees in  $\mathbb{N}^{\mathcal{U}}$ . For a multidegree supported on  $\mathcal{S} \subseteq \mathcal{U}$ , the corresponding parts of  $A$ ,  $\mathfrak{B}$ ,  $K_n$  and  $C$  involve only words in  $\mathcal{S}$ , the relations  $uu' - \lambda_u u' u$  with  $u, u' \in \mathcal{S}$  and the braiding (7) on  $\mathcal{S}$ ; hence they depend only on  $(\lambda_u)_{u \in \mathcal{S}}$ .

Take  $\mathcal{S} = \{u_0, u_1\}$  with eigenvalues 1 and  $\lambda \neq 1$ , so that  $c(u_a \otimes u_b) = q_{ab} u_b \otimes u_a$  with  $q_{0b} = -1$  and  $q_{1b} = -\lambda$ . As in Lemma 3.1,  $u_1 u_1$ ,  $u_0 u_1$  and  $u_1 u_0$  lie in  $R$ . The coefficient of a word  $v$  in  $\text{QS}_n(w)$  is the sum, over the permutations carrying  $w$  to  $v$ , of the products of  $q_{ab}$  over the pairs of letters  $a$

| $m$ | $\dim \mathfrak{B}_1$ | $\dim \mathfrak{B}_2$ | $\dim \mathfrak{B}_3$ | $\dim \mathfrak{B}_4$ | first term of $H_A(t)H_{\mathfrak{B}}(-t) - 1$ |
|-----|-----------------------|-----------------------|-----------------------|-----------------------|------------------------------------------------|
| 2   | 2                     | 3                     | 5                     | 8                     | none up to $t^4$                               |
| 3   | 3                     | 7                     | 16                    | 33                    | $-3t^4$                                        |
| 4   | 4                     | 14                    | 47                    | 152                   | $2t^3$                                         |
| 5   | 5                     | 22                    | 96                    | 407                   | $-11t^4$                                       |
| 6   | 6                     | 33                    | 173                   | 883                   | $8t^3$                                         |
| 7   | 7                     | 45                    | 286                   | 1788                  | $2t^3$                                         |
| 8   | 8                     | 60                    | 435                   | 3109                  | $14t^3$                                        |

Table 2: Cyclic racks  $\mathbb{Z}/m$ ,  $x \triangleleft y = x + 1$ , over  $\mathbb{Q}$  (exact computation in the eigenbasis over  $\mathbb{Q}(e^{2\pi i/m})$ , weights  $\leq 4$ ). In all cases  $\dim A_n = 1, m, \lceil m/2 \rceil, 1, 1$  for  $n = 0, \dots, 4$  and  $H_A(t)H_K(-t) = 1 + O(t^5)$ .

(originally on the left) and  $b$  that are interchanged. On  $W = \langle u_1 u_1 u_0, u_1 u_0 u_1, u_0 u_1 u_1 \rangle$  this gives

$$\text{QS}_3|_W = (1 - \lambda) \begin{pmatrix} 1 & -1 & 1 \\ -\lambda & 1 + \lambda & -1 \\ \lambda^2 & -\lambda & 1 \end{pmatrix}, \quad \det \text{QS}_3|_W = (1 - \lambda)^4(1 + \lambda^2), \quad (8)$$

which is (3) for  $\lambda = i$ .

*Proof of Theorem 1.4.* As in Section 3,  $C_L = C \otimes_k L$ , so it suffices to find non-zero classes over  $L$ . If 4 divides the length of a cycle  $Z$ , take  $u_0, u_1$  with eigenvalues  $1, i$  on  $Z$ . The  $(\alpha_0 + 2\alpha_1)$ -part of the complex depends only on  $(1, i)$ , so it is isomorphic to that for  $\mathbb{Z}/4$ , which contains the non-trivial class  $[z]$  of Section 3.

