Essential Self-Adjointness of the Laplace–Beltrami Operator for a Family of Non-Regular Almost-Riemannian Structures

Alper Ferudun*

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Abstract

Oberwolfach Report 47/2019 records the following question, posed in the abstract of L. Rizzi's talk (joint work with V. Franceschi, D. Prandi and M. Seri): is the Laplace–Beltrami operator of the almost-Riemannian structure on $\mathbb{R}^2$ with orthonormal frame $X_1 = \partial_x$, $X_2 = x(x^{2\ell} + z^2)\partial_z$ essentially self-adjoint on the regular region for every $\ell \geq 1$? The case $\ell = 1$, and the cases $\ell \leq n/2$ of an n-dimensional version, had been settled by Prandi, Rizzi and Seri in the complete (torus) setting of their Example 7.7; their effective-potential criterion does not apply for larger ℓ. We answer the question affirmatively for every ℓ in the complete setting assumed in the report: the structures $X_1 = \partial_x$, $X_2 = c(z)\,x(x^{2\ell} + \rho(z))\partial_z$ with $c > 0$, $\rho \geq 0$ and c, $c\rho$ bounded, all of which are complete, have an essentially self-adjoint Laplace–Beltrami operator. They include structures that coincide with the model on a strip around the non-regular point; the same holds for the n-dimensional examples of Prandi, Rizzi and Seri for all ℓ and n. We do not prove a localisation theorem for arbitrary complete structures that agree with the model near that point. The proof combines a one-dimensional Hardy inequality obtained with the multiplier δ (the distance from the singular set), whose weight $-\delta\Delta\delta$ is at least 1 for this family, with a standard Agmon-type argument; the effective potential, in contrast, is not bounded below by $3/(4\delta^2)$ when $\ell \geq 2$. The same argument shows that the effective-potential hypothesis in the criteria of Prandi–Rizzi–Seri and Franceschi–Prandi–Rizzi can be replaced by $-\Delta_\omega\delta \geq 1/\delta - \kappa$, which is an alternative sufficient condition rather than a strengthening. Taken literally on all of $\mathbb{R}^2$, the model is incomplete, and its Laplace–Beltrami operator is not essentially self-adjoint for reasons unrelated to the singular set. The more general real-analytic conjecture of the report remains open; we give a real-analytic example without tangency points for which the weak Hardy inequality underlying all these criteria fails. This is an unrefereed note.

1 Introduction

Let $X_1, \dots, X_K$ be smooth vector fields on a manifold M satisfying Hörmander's condition, and declare them orthonormal. If they span T_qM for q in an open dense set, they define an *almost-Riemannian structure* (ARS): the sub-Riemannian distance d is induced by a Riemannian metric g on the regular set $\mathcal{R}$, and the singular set is $\mathcal{S} = M \setminus \mathcal{R}$. In the abstract of L. Rizzi's talk (joint work with V. Franceschi, D. Prandi and M. Seri) in Oberwolfach Report 47/2019 [BKP19], one assumes that (M, d) is complete and asks whether the Laplace–Beltrami operator Δ_g with domain $C_c^\infty(\mathcal{R})$ is essentially self-adjoint in $L^2(\mathcal{R}, \mu_g)$, μ_g being the Riemannian measure. An affirmative answer means that quantum particles cannot cross $\mathcal{S}$ [BL13, PRS18]. The answer is known to be yes under various assumptions [BL13, PRS18, FPR20]. Theorem, example and equation numbers of [PRS18] and [FPR20] refer to the arXiv versions 1609.01724v3 and 1708.09626v3, against which they were

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.
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checked; the numbering of the journal versions may differ. Problem 1 of [BKP19, §3.2] concerns the structure on $\mathbb{R}^2$ with orthonormal frame

$$X_1 = \partial_x, \qquad X_2 = x\,(x^{2\ell} + z^2)\,\partial_z, \qquad \ell \in \mathbb{N}. \tag{1}$$

Here $\mathcal{S} = \{x = 0\}$, the distance from $\mathcal{S}$ is $\delta = |x|$, and $\mu_g = \mathrm{d}x\,\mathrm{d}z/(|x|(x^{2\ell} + z^2))$ is not of the form $\delta^{-a}\times$(smooth measure). The report notes that the case $\ell = 1$ is covered by Example 7.7 of Prandi, Rizzi and Seri [PRS18]; that example is set in the complete (torus) setting described below, and on the literal $\mathbb{R}^2$ even the case $\ell = 1$ fails (Proposition 6.1). About general ℓ the report says: "We expect this fact to be true for all $\ell \geq 1$", adding that the authors could not prove it. It then proposes, more generally, that the Laplace–Beltrami operator of every real-analytic two-dimensional ARS without tangency points is essentially self-adjoint. The question is record OWR-17290-002 of the UnsolvedMath dataset [ula26]. In [PRS18, Example 7.7] the model appears on $\mathbb{R}_t \times \mathbb{T}^{n-1}$ with z^2 replaced by a smooth function $f \geq 0$; for $\ell > n/2$ the authors state that they do not know the answer, even for $n = 2$ and $\ell = 2$.

Our main result answers the ℓ-question affirmatively in the complete setting of the report, for the following family of complete structures.

Theorem 1.1. *Let $\ell \geq 0$ be an integer, let N_1 be $\mathbb{R}$ or a circle $\mathbb{R}/L\mathbb{Z}$, and let $c, \rho \in C^\infty(N_1)$ satisfy $c > 0$, $\rho \geq 0$, $\sup c < \infty$ and $\sup c\rho < \infty$. Then*

$$X_1 = \partial_x, \qquad X_2 = c(z)\,x\left(x^{2\ell} + \rho(z)\right)\partial_z \tag{2}$$

is a complete ARS on $M = \mathbb{R} \times N_1$ with singular set $\mathcal{S} = \{x = 0\}$ and $\delta = |x|$, and its Laplace–Beltrami operator with domain $C_c^\infty(M \setminus \mathcal{S})$ is essentially self-adjoint in $L^2(M \setminus \mathcal{S}, \mu_g)$. In particular this holds for:

