

Loose Embeddings in a Fixed Euclidean Space: Countable Counterexamples and Flat Tori

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Abstract

A loose embedding of a metric space (X, d) in $\mathbb{R}^N$, in the sense of Chirvasitu, is a continuous map φ such that $|\varphi x - \varphi y| = |\varphi x' - \varphi y'|$ if and only if $d(x, y) = d(x', y')$. In his abstract in an Oberwolfach report of 2019, Chirvasitu asked whether a compact metric space all of whose finite subspaces are loosely embeddable in $\mathbb{R}^N$, for one fixed N , is itself loosely embeddable there, and whether every compact Riemannian manifold with its geodesic distance satisfies this hypothesis. We answer both questions negatively, the first in the reading in which the conclusion asks for a loose embedding in the same $\mathbb{R}^N$, which is the reading of the problem record we work from. For every $N \geq 1$ we construct a countable compact metric space all of whose finite subspaces are loosely embeddable in $\mathbb{R}^N$ but which is not loosely embeddable in $\mathbb{R}^N$. This does not answer Chirvasitu's later Question 2.1 of arXiv:2101.01242, which asks only for a loose embedding in some finite-dimensional Euclidean space: our example for $N = 1$ is loosely embeddable in $\mathbb{R}^2$, and that question remains open. For the second question we prove that every flat torus of dimension at least 2 has, for every N , a finite subset that is not loosely embeddable in $\mathbb{R}^N$. This proof is non-constructive: for general N it gives no explicit finite subset and no bound on its size (for $N \leq 2$ we give an explicit four-point subset of the hexagonal torus). The argument also shows that no flat torus of dimension at least 2 is loosely embeddable in any Euclidean space, whereas round spheres and real and complex projective spaces are. The key lemma for tori is the flat-torus analogue of a theorem of Heitzig on maps between Euclidean spaces that preserve equality of distances, and its proof follows his method. The definitions and the questions are Chirvasitu's. We found no printed answer to either question in the sources searched; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The questions

Definition 1.1 (Chirvasitu [Chi19, Chi21, Chi22]). Let (X, d) be a metric space. A *loose embedding* of X in $\mathbb{R}^n$ is a continuous map $\varphi: X \rightarrow \mathbb{R}^n$ such that, for all $x, y, x', y' \in X$,

$$|\varphi x - \varphi y| = |\varphi x' - \varphi y'| \iff d(x, y) = d(x', y').$$

X is *loosely embeddable* (LE) in $\mathbb{R}^n$ if it has a loose embedding in $\mathbb{R}^n$.

This is the definition on p. 2860 of [Chi19] and Definition 2.2 of [Chi21]; Definition 0.1 of [Chi22] states it in an equivalent form. Numbers cited from [Chi21] are those of arXiv:2004.09962v2, the arXiv version that carries the journal DOI; numbers cited from [Chi22] are those of arXiv:2101.01242v1. The

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first arXiv version of [Chi21] and the journal version of [Chi22] number their statements differently (see the bibliography). Taking $x' = y'$ shows that a loose embedding is injective; for finite X continuity is automatic. Equalities and inequalities of distances are preserved, their order need not be. Loose embeddings arose from quantum isometry groups: by a slight generalisation of [Gos15, Corollary 4.9], compact metric spaces that are LE in some $\mathbb{R}^n$ have quantum isometry groups [Chi21, Theorem 0.1].

For a metric space X and $N \geq 1$ write

$$H(N) : \quad \text{every finite subset of } X, \text{ with the restricted metric, is LE in } \mathbb{R}^N.$$

The first two of the four questions in Chirvasitu’s abstract [Chi19, p. 2860] are the following.

Question 1.2. Let (X, d) be a compact metric space satisfying $H(N)$ for some N . Is X loosely embeddable in $\mathbb{R}^N$, for the same N ?

Question 1.3. Does every compact Riemannian manifold, with its geodesic distance, satisfy $H(N)$ for some N ?