If 3 divides the length of  $Z$ , take eigenvalues 1 and a primitive cube root of unity  $\omega$ , and the multidegree  $\alpha_0 + 3\alpha_1$ . First,  $u_1 u_1 u_1 \in \mathfrak{B}_3$ , since  $\text{QS}_3(u_1^3) = (1 + q)(1 + q + q^2)u_1^3$  with  $q = -\omega$  and  $(1 - \omega)(1 - \omega + \omega^2) = -2\omega(1 - \omega) \neq 0$ . Second,  $u_0 u_1 u_1 \in \mathfrak{B}_3$ , since the determinant (8) is  $(1 - \omega)^4(1 + \omega^2) = -9$ . Third,  $v_4 = u_0 u_1 u_1 u_1 \notin \mathfrak{B}_4$ : in the basis  $v_1 = u_1 u_1 u_1 u_0$ ,  $v_2 = u_1 u_1 u_0 u_1$ ,  $v_3 = u_1 u_0 u_1 u_1$ ,  $v_4$ , the matrix of  $\text{QS}_4$  is

$$\begin{pmatrix} -2 - 4\omega & 2 + 4\omega & -2 - 4\omega & 2 + 4\omega \\ -4 - 2\omega & 4 - \omega & -1 + 4\omega & -2 - 4\omega \\ -2 + 2\omega & -4 - 5\omega & 4 - \omega & 2 + 4\omega \\ 2 + 4\omega & -2 + 2\omega & -4 - 2\omega & -2 - 4\omega \end{pmatrix},$$

and  $\ell'(\sum_j b_j v_j) = -b_1 + 2\omega b_2 + (2 + 2\omega)b_3 + b_4$  vanishes on its columns, while  $\ell'(v_4) = 1$ . Let  $z' = u_0 \otimes u_1 u_1 u_1 \otimes 1 + 1 \otimes u_0 u_1 u_1 \otimes u_1 \in C_3$ . By (1),  $dz' = u_0 u_1 \otimes u_1 u_1 \otimes 1 - u_0 \otimes u_1 u_1 \otimes u_1 + u_0 \otimes u_1 u_1 \otimes u_1 - 1 \otimes u_0 u_1 \otimes u_1 u_1 = 0$ . The boundaries of weight 4 in degree 3 are  $d(1 \otimes t \otimes 1) = \theta(t) + \rho(t)$ ,  $t \in \mathfrak{B}_4$ , with  $\theta(y) = y_1 \otimes y_2 y_3 y_4 \otimes 1$  and  $\rho(y) = 1 \otimes y_1 y_2 y_3 \otimes y_4$  for a word  $y = y_1 y_2 y_3 y_4$ . As  $z' = \theta(v_4) + \rho(v_4)$ , the argument of Theorem 1.3(a) shows that  $z'$  is not a boundary.  $\square$

*Remark 4.1.* The kernel of the  $4 \times 4$  matrix is spanned by the quantum Serre element  $(\text{ad}_c u_1)^3(u_0)$ : here  $q_{11}^2 q_{10} q_{01} = 1$ , while the other factors in [AS02, Lemma 3.7(b)] do not vanish. In the two multidegrees used above the complex has dimensions  $(C_1, C_2, C_3) = (3, 6, 2)$  for  $\lambda = i$  and  $(C_2, C_3, C_4) = (4, 8, 3)$  for  $\lambda = \omega$ , and one-dimensional homology (exact computation). For  $\lambda = -1$  and for primitive 5th, 6th, 7th, 8th and 12th roots of unity these parts are exact. The classes of  $\mathbb{Z}/5$  and  $\mathbb{Z}/7$  lie in other multidegrees: with  $u_j$  of eigenvalue  $e^{2\pi i j/m}$ , the eleven classes of  $\mathbb{Z}/5$  in weight 4 lie, for example, in  $3\alpha_1 + \alpha_2$  and  $2\alpha_0 + \alpha_1 + \alpha_4$ , and the two classes of  $\mathbb{Z}/7$  in weight 3 in  $\alpha_1 + \alpha_2 + \alpha_4$  and  $\alpha_3 + \alpha_5 + \alpha_6$ .