- *(a) $N_1 = \mathbb{R}$, $c \equiv 1$, and any bounded smooth $\rho \geq 0$ with $\rho(z) = z^2$ for $|z| \leq 1$; this structure coincides with* (1) *on the strip $\mathbb{R} \times (-1, 1)$;*
- *(b) $N_1 = \mathbb{R}$, $c(z) = (1 + z^2)^{-1}$, $\rho(z) = z^2$, a real-analytic structure whose field X_2 is the field of* (1) *multiplied by the positive function $(1 + z^2)^{-1}$;*
- *(c) $N_1 = \mathbb{R}/2\pi\mathbb{Z}$, $c \equiv 1$, $\rho = \sin^2$, which is [PRS18, Example 7.7] with $n = 2$ and $f = \sin^2$.*

Theorem 1.1 is a statement about the family (2). We do not prove a localisation theorem: our results apply to a complete ARS that agrees with (1) near the origin when it belongs to this family (as in (a)) or to the class of Corollary 7.1 below, and say nothing about other such structures. The n-dimensional Examples 7.7 and 7.8 of [PRS18] are covered for all ℓ and n (Corollary 4.3). Theorem 1.1 is a special case of a criterion for warped products (Theorem 3.2). The observation behind it is simple. For $f = x(x^{2\ell} + z^2)$ and $\omega = \mu_g$,

$$a := -\delta\,\Delta_\omega\delta = x\,\partial_x \log|f| = 1 + 2\ell u, \qquad u = \frac{x^{2\ell}}{x^{2\ell} + z^2} \in [0, 1],$$

so $a \geq 1$. By a one-dimensional identity (Lemma 2.1), $a \geq 1$ gives the Hardy inequality $\int|\partial_x w|^2 \mathrm{d}\omega \geq \int|w|^2\delta^{-2}\mathrm{d}\omega$ with the critical constant 1, and the Agmon-type argument of [NN09, PRS18, FPR20] then applies. The effective potential $V_{\mathrm{eff}} = \frac{1}{4}(\Delta_\omega\delta)^2 + \frac{1}{2}\partial_\delta(\Delta_\omega\delta)$ used in [PRS18, FPR20] satisfies instead

$$\delta^2 V_{\mathrm{eff}} = \tfrac{3}{4} + 2\ell(1 - \ell)u + 3\ell^2u^2, \qquad \min_{u \in [0,1]}\left(\tfrac{3}{4} + 2\ell(1 - \ell)u + 3\ell^2u^2\right) = \tfrac{3}{4} - \tfrac{(\ell-1)^2}{3}, \tag{3}$$

which is below the required $3/4$ along the curves $z = \gamma x^\ell$ with $(1+\gamma^2)^{-1} = (\ell-1)/(3\ell)$ as soon as $\ell \geq 2$, and negative for $\ell \geq 3$.

The same substitution works in the general criteria of [PRS18, FPR20]: their hypothesis $V_{\text{eff}} \geq 3/(4\delta^2) - \kappa/\delta$ may be replaced by $-\Delta_\omega \delta \geq 1/\delta - \kappa$ (Theorem 5.1). This is an alternative condition, not a strengthening: for $\mathrm{d}\omega = t^\alpha \mathrm{d}t\,\mathrm{d}\mu$ it holds only when $\alpha \leq -1$, whereas the effective-potential condition also holds for $\alpha \geq 3$ (Remark 5.2).

Two further points concern the literal wording of the problem. First, on all of $\mathbb{R}^2$ the structure (1) is not complete (the ends $z \to \pm\infty$ lie at finite distance), and its Laplace–Beltrami operator is not essentially self-adjoint (Proposition 6.1). This defect comes from those ends, not from $\mathcal{S}$; the report assumes completeness, and so Theorem 1.1 answers what we take to be the intended question. Second, the "more general" conjecture remains open. Corollary 7.1 proves it for structures with compact singular set satisfying $x\partial_x \log|f| \geq 1 - \kappa|x|$ in normal coordinates (the report does not assume that the singular set is compact), and Proposition 7.2 gives a complete real-analytic example without tangency points for which the weak Hardy inequality used by all these methods fails.

2 Two lemmas

Lemma 2.1 (weighted Hardy identity)**.** *Let $I \subset \mathbb{R}\setminus\{0\}$ be an open interval, let $m \in C^1(I)$ be positive and $b \in C^1(I)$ real, and put $a := -t\,m'(t)/m(t)$. Then for every complex $w \in W^{1,2}(I)$ with compact support in I,*

$$\int_I m|w'|^2 \mathrm{d}t = \int_I m\left|w' - \frac{b}{t}w\right|^2 \mathrm{d}t + \int_I \big(b(1+a-b) - t\,b'\big)\frac{m|w|^2}{t^2}\mathrm{d}t. \tag{4}$$

In particular ($b \equiv 1$), $\int_I m|w'|^2\mathrm{d}t \geq \int_I a\,m|w|^2 t^{-2}\mathrm{d}t$.

Proof. Expand $m|w' - bw/t|^2 = m|w'|^2 - (mb/t)(|w|^2)' + mb^2|w|^2/t^2$. Since $|w|^2 \in W^{1,1}(I)$ has compact support and $mb/t \in C^1(I)$, integration by parts gives $-\int (mb/t)(|w|^2)' = \int (mb/t)'|w|^2$, and $(mb/t)' = (m/t^2)(-ab + tb' - b)$. Collecting terms gives (4). $\square$

Remark 2.2. With $m = e^{2\vartheta}$, the choice $b = (1+a)/2$ turns the last weight into $(1+a)^2/4 - ta'/2 = \frac{1}{4} + t^2 V$ with $V = (\partial_t\vartheta)^2 + \partial_t^2\vartheta$; this is the effective potential of [PRS18, FPR20], and the resulting inequality is the one obtained there from the ground-state substitution $v = e^\vartheta u$ and the classical Hardy constant $1/4$. The choice $b \equiv 1$ corresponds to the trial function δ (the vector field $\nabla\delta/\delta$) and gives the weight a. For constant a one has $(1+a)^2/4 \geq a$; when a varies, the term $-ta'/2$ can make the first choice much worse, and this is what happens for (1). Hardy inequalities obtained from a multiplier, or a ground state, are classical; see for instance [BFT04], and [Bru04, NN22, Rob21] for their use in proofs of essential self-adjointness.