Reading. The first question of the abstract ends by asking whether X is itself “uniformly embeddable” in $\mathbb{R}^N$. This word is not defined in the abstract; the only notion defined there is loose embeddability, and in its hypothesis N is called “uniformly-chosen”. In the introduction of [Chi22] the phrase “uniformly loosely embeddable” refers to finite subspaces that are LE in spaces of uniformly bounded dimension. We read “uniformly embeddable” as a slip for “loosely embeddable”, as the problem record does: record OWR-17288-004 of the UnsolvedMath collection [ula26], listed there as open, asks whether X “is itself loosely embeddable there”, that is, in the same $\mathbb{R}^N$. This is Question 1.2. The second question of the abstract asks whether every compact Riemannian manifold satisfies “the hypothesis of the preceding question”; this is Question 1.3 (the record writes “have this property”). The other conceivable reading, that $H(N)$ implies loose embeddability for such manifolds, would be satisfied vacuously by a manifold that fails $H(N)$ for every N ; the wording of the source rules it out. As in [Chi22, Definition 1.1], Riemannian manifolds are taken connected, so that the geodesic distance is finite.

Chirvasitu also posed both questions in writing: as Questions 3.16 and 3.17 of the first arXiv version of [Chi21] (arXiv:2004.09962v1, April 2020), together with a weak form of loose embeddability under $H(N)$ (Theorem 3.19 there); in the second version this material was moved to [Chi22], where the questions are Questions 2.1 and 2.2 of Section 2. Question 2.2 of [Chi22] is Question 1.3. Question 2.1 of [Chi22] has the hypothesis of Question 1.2, but asks only whether X is LE in *some* finite-dimensional Euclidean space [Chi22, Definition 0.1]. A negative answer to Question 1.2 need not answer Question 2.1 of [Chi22].

1.2 Results

Main Theorem. (a) *For every $N \geq 1$ there is a countable compact metric space that satisfies $H(N)$ and is not loosely embeddable in $\mathbb{R}^N$ (Theorems A and B). So Question 1.2, in the reading in which the conclusion asks for a loose embedding in the same $\mathbb{R}^N$, has a negative answer for every $N \geq 1$. This does not answer [Chi22, Question 2.1], which asks only for a loose embedding in some finite-dimensional Euclidean space: the example for $N = 1$ is loosely embeddable in $\mathbb{R}^2$ (Proposition 5.1), and [Chi22, Question 2.1] remains open.*

(b) *No flat torus $\mathbb{R}^n/\Lambda$ of dimension $n \geq 2$, with its geodesic distance, satisfies $H(N)$ for any N (Theorem C). In particular the square torus $\mathbb{R}^2/\mathbb{Z}^2$ does not, and Question 1.3, which is [Chi22, Question 2.2], has a negative answer. The proof of (b) is non-constructive: for general N it gives no explicit finite subset that is not loosely embeddable in $\mathbb{R}^N$ and no bound on its size.*

(c) *No flat torus of dimension at least 2 is loosely embeddable in any Euclidean space (Corollary D).*

For $N = 1$ the example is $X_1 = \{p, x_1, x_2, \dots\}$ with $d(p, x_k) = 2^{-k}$, $d(x_k, x_{k+1}) = 2^{-k}$ and $d(x_j, x_k) = 2^{-j} + 2^{-k-1}$ for $k \geq j + 2$; a loose embedding in $\mathbb{R}$ would have to double $|\varphi x_k - \varphi p|$ at each step (Section 3). For $N \geq 2$ we glue regular simplices with centres so that, by an elementary fact on equidistant points in $\mathbb{R}^N$, the edge lengths of a loose embedding in $\mathbb{R}^N$ must grow geometrically (Section 4); that every finite subset is LE in $\mathbb{R}^N$ is proved by a genericity argument, not by an explicit map. For flat tori we pass from loose embeddings of finite subsets to a continuous non-constant map whose chord lengths depend only on the distance (Lemma 6.1), and show with the translations of the torus and a character expansion that such a map is constant (Lemma 6.2, the flat-torus analogue of a theorem of Heitzig [Hei02] for maps between Euclidean spaces, proved by his method). For the hexagonal torus four points with equal mutual distances give an explicit finite witness for $N \leq 2$ (Proposition 6.3); for the square torus and for $N \geq 3$ we know no explicit witness. On the positive side, the circle, round spheres and real and complex projective spaces are LE in a Euclidean space and so satisfy $H(N)$ for a suitable N (Section 7).