## 5 Finite-dimensional Nichols algebras and $R_3$

*Proof of Corollary 1.5.* This is the remark after [FG16, Question 44] combined with [JKV19]. Suppose  $C \rightarrow A$  is exact and  $\mathfrak{B}_n = 0$  for  $n > N$ . As  $C$  is split exact as a complex of right  $A$ -modules, for every left  $A$ -module  $L'$  the complex  $C \otimes_A L' \rightarrow L'$  is a resolution by the free modules  $A \otimes \mathfrak{B}_n \otimes L'$ , of length at most  $N$ ; similarly for right modules. So  $A$  has finite global dimension, and for a finite bijective left non-degenerate solution this implies that  $\sigma$  is involutive [JKV19, arXiv v2, Thm. 4.6]; see also [CJKV26, Cor. 7.11].  $\square$

Let  $R_3$  be the dihedral quandle of order 3:  $X = \mathbb{Z}/3$ ,  $x \triangleleft y = 2y - x$ ,  $\sigma(x, y) = (y, 2y - x)$ , the conjugation quandle of the transpositions in  $S_3$ . It is bijective, non-degenerate and square-free, and  $\sigma^2 \neq \text{id}$ . The  $\sigma$ -orbits on  $X \times X$  are the three pairs  $(x, x)$  and the 3-cycles  $\{01, 12, 20\}$  and  $\{02, 21, 10\}$ . Since  $\ker \text{QS}_2 = \ker(1 - \sigma)$  is spanned by the orbit sums,  $\mathfrak{B}$  is a quotient of  $E = T(V)/(x_0^2, x_1^2, x_2^2, x_0x_1 + x_1x_2 + x_2x_0, x_0x_2 + x_2x_1 + x_1x_0)$ . Identifying  $V$  with  $V^*$  by the basis  $X$ ,  $E$  is the quadratic dual  $A^!$ ; with  $a = x_0$ ,  $b = x_1$ ,  $c = x_2$  its relations  $a^2 = b^2 = c^2 = 0$ ,  $ab + bc + ca = 0 = ba + cb + ac$  are those of the Fomin–Kirillov algebra  $\text{FK}_3$  [FK99] in the generators  $x_{12}$ ,  $x_{23}$ ,  $-x_{13}$  of [WZ19, Def. 0.1]. Exact computation gives  $\dim \mathfrak{B}_n = \dim E_n = 1, 3, 4, 3, 1, 0$  for  $n = 0, \dots, 5$ , so  $\mathfrak{B} = E \cong \text{FK}_3$ , of dimension 12, and  $\mathfrak{B} = K$ . (That  $\text{FK}_3$  is a Nichols algebra is known [MS00]; here it follows from the computation.) So Corollary 1.5 applies to  $R_3$ .

Two further published routes give the same conclusion. Since  $\mathfrak{B} = K$ , Proposition 2.3 says that  $C \rightarrow A$  is a resolution if and only if  $A \cong \text{FK}_3^!$  is Koszul, and it is not:  $\text{FK}_3$  is not Koszul by a result of Roos, hence neither is its quadratic dual [Roo99], [WZ19, Introduction]. Moreover  $H_A(t) = (1+t)(1+t+t^2)/(1-t)$  [WZ19, Thm. 3.10(1)], which we confirmed by counting monoid classes up to length 8, and  $H_{\mathfrak{B}}(t) = (1+t)^2(1+t+t^2)$ , so  $H_A(t)H_{\mathfrak{B}}(-t) = (1-t^2)(1+t^2+t^4) = 1-t^6$ , and by Lemma 2.2 the complex is not exact in weight 6. Exact computation shows that it is exact in weights  $\leq 5$  and that  $H_3(C) \cong \mathbb{Q}$  in weight 6. Finally, as  $A$  is not Koszul,  $\text{Tor}_n^A(k, k)$  has a non-zero part in some weight  $w \neq n$ , which by Lemma 2.4 is not reached from any quotient or subcomplex of the braided complex with coefficients in  $k_0$ ; so, as in Theorem 1.3(b), the map induced by QS is not surjective in some degree.