Throughout, (Ω, g) is a Riemannian manifold, ω a smooth positive density, $\Delta_\omega = \operatorname{div}_\omega \circ \nabla$, and $\langle X, Y\rangle = g(X, \bar{Y})$. For $u \in C^\infty(\Omega)$ and v Lipschitz with compact support, Green's formula reads $\int (\Delta_\omega u)\bar{v}\,\mathrm{d}\omega = -\int \langle \nabla u, \nabla v\rangle \mathrm{d}\omega$.

Lemma 2.3. *(i) $T = -\Delta_\omega$ with domain $C_c^\infty(\Omega)$ is symmetric and nonnegative in $L^2(\Omega,\omega)$. It is essentially self-adjoint if and only if, for some $\lambda > 0$, the only $\psi \in L^2(\Omega,\omega)$ with $\Delta_\omega\psi = \lambda\psi$ in the sense of distributions is $\psi = 0$. Every such ψ is smooth.*

(ii) If $\psi \in C^\infty(\Omega)$ satisfies $\Delta_\omega\psi = \lambda\psi$ and h is real, Lipschitz and compactly supported in Ω, then

$$\int |\nabla(h\psi)|^2 \mathrm{d}\omega + \lambda \int h^2|\psi|^2 \mathrm{d}\omega = \int |\nabla h|^2 |\psi|^2 \mathrm{d}\omega. \tag{5}$$

Proof. (i) Symmetry and nonnegativity follow from Green's formula. For a nonnegative symmetric operator, essential self-adjointness is equivalent to $\ker(T^*+\lambda)=\{0\}$ for some $\lambda>0$ [RS75, Theorem X.1 and its corollary]. Since Δ_ω is formally symmetric with respect to ω, $\psi\in\ker(T^*+\lambda)$ means $\Delta_\omega\psi=\lambda\psi$ in $\mathcal{D}'(\Omega)$, and ψ is smooth by elliptic regularity. (ii) Apply Green's formula with $u=\psi$, $v=h^2\psi$, and use the pointwise identity $\operatorname{Re}\langle\nabla\psi,\nabla(h^2\psi)\rangle=|\nabla(h\psi)|^2-|\nabla h|^2|\psi|^2$ (cf. [PRS18, Lemma 3.5], [FPR20, Lemma 4.4]). □

3 A criterion for warped products

Setting 3.1. N is a smooth manifold without boundary with a smooth positive density μ; $\Omega=(\mathbb{R}\setminus\{0\})\times N$ and $\Omega_\pm=\mathbb{R}_\pm\times N$, with coordinates (x,z). The metric is $g=\mathrm{d}x^2+h_x$, where (h_x) is a family of Riemannian metrics on N depending smoothly on $(x,z)\in\Omega$, and $\omega=m\,\mathrm{d}x\,\mathrm{d}\mu$ with $m\in C^\infty(\Omega)$ positive. Then $\nabla x=\partial_x$, and for a Lipschitz function u one has $|\nabla u|^2=|\partial_x u|^2+|\mathrm{d}_N u|^2_{h_x}\geq|\partial_x u|^2$ almost everywhere. If φ depends on x only and η on z only, then $\langle\nabla\varphi,\nabla\eta\rangle=\varphi'\,\partial_x\eta=0$.

Theorem 3.2. *In Setting 3.1 assume:*

(A1) $a:=-x\,\partial_x\log m\geq 1$ *on* Ω*;*

(A2) for every $R>0$ *there are Lipschitz functions* $\eta_k:N\to[0,1]$ *with compact support such that* $\eta_k\to 1$ *pointwise on* N *and* $\varepsilon_k(R):=\sup\{|\nabla\eta_k|^2(x,z):0<|x|\leq 2R,\ z\in N\}\to 0$ *as* $k\to\infty$*.*

Then Δ_ω *with domain* $C_c^\infty(\Omega)$ *is essentially self-adjoint in* $L^2(\Omega,\omega)$*, and so is its restriction to* $C_c^\infty(\Omega_\pm)$ *in* $L^2(\Omega_\pm,\omega)$*. If* N *is compact, (A2) holds with* $\eta_k\equiv 1$*.*

Proof. We treat Ω; the same proof applies to Ω_+ and Ω_-. By Lemma 2.3(i) it suffices to show that a smooth $\psi\in L^2(\Omega,\omega)$ with $\Delta_\omega\psi=\lambda\psi$, $\lambda>0$, vanishes.

Hardy inequality. Let w be Lipschitz with compact support in Ω. For each $z\in N$ the function $x\mapsto w(x,z)$ is Lipschitz with compact support in $\mathbb{R}\setminus\{0\}$. Lemma 2.1 with $b\equiv 1$ on each half-line, (A1), Fubini's theorem and $|\nabla w|\geq|\partial_x w|$ give

$$\int_\Omega |\nabla w|^2\mathrm{d}\omega\ \geq\ \int_\Omega\frac{|w|^2}{x^2}\mathrm{d}\omega. \tag{6}$$

Cut-offs. Fix $0<r_1<r_0<1\leq R$ and define $G:[0,\infty)\to[0,1]$ by $G=0$ on $[0,r_1/2]$, $G(r)=(2r-r_1)/r_0$ on $[r_1/2,r_1]$, $G(r)=r/r_0$ on $[r_1,r_0]$ and $G=1$ on $[r_0,\infty)$. Then, almost everywhere, $G'(r)^2-G(r)^2/r^2$ is 0 on $[0,r_1/2]\cup[r_1,r_0]$, at most $4/r_0^2$ on $[r_1/2,r_1]$, and negative on $[r_0,\infty)$. Let $\xi\in C_c^\infty(\mathbb{R})$ take values in $[0,1]$, with $\xi=1$ on $[-1,1]$ and $\operatorname{supp}\xi\subset[-2,2]$, and put $\xi_R(x)=\xi(x/R)$. For the functions η_k of (A2) let $h_k(x,z)=G(|x|)\,\xi_R(x)\,\eta_k(z)$, which is Lipschitz with compact support in Ω. As G' vanishes for $|x|\geq r_0$ and ξ_R' vanishes for $|x|\leq R$, Setting 3.1 gives

$$|\nabla h_k|^2=G'^2\xi_R^2\eta_k^2+G^2\xi_R'^2\eta_k^2+G^2\xi_R^2|\nabla\eta_k|^2.$$