1.3 Known results used or compared

Finite metric spaces are LE in some Euclidean space [Chi21, Remark 2.4], by [DM90, Corollary 3]; so $H(N)$ is a condition on the dimension only. Countable compact spaces containing regular simplices of every size are not LE in any $\mathbb{R}^d$ [Chi21, Example 2.3]; they fail $H(N)$ for every N , unlike our examples. Generic 0-dimensional compact metric spaces, and those with an injective distance function, are LE in $\mathbb{R}$ [Chi21, Theorem 3.1 and Proposition 3.3]; our examples are 0-dimensional but have designed equal distances. The circle is LE in $\mathbb{R}^2$ [Chi22, Lemma 1.2]; a compact Riemannian manifold contains no arbitrarily large regular simplices [Chi22, Proposition 1.3] and no arbitrarily long flags of median hyperplanes [Chi22, Theorem 1.6]. Flat tori, being compact Riemannian manifolds, satisfy these necessary conditions; Theorem C rests on a global obstruction instead. Under $H(N)$ a compact Riemannian metric space is weakly LE in $\mathbb{R}^N$ [Chi22, Theorem 2.4]; we do not use this theorem. Heitzig [Hei02, Theorem 1] proved that a continuous map $\mathbb{R}^m \rightarrow \mathbb{R}^n$, $m \geq 2$, with $|x - y| = |z - w| \Rightarrow |hx - hy| = |hz - hw|$ is a similarity; his Lemma 1 writes such a map on a monoid with a left-invariant distance function as an orbit of a homomorphism into the motions of $\mathbb{R}^n$. Lemma 6.2 is the analogue of his theorem for flat tori; Steps 1–2 of its proof are his Lemma 1 in our setting, and Steps 3–5 follow the route of his proof (representation by motions, trigonometric expansion of the squared chord, independence of the direction). His theorem does not apply to tori directly, because the distance of a flat torus is a function of $|x - y|$ only locally. Standard tools are used without further reference: ultrafilter limits, the extension of isometries between subsets of a Euclidean space, the decomposition of continuous unitary representations of compact abelian groups into characters, and the identity theorem for real-analytic functions.

We found no printed answer to either question in the sources searched (the arXiv, Crossref and zbMATH Open databases, the citation data of Semantic Scholar and OpenAlex, and web searches; see the paragraph “Scope and priority” in Section 9). A search that finds nothing is not a proof of novelty.

2 Preliminaries

For a metric space (X, d) let $D(X) = \{d(x, y) : x, y \in X\}$ be its set of distances, and $|\cdot|$ the Euclidean norm.

Lemma 2.1. *Let (X, d) be a metric space.*

(a) *A continuous map $\varphi : X \rightarrow \mathbb{R}^n$ is a loose embedding if and only if there is an injective function*

$\Phi: D(X) \rightarrow [0, \infty)$ with $|\varphi x - \varphi y| = \Phi(d(x, y))$ for all $x, y \in X$. Then $\Phi(0) = 0$, $\Phi(t) > 0$ for $t > 0$, and φ is injective.

- (b) The restriction of a loose embedding of X to a subset Y is a loose embedding of Y . Hence, if every finite subset of X is contained in a subset that is LE in $\mathbb{R}^N$, then X satisfies $H(N)$; and if X is LE in $\mathbb{R}^N$, then X satisfies $H(N)$.
- (c) If φ is a loose embedding and $\sigma(u) = \lambda Au + b$ with $\lambda > 0$, A orthogonal and $b \in \mathbb{R}^n$, then $\sigma \circ \varphi$ is a loose embedding.

