

Orbit Lattice Formulas for Second Homology of Finite Abelian Quandles

Alper Ferudun*

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Abstract

Let Q be a finite quandle whose right translations commute, with r orbits. Its orbit action determines finite-index lattices $H_i \leq L = \mathbb{Z}^r$ containing the respective coordinate vectors e_i . We give an explicit presentation for the torsion in the second integral rack and quandle homology: the summand associated with the i th orbit is

$$T_i = \bigwedge^2 L / \left(\sum_j e_j \wedge H_j + \bigwedge^2 H_i \right).$$

The free ranks are r^2 and $r(r-1)$, respectively. Each orbit size annihilates its torsion summand, and this bound is sharp. A twenty-element example shows that the exponent of the orbit group need not annihilate that summand. Smith normal form computes all the groups from matrices whose dimensions depend only on r . The universal coefficient theorem gives second cohomology with any constant abelian coefficient group. The proof makes explicit the consequences of the structure-group and covering methods of Darné, Eisermann, and Lebed–Mortier.

1 Scope and the orbit lattices

A right quandle is a nonempty set Q with a binary operation $\triangleright$ such that $a \triangleright a = a$, each map $R_b : a \mapsto a \triangleright b$ is bijective, and

$$(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c).$$

Throughout, Q is finite and its right translations commute. We use the term *abelian quandle* in this sense, as in [2, 3]; we do not mean the more general class of medial quandles.

Let $O_1, \dots, O_r$ be the orbits of the inner action and choose $q_i \in O_i$. Self-distributivity says that translations indexed by points in the same orbit are conjugate. Since all translations commute, they are equal. Hence $L = \mathbb{Z}^r$, with basis $e_1, \dots, e_r$, acts on Q by letting e_j act as R_{q_j} . Write

$$H_i = \text{Stab}_L(q_i).$$

Then $[L : H_i] = |O_i|$ and $e_i \in H_i$. The stabilizer is constant on O_i , because the acting group is abelian. Thus

$$Q = \prod_{i=1}^r L/H_i, \quad (u + H_i) \triangleright (v + H_j) = u + e_j + H_i. \quad (1)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing). Unrefereed preprint. CC BY 4.0.

Conversely, any finite-index lattices $H_i \leq L$ containing e_i define a finite abelian quandle by (1). This is the orbit-lattice form of the established classification; see [3, Theorem 6.2] and [2].

Lebed's Problem 1 in [1, pp. 3220–3221] asks for the torsion of second *cohomology*, without specifying coefficients. Integral second cohomology and integral second homology must not be conflated: the former is torsion-free here. We compute integral homology and then give the complete constant-coefficient cohomology formula. Lebed–Mortier [2, Sections 6–8] describe the orbit torsion groups as quotients of their structure-group parameter group and compute several families. Darné [3, Theorem 6.4] gives the class-two structure-group presentation used below. Eisermann's covering correspondence includes disconnected quandles [4, Theorem 9.9]. More recent computations for conjugation and cyclic Alexander quandles appear in [5, 6]. We credit these methods and results; no absolute priority claim is made for the formula derived here.

2 The integral formula

Put

$$W = \bigwedge^2 L, \quad R = \sum_{j=1}^r e_j \wedge H_j, \quad T_i = W / (R + \bigwedge^2 H_i). \quad (2)$$

Here $\bigwedge^2 H_i$ means its image in W . All subgroups in (2) are integral sublattices: no saturation is taken.

Theorem 1. *For the quandle (1), every T_i is finite, and*

$$H_2^R(Q; \mathbb{Z}) \cong \mathbb{Z}^{r^2} \oplus \bigoplus_{i=1}^r T_i, \quad (3)$$

$$H_2^Q(Q; \mathbb{Z}) \cong \mathbb{Z}^{r(r-1)} \oplus \bigoplus_{i=1}^r T_i. \quad (4)$$

The superscripts R and Q denote rack homology and normalized quandle homology. The free splittings need not be canonical, but T_i is the torsion subgroup of the summand belonging to O_i .

2.1 The structure group

Use the right-convention structure group

$$G = \langle g_x \ (x \in Q) \mid g_x g_y = g_y g_{x \triangleright y} \rangle.$$

Its abelianization is the orbit-degree map $\pi : G \rightarrow L$, with $\pi(g_x) = e_i$ for $x \in O_i$. Indeed, abelianizing its defining relations identifies exactly the generators belonging to the same orbit.

We recall Darné's structure-group description, including its short proof to fix conventions. Let $F_{r,2}$ be the free nilpotent group of class at most two on $x_1, \dots, x_r$. Its commutator group is naturally $\bigwedge^2 L$. Fix the convention $[x, y] = x^{-1}y^{-1}xy$ and identify $[x_a, x_b]$ with $e_a \wedge e_b$. Then

$$G \cong F_{r,2}/R, \quad [G, G] \cong C := W/R. \quad (5)$$

The kernel of the action of a structure group on its quandle is central: an element acting trivially conjugates each g_x to itself. The inner action in our case is abelian, so $[G, G]$ is central. Moreover, the g_{q_i} generate G . Every point in an orbit is reached by translations indexed by the q_j , and its generator is the corresponding conjugate of g_{q_i} . If a word has degree $h \in H_i$, it stabilizes q_i and

commutes with g_{q_i} . In a class-two group these relations are precisely $e_i \wedge h = 0$. They define a surjection $F_{r,2}/R \rightarrow G$.