Conclusion. Apply (5) with $h=h_k$ and (6) with $w=h_k\psi$:

$$\lambda\int h_k^2|\psi|^2\mathrm{d}\omega\leq\int\Big(|\nabla h_k|^2-\frac{h_k^2}{x^2}\Big)|\psi|^2\mathrm{d}\omega\leq\frac{4}{r_0^2}\int_{r_1/2\leq|x|\leq r_1}|\psi|^2\mathrm{d}\omega+\Big(\frac{\|\xi'\|_\infty^2}{R^2}+\varepsilon_k(R)\Big)\|\psi\|^2.$$

On the left, $h_k^2\geq\mathbf{1}_{\{r_0\leq|x|\leq R\}}\eta_k^2$. Let $k\to\infty$; by dominated convergence (the integrands are bounded by $|\psi|^2$), $\lambda\int_{r_0\leq|x|\leq R}|\psi|^2\mathrm{d}\omega\ \leq\ (4/r_0^2)\int_{r_1/2\leq|x|\leq r_1}|\psi|^2\mathrm{d}\omega+\|\xi'\|_\infty^2R^{-2}\|\psi\|^2$. Let $R\to\infty$ (monotone convergence on the left) and then $r_1\to 0$ (dominated convergence on the right, since $\mathbf{1}_{\{r_1/2\leq|x|\leq r_1\}}\to 0$ pointwise on Ω). This gives $\lambda\int_{|x|\geq r_0}|\psi|^2\mathrm{d}\omega\leq 0$ for every $r_0\in(0,1)$, so $\psi=0$. □

Remark 3.3. This is the Agmon-type argument of [NN09] as in [PRS18, Prop. 3.4], [FPR20, Prop. 4.7]; G is the function of the proof of [FPR20, Prop. 4.7, eq. (80)] (with $\zeta = r_1/2$ and $\eta = r_0$; equivalently, [FPR20, Remark 4.2] with $a = 1$), and the cancellation $G'^2 = G^2/r^2$ on $[r_1, r_0]$ makes 1 the critical constant in (6).

4 Two-dimensional structures and Problem 1

Let N_1 be $\mathbb{R}$ or $\mathbb{R}/L\mathbb{Z}$, and let $f \in C^\infty(\mathbb{R} \times N_1)$ be real with $f(0, z) = 0$ and $f(x, z) \neq 0$ for $x \neq 0$. Consider $X_1 = \partial_x$, $X_2 = f\partial_z$ on $M = \mathbb{R} \times N_1$, and assume Hörmander's condition (at $(0, z)$ it holds, for instance, if $\partial_x^j f(0, z) \neq 0$ for some j, since the j-fold bracket $[X_1, [X_1, \ldots, [X_1, X_2]]]$ equals $(\partial_x^j f)\partial_z$). On $\Omega = \{x \neq 0\}$,

$$g = \mathrm{d}x^2 + f^{-2}\mathrm{d}z^2, \qquad \mu_g = \frac{\mathrm{d}x\,\mathrm{d}z}{|f|}, \qquad \Delta_g u = |f|\,\partial_x\big(|f|^{-1}\partial_x u\big) + |f|\,\partial_z\big(|f|\,\partial_z u\big).$$

Lemma 4.1. *The distance from $\mathcal{S} = \{x = 0\}$ is $\delta = |x|$. If $\Lambda_R := \sup\{|f(x, z)| : |x| \leq R,\ z \in N_1\}$ is finite for every R, then (M, d) is complete.*

Proof. Along a horizontal curve $\dot\gamma = u_1 X_1 + u_2 X_2$ with $u_1^2 + u_2^2 \leq 1$ we have $|\dot x| \leq 1$ and $|\dot z| \leq |f(\gamma)|$. Hence x is 1-Lipschitz for d, so $d(q, \mathcal{S}) \geq |x(q)|$, and the segment $s \mapsto (s\,x(q), z(q))$ has length $|x(q)|$. If B is contained in the d-ball of radius r around q_0, each $q \in B$ is joined to q_0 by a horizontal curve of length less than $r + 1$, along which $|x| \leq |x(q_0)| + r + 1 =: R$; so $|z(q) - z(q_0)| \leq (r + 1)\Lambda_R$. Thus d-bounded sets are bounded. Since d induces the manifold topology [ABB20, Ch. 3], closed d-balls are compact, and (M, d) is complete. □

Corollary 4.2. *Assume* (A1′) $x\,\partial_x \log|f| \geq 1$ *on* Ω, *equivalently, for each z the function $|f(x, z)|/|x|$ is nondecreasing in $|x|$ on both half-lines; and* (A2′) *N_1 is compact or $\Lambda_R < \infty$ for every R. Then Δ_g with domain $C_c^\infty(\Omega)$ (or $C_c^\infty(\Omega_\pm)$) is essentially self-adjoint in $L^2(\Omega, \mu_g)$.*

Proof. Apply Theorem 3.2 with $N = N_1$, $\mu = \mathrm{d}z$, $h_x = f^{-2}\mathrm{d}z^2$ and $m = |f|^{-1}$, so that $a = x\partial_x \log|f|$. For (A2) take $\eta_k(z) = \eta(z/k)$ with $\eta \in C_c^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, $\eta = 1$ on $[-1, 1]$; then $|\nabla\eta_k|^2 = f^2\eta'(z/k)^2 k^{-2} \leq \Lambda_{2R}^2 \|\eta'\|_\infty^2 k^{-2}$ for $|x| \leq 2R$. □

Proof of Theorem 1.1. For $f = c(z)x(x^{2\ell} + \rho(z))$ we have $f \neq 0$ for $x \neq 0$ and

$$x\,\partial_x \log|f| = 1 + \frac{2\ell x^{2\ell}}{x^{2\ell} + \rho(z)} \ \geq\ 1, \qquad |f| \leq R^{2\ell+1}\sup c + R\sup(c\rho) \ \text{ for } |x| \leq R.$$