Proof. (a) If φ is a loose embedding, $\Phi(d(x, y)) := |\varphi x - \varphi y|$ is well defined by the implication “ $\Leftarrow$ ” of Definition 1.1 and injective by “ $\Rightarrow$ ”. Conversely such a Φ gives both implications. $\Phi(0) = |\varphi x - \varphi x| = 0$, and $\Phi(t) \neq \Phi(0)$ for $t > 0$ by injectivity; so $\varphi x \neq \varphi y$ for $x \neq y$. (b) and (c) are immediate from (a). $\square$

Lemma 2.2. Let $N \geq 1$ and let $P_0, \dots, P_N \in \mathbb{R}^N$ be pairwise at distance $S > 0$.

- (a) The vectors $P_i - P_0$, $1 \leq i \leq N$, form a basis of $\mathbb{R}^N$. No $N + 2$ points of $\mathbb{R}^N$ are pairwise at the same positive distance.
- (b) If $Q \in \mathbb{R}^N$ and $|Q - P_i| = R$ for $0 \leq i \leq N$, then Q is the centroid of $P_0, \dots, P_N$ and $R = c_N S$, where

$$c_N = \left(\frac{N}{2N+2} \right)^{1/2}, \quad 0 < c_N < 2^{-1/2}.$$

Proof. (a) By $2u \cdot v = |u|^2 + |v|^2 - |u - v|^2$, the Gram matrix of $P_1 - P_0, \dots, P_N - P_0$ is $\frac{S^2}{2}(I + J)$, where J is the all-ones matrix. Its eigenvalues are $\frac{S^2}{2}$ and $\frac{(N+1)S^2}{2}$, so it is positive definite and the vectors are linearly independent. If there were a further point P_{N+1} at distance S from $P_0, \dots, P_N$, the same computation would give $N + 1$ linearly independent vectors in $\mathbb{R}^N$.

(b) The equations $|Q - P_i|^2 = |Q - P_0|^2$, $1 \leq i \leq N$, read $2Q \cdot (P_0 - P_i) = |P_0|^2 - |P_i|^2$; by (a) their normals are linearly independent, so they have exactly one solution. The centroid $G = \frac{1}{N+1} \sum_j P_j$ is a solution: $G - P_0 = \frac{1}{N+1} \sum_{j \geq 1} (P_j - P_0)$, so by the Gram matrix

$$|G - P_0|^2 = \frac{1}{(N+1)^2} \left(NS^2 + N(N-1) \frac{S^2}{2} \right) = \frac{N}{2N+2} S^2,$$

and the same computation with P_i in the role of P_0 gives the same value for $|G - P_i|^2$. Hence $Q = G$ and $R = c_N S$. $\square$

3 The case $N = 1$

Let $X_1 = \{p, x_1, x_2, \dots\}$, with the symmetric function d that vanishes on the diagonal and, for $1 \leq j < k$, is given by

$$d(p, x_k) = 2^{-k}, \quad d(x_j, x_{j+1}) = 2^{-j}, \quad d(x_j, x_k) = 2^{-j} + 2^{-k-1} \quad (k \geq j + 2).$$

Lemma 3.1. (a) d is a metric on X_1 , and X_1 is compact, with p as its only accumulation point. (b) The only equalities between distances of distinct pairs of points of X_1 are