## 6 Solutions on two and three points

We enumerated the bijective solutions on two and three points with three independent programs (items 2 and 3 of the verification below) and applied Lemma 2.2, with  $\dim A_n$  from exact counts of monoid classes and  $\dim \mathfrak{B}_n$  from exact ranks for  $n \leq 5$  (for  $n = 6$ , from ranks modulo large primes). A class *fails* if  $H_A(t)H_{\mathfrak{B}}(-t) - 1$  has a non-zero coefficient in weight  $\leq 6$ ; the first such coefficient only involves exact data, so this proves non-exactness. A class that does not fail has only passed a finite test. On two points there are 5 solutions, pairwise non-isomorphic, 3 of them involutive; none fails up to weight 8. On three points there are 73 solutions in 29 isomorphism classes, 8 of them involutive, which pass, as they must by [FG16, Prop. 24]. Of the other 21 classes, 13 fail: 4 with  $H_A(t)H_{\mathfrak{B}}(-t) = 1 - t^4 + \dots$  and  $H_{\mathfrak{B}}(t) = 1 + 3t + 4t^2 + 4t^3 + 3t^4 + t^5$ , among them the quandle on  $\{0, 1, 2\}$  in which 0 acts by the transposition (12) and 1, 2 act trivially; 3 with  $1 - 3t^4 + \dots$ , among them the rack  $\mathbb{Z}/3$ ; and 6 with  $1 - t^6 + \dots$  and  $\dim \mathfrak{B} = 12$ , among them  $R_3$ . The remaining 8 classes pass the test up to weight 6; for them  $\dim \mathfrak{B}_6$  was also certified exactly (item 3 of the verification below), so they pass it with exact data. All 13 failing classes are left and right non-degenerate, so Corollary 1.5 already covers the ten with  $\dim \mathfrak{B} < \infty$ , and Theorem 1.4 covers the rack  $\mathbb{Z}/3$ .

## 7 Concluding remarks

By Proposition 2.3, Problem 1.1 has a positive answer exactly when  $A$  is Koszul and the Nichols algebra of  $-\sigma$  is the quadratic algebra dual to  $A$ . It would be interesting to characterise the solutions with this property; the involutive ones have it in characteristic zero [FG16], and the 8 non-involutive classes on three points that pass our test are candidates. As far as we know, two related questions of [FG16] remain open: the analogue for the group algebra of the structure group [FG16, Remark 33], and the involutive case in positive characteristic. For the constant character  $k_1$  we only showed that one particular quotient does not work (Remark 3.3).