Hörmander's condition holds because $\partial_x^{2\ell+1} f(0, z) = (2\ell + 1)!\,c(z) \neq 0$ for $\ell \geq 1$, and $\partial_x f(0, z) = c(z)(1 + \rho(z)) \neq 0$ for $\ell = 0$. Lemma 4.1 and Corollary 4.2 give the claims. In (a) a bounded smooth $\rho \geq 0$ equal to z^2 on $[-1, 1]$ exists (for instance $\rho = z^2\chi + 1 - \chi$ with a cut-off χ); (b) and (c) satisfy the hypotheses directly. □

Corollary 4.3. *Let $n \geq 2$, $\ell \geq 1$ and let $f \geq 0$ be smooth on $\mathbb{T}^{n-1}$. For the structures of [PRS18, Examples 7.7 and 7.8],*

$$\begin{aligned} &M = \mathbb{R}_t \times \mathbb{T}_x^{n-1} : && X_0 = \partial_t,\quad X_i = F\partial_{x_i} \quad (1 \leq i \leq n-1), \\ &M = \mathbb{R}_t \times \mathbb{T}_x^{n-1} \times \mathbb{T}_y^{n-1} : && X_0 = \partial_t,\quad X_i = F\partial_{x_i},\quad Y_i = \partial_{y_i} + t\partial_{x_i} \quad (1 \leq i \leq n-1), \end{aligned}$$

with $F = t(t^{2\ell} + f(x))$, the Laplace–Beltrami operator with domain $C_c^\infty(\{t \neq 0\})$, or $C_c^\infty(\{\pm t > 0\})$, is essentially self-adjoint.

Proof. In both cases $\partial_t = X_0$ is a unit vector orthogonal to the remaining fields, which span the tangent space of the torus factor where $t \neq 0$; so $g = \mathrm{d}t^2 + h_t$ as in Setting 3.1, with the compact fibre $N = \mathbb{T}^{n-1}$ (resp. $\mathbb{T}^{n-1} \times \mathbb{T}^{n-1}$). The frame determinant is F^{n-1} in both cases (in the second, the matrix of (X_i, Y_i) is block triangular with diagonal blocks $F\,I$ and I), so $\mu_g = |F|^{-(n-1)}\mathrm{d}t\,\mathrm{d}x$ (resp. $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$) and $a = -t\partial_t \log|F|^{-(n-1)} = (n-1)\big(1 + 2\ell t^{2\ell}/(t^{2\ell} + f)\big) \geq 1$. Apply Theorem 3.2.[1] □

For comparison, [PRS18, eqs. (241)–(242)] give $\delta^2 V_{\text{eff}}$ for Example 7.7; for $n = 2$ it agrees with (3) (with z^2 replaced by f), and the effective-potential criterion applies exactly when $\ell \leq n/2$.

5 A first-order variant of the effective-potential criteria

Theorem 5.1. *Consider either the Riemannian setting of [PRS18, Theorem 3.1] with $V \equiv 0$ (a Riemannian manifold Ω with smooth measure ω for which the distance δ from the metric boundary is C^2 on $\Omega_\varepsilon = \{0 < \delta \leq \varepsilon\}$), in particular the almost-Riemannian setting of [PRS18, Theorem 7.9], or the sub-Riemannian setting of [FPR20, Theorem 1.1]; in the last two, $\Omega = M \setminus \mathcal{S}$ is the complement of the singular set $\mathcal{S}$ in the ambient manifold M. In each of these theorems replace the hypothesis $V_{\text{eff}} \geq 3/(4\delta^2) - \kappa/\delta$ on Ω_ε by*

$$-\Delta_\omega \delta \;\geq\; \frac{1}{\delta} - \kappa \qquad \text{on } \{0 < \delta \leq \varepsilon\}, \text{ for some } \kappa \geq 0. \tag{7}$$

Then Δ_ω with domain $C_c^\infty(\Omega)$ is essentially self-adjoint in $L^2(\Omega, \omega)$ (on Ω or on any of its connected components, as in those theorems).

Proof. In the proofs of [PRS18, Theorem 3.1] (with $V \equiv 0$, $\nu \equiv 0$) and [FPR20, Theorem 1.1], the hypothesis on V_{eff} is used only in Step 1 of the proof of the weak Hardy inequality [PRS18, Prop. 3.3, eq. (57)], [FPR20, Prop. 4.6, eq. (64)]. The conclusion of that step is

$$\int |\nabla u|^2 \mathrm{d}\omega \;\geq\; \int \Big(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\Big)|u|^2 \mathrm{d}\omega \tag{8}$$

for the functions $u \in W^1_{\text{comp}}$ supported in the collar considered there. Step 2 of those propositions, the Agmon-type estimates [PRS18, Prop. 3.4], [FPR20, Prop. 4.7], and the final deduction use the hypothesis only through (8) and its consequences. It therefore suffices to derive (8) from (7). In both papers the collar is identified with $(0, \varepsilon) \times X$, with $\delta(t, q) = t$, $\mathrm{d}\omega = e^{2\vartheta}\mathrm{d}t\,\mathrm{d}\mu(q)$ with ϑ smooth in t, and $\Delta_\omega \delta = 2\partial_t \vartheta$ ([PRS18, proof of Prop. 2.2, eq. (51)], [FPR20, eq. (61)]). Moreover $|\nabla u| \geq |\partial_t u|$, and for almost every q the function $t \mapsto u(t, q)$ lies in $W^{1,2}$ with compact support in $(0, \varepsilon)$, as noted there. Apply Lemma 2.1 with $m = e^{2\vartheta(\cdot, q)}$ and $b \equiv 1$: here $a = -t\partial_t \log m = -\delta\Delta_\omega \delta \geq 1 - \kappa t$ by (7), so $\int_0^\varepsilon |\partial_t u|^2 e^{2\vartheta} \mathrm{d}t \geq \int_0^\varepsilon (t^{-2} - \kappa t^{-1})|u|^2 e^{2\vartheta}\mathrm{d}t$. Integration over q gives (8). □

The compact-resolvent parts of those theorems are derived there from the same weak Hardy inequality, so they carry over as well; we do not need them.