$$d(p, x_k) = d(x_k, x_{k+1}) = 2^{-k} \quad (k \geq 1). \quad (3.1)$$

Proof. (a) For $j < k$ we have $2^{-j} \leq d(x_j, x_k) \leq 2^{-j} + 2^{-k-1}$. For a triangle p, x_j, x_k with $j < k$: $d(x_j, x_k) \leq 2^{-j} + 2^{-k-1} < d(p, x_j) + d(p, x_k)$; $d(p, x_j) = 2^{-j} \leq d(x_j, x_k)$; and $d(p, x_k) < 2^{-j} = d(p, x_j)$. For a triangle x_i, x_j, x_k with $i < j < k$: $d(x_i, x_k) = 2^{-i} + 2^{-k-1} \leq 2^{-i} + 2^{-j} \leq d(x_i, x_j) + d(x_j, x_k)$; $d(x_i, x_j) \leq 2^{-i} + 2^{-j-1} < 2^{-i} + 2^{-j} \leq d(x_i, x_k) + d(x_j, x_k)$; and $d(x_j, x_k) \leq 2^{-j} + 2^{-k-1} < 2^{-j+1} \leq 2^{-i} \leq d(x_i, x_j)$. So the triangle inequality holds, and d is positive off the diagonal. Since $d(p, x_k) \rightarrow 0$, every x_k is isolated, $x_k \rightarrow p$, and every sequence in X_1 has a convergent subsequence.

(b) The values 2^{-k} are taken exactly by the pairs (p, x_k) and (x_k, x_{k+1}) . A value $2^{-j} + 2^{-k-1}$ with $k \geq j + 2$ has exactly two digits 1 in its finite binary expansion, at the distinct positions j and $k + 1$; so it is not a power of 2, and it determines (j, k) . $\square$

Theorem A. X_1 is a countable compact metric space that satisfies $H(1)$ and is not loosely embeddable in $\mathbb{R}$.

Proof. X_1 is not LE in $\mathbb{R}$. Let $f: X_1 \rightarrow \mathbb{R}$ be a loose embedding; by Lemma 2.1(c) we may assume $f(p) = 0$. Put $g_k = f(x_k)$. By (3.1), $|g_{k+1} - g_k| = |g_k|$, so $g_{k+1} \in \{0, 2g_k\}$. Since f is injective, $g_{k+1} \neq 0 = f(p)$; hence $g_k = 2^{k-1}g_1$ with $g_1 \neq 0$, and $|g_k| \rightarrow \infty$. But $x_k \rightarrow p$ and f is continuous, so $g_k \rightarrow f(p) = 0$, a contradiction.

X_1 satisfies $H(1)$. Let $Y_K = \{p, x_1, \dots, x_K\}$ and define $f(p) = 0$, $f(x_k) = 2^{k-1}$. The image distances are 2^{k-1} for (p, x_k) and $2^{l-1} - 2^{k-1} = 2^{k-1}(2^{l-k} - 1)$ for (x_k, x_l) , $k < l$; all are positive. Comparing 2-adic valuations and odd parts, $2^{k-1} = 2^{k'-1}$ only if $k = k'$; $2^{k-1}(2^{l-k} - 1) = 2^{k'-1}(2^{l'-k'} - 1)$ only if $(k, l) = (k', l')$; and $2^{k-1}(2^{l-k} - 1) = 2^{j-1}$ only if $j = k$ and $l = k + 1$. So the equalities between image distances of distinct pairs are exactly the instances of (3.1) inside Y_K , and by Lemma 3.1(b) f is a loose embedding of Y_K in $\mathbb{R}$. Every finite subset of X_1 lies in some Y_K , so X_1 satisfies $H(1)$ by Lemma 2.1(b). $\square$