For the inverse, set $K = F_{r,2}/R$. Given $u \in L$, choose a lift $\tilde{u} \in K$ and assign

$$g_{u+H_i} \mapsto \tilde{u}^{-1} x_i \tilde{u}. \quad (6)$$

The assignment is independent of the lift because the kernel of $K \rightarrow L$ is central. It is independent of the coset representative because $e_i \wedge H_i = 0$ in K . The assigned image of $v + H_j$ differs from x_j by a central element. Conjugating by it therefore changes u to $u + e_j$, as required by (1). Thus all defining relations hold. The two homomorphisms are inverse on the orbit generators, proving (5). In particular the commutator pairing on G is the image of $u \wedge v$ for elements of degrees $u, v \in L$.

A common positive integer N satisfies $NL \subseteq H_j$ for every j . Consequently $NW \subseteq R$, so C and all its quotients T_i are finite.

2.2 Abelianizing an orbit stabilizer

Let $G_i = \text{Stab}_G(q_i)$. The action factors through L , and hence

$$G_i = \pi^{-1}(H_i), \quad 1 \longrightarrow C \longrightarrow G_i \longrightarrow H_i \longrightarrow 1.$$

The commutator pairing gives

$$[G_i, G_i] = (R + \bigwedge^2 H_i)/R. \quad (7)$$

Every commutator of two stabilizer elements has this form. Conversely, every wedge of two elements of H_i is obtained by lifting those elements to G_i . Centrality ensures that products of such commutators add their wedge classes without further terms. Abelianization therefore gives an exact sequence

$$0 \longrightarrow T_i \longrightarrow G_i^{\text{ab}} \longrightarrow H_i \longrightarrow 0. \quad (8)$$

Since H_i is free of rank r , this sequence splits as a sequence of abelian groups. Thus $G_i^{\text{ab}} \cong \mathbb{Z}^r \oplus T_i$.

2.3 The homology identification and normalization

For clarity, the stabilizer interpretation can be obtained directly from a presentation two-complex. Take one vertex, an edge labelled by every $b \in Q$, and a square for every relation

$$g_b g_c = g_c g_{b \triangleright c}.$$

The covering corresponding to the right G -set Q has vertices $a \in Q$, edges $(a, b) : a \rightarrow a \triangleright b$, and squares indexed by triples (a, b, c) . The edge boundary is

$$d_2(a, b) = a \triangleright b - a,$$

and the boundary of a square is

$$d_3(a, b, c) = (a \triangleright b, c) - (a, c) - (a \triangleright c, b \triangleright c) + (a, b).$$

These are the rack boundaries, with a harmless choice of signs for the chain differential. The first cellular homology of the component on O_i is consequently the i th orbit summand of $H_2^R(Q; \mathbb{Z})$. Its fundamental group is G_i by covering theory, and degree-one Hurewicz gives G_i^{ab} . This proves (3); it also describes the path-map mechanism in [2, Section 6].

Normalization kills the diagonal edges (a, a) . If a path labelled by w goes from q_i to $a = q_i \cdot w$, the corresponding based diagonal loop has label

$$wg_a w^{-1} = g_{q_i}.$$

All diagonals in the component therefore have the same class. This element is central in G_i , since each element of G_i fixes q_i . Its degree is e_i , primitive in H_i : the total-coordinate map $H_i \rightarrow \mathbb{Z}$ takes e_i to 1. Quotienting (8) by this element removes one free factor and no torsion. Explicitly, the subgroup generated by its class intersects T_i trivially; the resulting exact sequence has quotient $H_i/\mathbb{Z}e_i$, which is free of rank $r - 1$, and so splits.

In chain terms this is normalized quandle homology. The quotient of rack degree two by diagonal pairs is C_2^Q ; boundaries of degenerate triples vanish in that quotient. Thus the remaining boundary image is precisely the normalized image. Summing over the components proves (4), with no connectedness hypothesis. This agrees with the disconnected covering theorem in [4, Theorem 9.9].

3 Cohomology and annihilators

Corollary 2. *Let A be any constant abelian coefficient group, and let $d_{i1}, \dots, d_{it_i} > 1$ be the invariant factors of T_i . With $A[d] = \{a \in A : da = 0\}$, one has*

$$H_R^2(Q; A) \cong A^{r^2} \oplus \bigoplus_{i,k} A[d_{ik}], \quad (9)$$

$$H_Q^2(Q; A) \cong A^{r(r-1)} \oplus \bigoplus_{i,k} A[d_{ik}]. \quad (10)$$

Their torsion subgroups are obtained by replacing the displayed copies of A by $\text{Tor}(A)$; each $A[d_{ik}]$ is already torsion.