Remark 5.2 (an alternative, not a strengthening). If $\mathrm{d}\omega = t^\alpha \mathrm{d}t\,\mathrm{d}\mu$ on the collar, then $-\Delta_\omega \delta = -\alpha/t$ and $V_{\text{eff}} = \alpha(\alpha - 2)/(4t^2)$. Thus (7) holds if and only if $\alpha \leq -1$, whereas $V_{\text{eff}} \geq 3/(4\delta^2)$ holds if and only if $\alpha \leq -1$ or $\alpha \geq 3$ [PRS18, Theorem 4.2]. Neither criterion implies the other: the family (1) with $\ell \geq 2$ satisfies (7) but not the effective-potential condition, while $\alpha \geq 3$ satisfies only the latter. Both are contained in one statement: by Lemma 2.1, the conclusion of Theorem 5.1 holds whenever there is a function b on the collar, C^1 in t, with $b(1 + a - b) - t\partial_t b \geq 1 - \kappa t$, where $a = -\delta\Delta_\omega \delta$. The choice $b \equiv 1$ gives (7); the choice $b = (1 + a)/2$ gives exactly $V_{\text{eff}} \geq 3/(4\delta^2) - \kappa/\delta$ (Remark 2.2).

[1] For Example 7.8, [PRS18] state that the effective potential is obtained from that of Example 7.7 by replacing $n - 1$ with $2(n - 1)$. Our computation of the frame determinant gives the exponent $n - 1$. Either way a is a positive integer multiple of $1 + 2\ell t^{2\ell}/(t^{2\ell} + f)$, so the conclusion does not depend on this point.

Example 5.3. In [PRS18, Example 4.1], $\mathrm{d}\omega = t^p(t^\ell + f(x))^k \mathrm{d}t\,\mathrm{d}\mu$ with $f \geq 0$ (we write p for the exponent called m there). In case 2 there ($p \leq -1$, $k < 0$) the effective-potential route requires $\ell \leq 1 - p$. Since $a = -p - k\ell\, t^\ell/(t^\ell + f) \geq -p \geq 1$, Theorem 5.1 gives essential self-adjointness for every $\ell \geq 0$ (in the setting of [PRS18, Theorem 3.1]).

6 The literal model on the whole plane

Proposition 6.1. *Let $\ell \geq 1$. The structure* (1) *on $\mathbb{R}^2$ is not complete, and its Laplace–Beltrami operator with domain $C_c^\infty(\mathbb{R}^2 \setminus \mathcal{S})$ is not essentially self-adjoint in $L^2(\mathbb{R}^2 \setminus \mathcal{S}, \mu_g)$.*

Proof. Write $F = x(x^{2\ell} + z^2)$ for $x > 0$. The curve $z \mapsto (1, z)$ has length $\int_0^\infty \mathrm{d}z/(1 + z^2) = \pi/2$, so $(1, j)_{j\in\mathbb{N}}$ is a Cauchy sequence without limit.

Let $\varphi \in C_c^\infty((1, 2))$ be real and not identically zero, and let $\chi \in C^\infty(\mathbb{R})$ be real with $\chi = 0$ on $(-\infty, 1]$ and $\chi = 1$ on $[2, \infty)$. Put $U = \varphi(x)\chi(z)$ and $W = \varphi(x)\chi(z)/z$. Both are smooth, supported in $[1, 2] \times [1, \infty)$, and in $L^2(\mu_g)$ because $1/F \leq 1/z^2$ there. For $z \geq 2$, $\Delta_g U = \varphi'' - (F_x/F)\varphi'$, and

$$\Delta_g W = \frac{\varphi'' - (F_x/F)\varphi'}{z} + \varphi\, F\partial_z\big(F\partial_z z^{-1}\big), \qquad F\partial_z\big(F\partial_z z^{-1}\big) = \frac{2x^{2\ell+2}(x^{2\ell} + z^2)}{z^3},$$

while F_x/F is bounded on $[1, 2] \times \mathbb{R}$. Hence $\Delta_g U, \Delta_g W \in L^2(\mu_g)$, and U, W belong to the domain of the adjoint T^* of $T = \Delta_g|_{C_c^\infty}$, which acts as Δ_g. In divergence form, $F^{-1}\Delta_g u = \partial_x(F^{-1}u_x) + \partial_z(F u_z)$. Integrating $(W\Delta_g U - U\Delta_g W)F^{-1}$ over $\{z < Z\}$ with $Z \geq 2$, the x-terms vanish and the z-terms leave the boundary term at $z = Z$:

$$\int_1^2 F(WU_z - UW_z)\big|_{z=Z}\mathrm{d}x = \int_1^2 \varphi^2 \frac{F(x, Z)}{Z^2}\mathrm{d}x = \int_1^2 x\varphi^2\Big(1 + \frac{x^{2\ell}}{Z^2}\Big)\mathrm{d}x.$$

Letting $Z \to \infty$ (dominated convergence on the left), $\langle T^*U, W\rangle - \langle U, T^*W\rangle = \int_1^2 x\varphi(x)^2\mathrm{d}x \neq 0$. So T^* is not symmetric, and T is not essentially self-adjoint. ☐

The failure comes from the ends $z \to \pm\infty$ at $x \neq 0$, far from $\mathcal{S}$; the same computation applies to $\ell = 1$. The completeness assumption of [BKP19, §3.1], and the complete setting of [PRS18, Example 7.7] cited there for $\ell = 1$, exclude it.

7 The general conjecture

Corollary 7.1. *Let M be a complete two-dimensional ARS whose singular set $\mathcal{S}$ is a compact embedded curve without tangency points. By [PRS18, Lemma 7.8], near each point of $\mathcal{S}$ there are coordinates (x, z) with $\delta = |x|$ and a local orthonormal frame $X_1 = \partial_x$, $X_2 = f\partial_z$. If, for some $\kappa \geq 0$, in each such chart near $\mathcal{S}$,*

$$x\,\partial_x \log|f(x, z)| \;\geq\; 1 - \kappa|x|, \tag{9}$$

then the Laplace–Beltrami operator on $M \setminus \mathcal{S}$ (or on any of its connected components) is essentially self-adjoint.