**Verification.** The claims were checked as follows.

1. A self-contained program by the author, `check_note.py` (standard library only, exact arithmetic over  $\mathbb{Q}$  and cyclotomic fields, under a minute), checks for  $\mathbb{Z}/4$  the properties of  $\sigma$ , Table 1 (in the  $X$ -basis for  $n \leq 4$ , in the eigenbasis for  $n \leq 5$ ), the homology of  $C$  in weights  $\leq 4$  in both bases, all statements of Section 3 from Lemma 3.1 to (6), and Remark 3.3; for Theorem 1.4, (8) as polynomials in  $\lambda$ , the  $4 \times 4$  matrix, the forms, the cycle  $z'$  and Remark 4.1; for  $R_3$ ,  $\dim A_n$  ( $n \leq 8$ ),  $\dim \mathfrak{B}_n = \dim K_n = \dim E_n$  ( $n \leq 5$ ),  $1 - t^6$  and the homology in weights  $\leq 6$ ; and Table 2.
2. The results were found with the finder's programs (Python with `numpy`), which work in the  $X$ -basis with exact fraction-free elimination and, for larger cases, modulo large primes. They check that  $f$  is a chain map on all basis elements of weight  $\leq 3$  for six solutions, and compute the homology of  $C$  and of  $\mathcal{D}/\langle \omega \rangle$  with exact ranks, the complexes for  $k_0$  and  $k_1$ , the Koszul bimodule complexes and the census; results obtained only modulo primes serve here only as evidence.
3. Two independent verification runs, which were AI-assisted, re-derived the proofs and recomputed the claims with separate code. The first worked in exact arithmetic over  $\mathbb{Q}(i)$ ,  $\mathbb{Q}(e^{2\pi i/3})$ ,  $\mathbb{Q}(e^{2\pi i/5})$  and  $\mathbb{Q}(e^{2\pi i/7})$ : Table 1 and the multidegree decomposition of the homology for  $\mathbb{Z}/4$ , the block (3) and the cycle  $z$ , Remark 3.3, the two-letter blocks, the results for  $R_3$ , the Euler coefficients of  $\mathbb{Z}/5$  and  $\mathbb{Z}/7$ , and the census, by an own enumeration with the same 13 failing classes. It also pointed out that Corollary 1.5, and so the case of  $R_3$ , follows from [FG16] and [JKV19]. The second used integer matrices in the  $X$ -basis, with ranks over  $\mathbb{Q}$  certified by the Hadamard bound from ranks modulo many primes, and its own arithmetic in  $\mathbb{Q}(e^{2\pi i/m})$ . It rechecked Table 1, the homology of  $C$  in weights  $\leq 4$  in both bases, (3),  $\ell$ ,  $\kappa$ ,  $z_{\text{re}}$  and  $z_{\text{im}}$  (also over  $\mathbb{F}_p$  for all odd primes  $p \leq 31$ ), (8) and its determinant as polynomials in  $\lambda$ , the  $4 \times 4$  matrix,  $\ell'$ ,  $z'$  and Remark 4.1, all of Table 2, Remark 3.3, the results for  $R_3$ , and the census by a third enumeration, with  $\dim \mathfrak{B}_n$  exact for  $n \leq 8$  on two points and, on three points, for  $n \leq 5$  and, for the classes that pass,  $n = 6$ .

**Scope and priority.** Theorem 1.3 answers Problem 1.1 negatively, and Problem 1.2 in the formulation described in the introduction, including the strongest one, which allows any quotient of the braided complex. The case of a finite-dimensional Nichols algebra, in particular  $R_3$ , is not new in substance: it follows from the remark of Farinati and García Galofre after their Question 44 together with [JKV19, arXiv v2, Thm. 4.6] (see also [CJKV26, Cor. 7.11]), and for  $R_3$  also from Roos' result and the Hilbert series of Walton and Zhang [Roo99, WZ19]; we did not find this implication stated in print. Proposition 2.3 is elementary. We found no earlier treatment of the example  $\mathbb{Z}/4$ , of Theorem 1.3(b) or of Theorem 1.4. We read [Leb19] in the published report and [FG16, Leb17, Leb18, Leb20, Yan16, JKV19, CJKV26, WZ19, AS02] in their arXiv versions, whose

numbering we use (for [JKV19], version 2 of arXiv:1812.02026, which incorporates its corrigendum; the published version numbers the corresponding characterisation of involutive solutions as Theorem 4.5 [CJKV26, §7]; we did not see it and cite version 2, whose Theorem 4.6 contains the global-dimension condition). We did not consult [Roo99, FK99, MS00, PP05, Pri70] directly: the facts about  $FK_3$  that we use are recorded in [WZ19] or checked by our computations, and those about Koszul algebras are standard. We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, including works citing [FG16, Leb17, Leb20, Leb19], and made web searches. We found no answer to Problem 1.1 or Problem 1.2 beyond the involutive and idempotent cases. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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