Proof. Both integral first homology groups are free of rank r . The degree-two universal coefficient sequence therefore has zero Ext term and identifies cohomology with $\text{Hom}(H_2, A)$. Apply Theorem 1. The torsion assertion follows because the direct sums are finite and $A[d]$ has exponent dividing d . $\square$

In particular, integral second cohomology is torsion-free. If $A = \mathbb{Z}/m$, the additional factors in (9)–(10) are cyclic of order $\gcd(m, d_{ik})$. The statement does not concern twisted or nonabelian coefficients.

Corollary 3. *The size of the i th orbit annihilates T_i :*

$$|O_i| T_i = 0.$$

More precisely, if $r \geq 2$ and $d_1 \mid \dots \mid d_r$ are the invariant factors of $H_i \subset L$, then $d_{r-1}d_r T_i = 0$.

Proof. For $r = 1$ the only quandle under consideration is the singleton and $T_1 = 0$. For $r \geq 2$, take a Smith basis of the lattice inclusion. Then

$$W / \bigwedge^2 H_i \cong \bigoplus_{a < b} \mathbb{Z}/(d_a d_b).$$

Each $d_a d_b$ divides $d_{r-1}d_r$ and also divides $\prod_k d_k = [L : H_i] = |O_i|$. The group T_i is a quotient of this group, proving both assertions. $\square$

The orbit-size bound improves the square-of-orbit-size annihilator of [2, Proposition 8.2]. It is sharp: with two orbits and

$$H_1 = \langle e_1, ne_2 \rangle, \quad H_2 = \langle ne_1, e_2 \rangle,$$

both orbits have size n and $T_1 = T_2 = \mathbb{Z}/n$. This is among the families already computed in [2, Section 8].

The exponent of the orbit group L/H_i is not, in general, an annihilator. Take $r = 3$ and

$$H_1 = \langle e_1, 2e_2, 2e_3 \rangle,$$

$$H_2 = \langle 2e_1, e_2, 4e_3 \rangle,$$

$$H_3 = \langle 2e_1, 4e_2, e_3 \rangle.$$

Their indices are 4, 8, 8, so the quandle has twenty elements. Relative to $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$, the relation subgroup R is diagonal with entries 2, 2, 4. Each $\bigwedge^2 H_i$ is contained in R . Hence

$$T_1 = T_2 = T_3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

The group of the first orbit is $(\mathbb{Z}/2)^2$, of exponent 2, whereas T_1 has exponent 4. The full second quandle homology in this example is

$$H_2^Q(Q; \mathbb{Z}) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/4)^3.$$

4 Smith matrices and reproducibility

Choose an integer basis $h_{j1}, \dots, h_{jr}$ of each H_j . Let M_i have rows indexed by $a < b$ in the basis $e_a \wedge e_b$, and the following columns:

$$e_j \wedge h_{jk} \quad (1 \leq j, k \leq r), \quad h_{ia} \wedge h_{ib} \quad (1 \leq a < b \leq r). \quad (11)$$

Then $T_i = \text{coker}_{\mathbb{Z}} M_i$. Thus its invariant factors are the nonunit Smith factors of a matrix with $\binom{r}{2}$ rows and $r^2 + \binom{r}{2}$ columns. This bound concerns matrix dimensions, not the bit size of their entries. In the singleton case the matrix has zero rows and presents the zero group.

For an operation-table input, first verify the quandle axioms and commuting translations and compute the orbits. For each $a \in O_i$, choose a path from q_i and record its degree $v_a \in L$. The vectors

$$v_a + e_j - v_{a \triangleright q_j} \quad (a \in O_i, 1 \leq j \leq r) \quad (12)$$

generate H_i . They are stabilizer elements. Conversely, traverse any stabilizing word, allowing inverse edges, and telescope the edge differences to express its degree as an integral combination of (12). Hermite normal form gives a basis for (11). If the lattices are supplied directly, constructing the operation table is unnecessary.

The accompanying Python scripts compare the formula with matrices of d_2 and d_3 constructed directly from operation tables, separately for each orbit and for both rack and normalized quandle chains. The tests comprise 52 distinct finite examples, including the twenty-element example above, with 58,625 explicit conditions in each of normal and optimized Python modes. The two modes agree. They share the same Smith-normal-form library; they are not independent implementations of that algorithm. For the boundary-matrix comparison, torsion can be read from $\text{coker } d_3$, because its quotient by H_2 is the free group $\text{im } d_2$. The free rank is adjusted by $\text{rank } d_2$. These finite tests check the implementation and examples, not the general theorem in place of its proof.

This unrefereed manuscript was prepared with AI assistance in research, drafting, and exact computational checks. The written arguments were self-audited; no independent expert review or proof-assistant verification is claimed. The scope is finite quandles with commuting right translations, second integral homology, and second cohomology with constant abelian coefficients.

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