Proof. In such a chart $\mu_g = |f|^{-1}\mathrm{d}x\,\mathrm{d}z$ and $\nabla\delta = \pm\partial_x$ on $\{\pm x > 0\}$, so $-\Delta\delta = \mathrm{sgn}(x)\,\partial_x \log|f|$, and (9) is (7). By compactness finitely many charts cover a collar of $\mathcal{S}$. Apply Theorem 5.1 in the setting of [PRS18, Theorem 7.9]. ☐

Condition (9) holds, for example, at Grushin points ($f = xe^{\phi}$, where the left side is $1 + x\phi_x$); for regular structures in the sense of [PRS18] ($f = \pm(x\phi)^k$, giving $k(1+x\phi_x/\phi)$); for $f = x^k h$ with $k \geq 1$ and $|h|\, e^{\kappa|x|}$ nondecreasing in $|x|$, for instance $h = \sum_j c_j(z)x^{2j}$ with $c_j \geq 0$ and $h \neq 0$ for $x \neq 0$; and for the family of Theorem 1.1 with N_1 a circle. The following example shows the limits of the method. For $f = x((z-x)^2 + x^4)$, the exact values $x\partial_x \log|f| = 3 - 1/x$ at $z = x + x^2$ and $\delta^2 V_{\text{eff}} = 35/4 - 1/x^2$ at $z = x$ show that neither (7) nor the effective-potential condition holds. More is true: the weak Hardy inequality through which both criteria act fails.

Proposition 7.2. *Let $N_1 \in \{\mathbb{R}, \mathbb{R}/L\mathbb{Z}\}$ with $L \geq 1$, let $c_1 \in (0,1]$, and let $\sigma \in C^\infty(\mathbb{R})$ satisfy $c_1 y^2 \leq \sigma(y) \leq y^2$ for $|y| \leq 1$. Let $X_1 = \partial_x$, $X_2 = f\partial_z$ be an ARS on $\mathbb{R} \times N_1$ as at the beginning of Section 4, with $f(x,z) = x(\sigma(z-x)+x^4)$ for $0 < x < 1/2$ and $0 < z < 1/2$. Then for all $a > 0$, $\kappa \geq 0$, $\eta > 0$ and $C \in \mathbb{R}$ there is a Lipschitz function w with compact support in $\{0 < x < 1/2,\ 0 < z < 1/2\}$ such that*

$$\int |\nabla w|^2 \mathrm{d}\mu_g \ < \ a \int_{\{0<\delta<\eta\}} \Big(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\Big)|w|^2 \mathrm{d}\mu_g + C\int |w|^2 \mathrm{d}\mu_g. \tag{10}$$

This applies to the complete real-analytic ARS $f = x(\sin^2(z-x) + x^4)$ on $\mathbb{R} \times \mathbb{R}/\pi\mathbb{Z}$ (with $c_1 = 4/\pi^2$), whose singular set $\{x = 0\}$ is compact and has no tangency points.

The inequality opposite to (10), for all w with some a, κ, η, C, is the weak Hardy inequality [PRS18, eq. (57)], [FPR20, eqs. (64), (84)]; each of the hypotheses of [PRS18, Theorem 7.9], [FPR20, Theorem 1.1], Theorem 5.1 and Remark 5.2 implies it with $a = 1$.

Proof. By Lemma 4.1, $\delta = |x|$. Define the piecewise linear functions W, E by $W = 0$ outside $(\frac{1}{4}, 2)$, $W(s) = 4s - 1$ on $[\frac{1}{4}, \frac{1}{2}]$, $W = 1$ on $[\frac{1}{2}, \frac{3}{2}]$, $W(s) = 4 - 2s$ on $[\frac{3}{2}, 2]$; and $E = 0$ outside $(1,2)$, $E(r) = 4(r-1)$ on $[1, \frac{5}{4}]$, $E = 1$ on $[\frac{5}{4}, \frac{7}{4}]$, $E(r) = 4(2-r)$ on $[\frac{7}{4}, 2]$. For $0 < z_0 < 1/8$ put $w(x,z) = W(x/z)E(z/z_0)$. On its support $z_0 \leq z \leq 2z_0 < 1/4$ and $z/4 \leq x \leq 2z < 1/2$, so w has compact support in the open square $\{0 < x < 1/2,\ 0 < z < 1/2\}$ and $|z - x| \leq z \leq 1$. Let $A = \int |\partial_x w|^2 |f|^{-1}$, $B = \int f^2 |\partial_z w|^2 |f|^{-1}$, $D = \int |w|^2 x^{-2}|f|^{-1}$ and $N = \int |w|^2 |f|^{-1}$ (integrals in $\mathrm{d}x\,\mathrm{d}z$), so that $\int |\nabla w|^2 \mathrm{d}\mu_g = A + B$.

Lower bound for D. On $P = \{\frac{5}{4}z_0 \leq z \leq \frac{7}{4}z_0,\ |x - z| \leq z^2\}$ we have $x/z \in [1-z, 1+z] \subset [\frac{1}{2}, \frac{3}{2}]$, so $w = 1$; moreover $x \leq \frac{5}{4}z$ and $\sigma(z-x) \leq z^4$, hence $x^2 f \leq (\frac{5}{4})^3(1 + (\frac{5}{4})^4)z^7 = \frac{110125}{16384}z^7$. Therefore

$$D \geq \int_{5z_0/4}^{7z_0/4} \frac{2z^2 \cdot 16384}{110125\, z^7}\mathrm{d}z = \frac{32768}{110125} \cdot \frac{113664}{1500625}\, z_0^{-4} > \frac{z_0^{-4}}{45}.$$

Upper bound for A. Here $\partial_x w = W'(x/z)E(z/z_0)/z$ vanishes except on the ramps. On $x/z \in (\frac{1}{4}, \frac{1}{2})$: $|W'| = 4$, $x \geq z/4$ and $|z - x| \geq z/2$, so $f \geq x\sigma(z-x) \geq c_1 z^3/16$; the ramp has length $z/4$. On $x/z \in (\frac{3}{2}, 2)$: $|W'| = 2$, $x \geq 3z/2$, $|z-x| \geq z/2$, so $f \geq 3c_1 z^3/8$; the ramp has length $z/2$. Hence the inner integral is at most $(64 + \frac{16}{3})c_1^{-1}z^{-4}$, and $A \leq \frac{208}{3c_1}\int_{z_0}^{2z_0} z^{-4}\mathrm{d}z = \frac{182}{9c_1}z_0^{-3} \leq 21\, c_1^{-1}z_0^{-3}$.

Upper bound for B. We have $\partial_z w = -xz^{-2}W'E + WE'/z_0$, so $|\partial_z w| \leq 8/z + 4/z_0 \leq 12/z_0$, and $f \leq 2z(z^2 + 16z^4) \leq 4z^3 \leq 32z_0^3$. The support has area $\int_{z_0}^{2z_0} \frac{7}{4}z\,\mathrm{d}z = \frac{21}{8}z_0^2$. Hence $B \leq 32 \cdot 144 \cdot \frac{21}{8}\, z_0^3 = 12096\, z_0^3$.

Bound for N. Since $x \leq 4z_0$ on the support, $N \leq 16z_0^2 D$.

Conclusion. Suppose (10) fails for all such w, that is, $A + B \geq a\int_{\delta<\eta}(\delta^{-2} - \kappa\delta^{-1})|w|^2\mathrm{d}\mu_g + CN$. Take $0 < z_0 < \min\{1/8, \eta/8\}$. Then $\delta = x \leq 4z_0 < \eta$ on the support, and $\delta^{-2} - \kappa\delta^{-1} \geq (1 - 4\kappa z_0)\delta^{-2}$ there, so $A + B \geq (a(1-4\kappa z_0) - 16|C|z_0^2)D$. With the bounds above and $D > z_0^{-4}/45$,

$$a(1 - 4\kappa z_0) - 16|C|z_0^2 \ \leq\ \frac{A+B}{D} \ \leq\ 45\Big(\frac{21}{c_1}z_0 + 12096\, z_0^7\Big).$$

Letting $z_0 \to 0$ gives $a \le 0$, a contradiction. For the last statement, $\sin^2(z-x)$ is π-periodic in z, $\frac{4}{\pi^2}y^2 \le \sin^2 y \le y^2$ for $|y| \le 1$, and $f \ne 0$ for $x \ne 0$. Hörmander's condition holds at $(0,z)$ because $\partial_x f(0,z) = \sin^2 z$ and $\partial_x^3 f(0,z) = 6\cos 2z$ do not vanish simultaneously. Completeness follows from Lemma 4.1, and $X_1 = \partial_x$ is transversal to $\{x=0\}$. □

Thus the real-analytic conjecture of [BKP19] remains open beyond the class of Corollary 7.1 (condition (9) and compact singular set): by Proposition 7.2, the structure $f = x(\sin^2(z-x)+x^4)$ cannot be decided through the weak Hardy inequality in δ, and other distance-like functions or the coupling in z would be needed. We claim neither a proof nor a counterexample for it.

Verification. The proofs are by hand; the computations below only corroborate them.

1. A program by the finder of the argument performs, in exact rational arithmetic, 114 exact checks (identities, and also a minimisation, finiteness and distance checks, and a scaling check): the formula for Δ_g, the identity $a = 1+2\ell u$, formula (3) and its agreement with [PRS18, eqs. (241)–(242)] for $n = 2,\dots,5$, the pointwise form of Lemma 2.1, the minimum in (3), the computations of Proposition 6.1, the weights in Theorem 1.1 and in [PRS18, Example 4.1], the values quoted before Proposition 7.2, and the two choices of b in Remark 2.2.

2. Numerical programs (not proofs) compute fibrewise Hardy constants (all ≥ 1 for $\ell \le 6$, although $\min \delta^2 V_{\text{eff}} < 0$ for $\ell \ge 3$), the boundary form of Proposition 6.1 (twelve digits, $\ell = 1,2,3$), and the quotients in the proof of Proposition 7.2 (about $13.3\,z_0$).

3. An independent verifier checked every step by hand, re-ran all programs, and made independent finite-element and quadrature checks. A referee program, written from the text of this paper, re-derives the identities with exact rational jets, checks the constants of Proposition 7.2 exactly and its four integrals numerically for $\sigma(y) = y^2$ and $\sin^2 y$, the frame determinant in Corollary 4.3, and the boundary form of Proposition 6.1 with other φ and χ. A second independent (internal) referee checked the proofs line by line, wrote separate code (311 exact checks, including four negative controls, and 48 numerical checks, all passed) and reproduced all recorded outputs byte for byte.

Scope and priority. Theorem 1.1 answers the ℓ-question of Problem 1 of [BKP19, §3.2] affirmatively, in the complete setting that the report assumes, for the family (2); Corollary 4.3 covers all ℓ and n in [PRS18, Examples 7.7 and 7.8], including those where the effective-potential criterion does not apply. Taken literally on $\mathbb{R}^2$, the operator is not essentially self-adjoint (Proposition 6.1), for a reason unrelated to the singular set. The real-analytic no-tangency conjecture of [BKP19] is proved only under (9) and for compact singular set (which the report does not assume), and otherwise remains open. The case $\ell = 1$ and the cases $\ell \le n/2$, in the complete (torus) setting of [PRS18, Example 7.7], are due to Prandi, Rizzi and Seri [PRS18]. The ingredients are classical: the Agmon-type argument of [NN09, PRS18, FPR20] and Hardy inequalities from multipliers or ground states [BFT04, Bru04, Rob21]; see also Ward's work on essential self-adjointness on domains and on the associated Hardy constant [War16, War17]. Nenciu and Nenciu [NN22] prove weak anisotropic Hardy inequalities by the vector-field method and apply them to two-dimensional ARS. Their results do not cover (1): their assumption on the density, $\rho = r\delta^\gamma$ with $(|\nabla \log r| + |\nabla\gamma|)\delta^s$ bounded for some $s < 1$, would give $|\partial_z \log\rho| \le C\delta^{-s}(1+|\log\delta|)$, whereas $\partial_z \log\rho = -x^{-\ell}$ at $z = x^\ell$ for (1). Their Hardy weight also corresponds to the ground-state choice rather than to $b \equiv 1$. Other work on quantum confinement in ARS [BL13, GMP19, Poz20, Bes23, BBP23, BQ23] treats two-step, generic, curvature-perturbed or α-Grushin structures, not this family. The only new input is the choice $b \equiv 1$ (the trial function δ). We searched arXiv, OpenAlex (all works citing [PRS18] and [FPR20]), zbMATH and

Crossref, and ran web searches. We found no resolution of Problem 1 or of [PRS18, Example 7.7(2)]. We cannot exclude that criterion (7) appears elsewhere in some form, and this negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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