Remark 3.2. The programs of Section 9 also check an explicit loose embedding defined directly on an arbitrary finite $F \subset X_1$. Split $\{k : x_k \in F\}$ into maximal runs $R_1 < \dots < R_s$ of consecutive integers, with first elements a_r ; let $m + 1$ be the length of the longest run and $M \geq 2^{m+3}$ any integer (for example a power of 2). Put $f(p) = 0$ and $f(x_k) = M^{r-1}2^{k-a_r}$ for $k \in R_r$. Each image distance is $|\sum_r L_r M^{r-1}|$ for an integer vector L of one of the forms $2^i e_r$, $(2^{i'} - 2^i)e_r$ with $i < i'$, or $2^{i'} e_{r'} - 2^i e_r$ with $r < r'$ ($0 \leq i, i' \leq m$, e_r the unit vectors). If $c \neq 0$ is an integer vector with all $|c_r| \leq M/2$, then $\sum_r c_r M^{r-1} \neq 0$: for the largest r with $c_r \neq 0$, $|c_r M^{r-1}| \geq M^{r-1} > \frac{M}{2} \cdot \frac{M^{r-1}-1}{M-1}$. Hence no image distance is 0; and since the coefficients of $L_1 \mp L_2$ are at most $2^{m+1} \leq M/4$ in absolute value, two image distances are equal only if $L_1 = \pm L_2$. Vectors with different numbers of non-zero entries are not equal up to sign. For vectors with two non-zero entries the positive entry sits at the larger index, so $L_1 = \pm L_2$ forces the same pair. For one non-zero entry both vectors are positive multiples of the same e_r , and the valuation argument of the proof above leaves only the instances of (3.1). So f is a loose embedding of F . For $F = Y_K$ it is the map used in the proof. (The argument uses only $2^{m+1} \leq M/2$, so $M \geq 2^{m+2}$ suffices as well.)

4 Every $N \geq 2$

Fix $N \geq 2$ and let $c = c_N$ be the constant of Lemma 2.2.

Gadgets. For $k \geq 1$ let $G_k = \{v_{k,0}, \dots, v_{k,N}, c_k\}$ with the metric $d_k(v_{k,i}, v_{k,j}) = 2^{-k}$ for $i \neq j$ and $d_k(c_k, v_{k,i}) = 2^{-k+1}$ (the only triangle inequalities to check are $2^{-k} \leq 2 \cdot 2^{-k+1}$ and $2^{-k+1} \leq 2^{-k+1} + 2^{-k}$). The pairs $\{v_{k,i}, v_{k,j}\}$ are the *edges* and the pairs $\{c_k, v_{k,i}\}$ the *radii* of G_k . Edges of G_k and radii of G_{k+1} have the same length 2^{-k} .

Heights. Choose for every gadget point $a \in G_k$ a number $e(a) = 2^{-k}(1 + \theta_a)$ with $\theta_a \in [0, \frac{1}{2})$ such that the family $\{1\} \cup \{e(a)\}$ is linearly independent over $\mathbb{Q}$. This is possible: enumerate the gadget

points as $a_1, a_2, \dots$; once $\theta_{a_1}, \dots, \theta_{a_{n-1}}$ are chosen, the $\mathbb{Q}$ -span of $1, e(a_1), \dots, e(a_{n-1})$ is countable, and we choose θ_{a_n} so that $e(a_n)$ lies outside it. (An explicit choice is $\theta_a = \frac{1}{2}(\sqrt{q_a} - \lfloor \sqrt{q_a} \rfloor)$ with pairwise distinct primes q_a ; then $e(a)$ is a rational number plus $2^{-k-1}\sqrt{q_a}$, and the independence follows from that of 1 and the square roots of distinct primes [Bes40]. Only the existence is needed.)

The space. The blocks are $\{p\}$ and the gadgets G_k . Put $h(a) = 2^{-k}$ for $a \in G_k$ and $h(p) = e(p) = 0$. Let $X_N = \{p\} \cup \bigcup_{k \geq 1} G_k$ with

$$d(a, b) = \begin{cases} d_k(a, b), & a, b \in G_k, \\ e(a) + |h(a) - h(b)| + e(b), & a, b \text{ in different blocks.} \end{cases}$$

Pairs in different blocks are called *cross pairs*.

Lemma 4.1. (a) d is a metric on X_N , and X_N is compact, with p as its only accumulation point.
(b) The classes of pairs of distinct points with equal distance are

$$E_j = \{\text{edges of } G_j\} \cup \{\text{radii of } G_{j+1}\} \quad (j \geq 1), \quad E_0 = \{\text{radii of } G_1\}, \quad (4.1)$$

with the distances 2^{-j} , and the singletons $\{\{a, b\}\}$ of the cross pairs.

Proof. (a) For distinct a, b in one gadget G_k ,

$$|e(a) - e(b)| < 2^{-k-1} < 2^{-k} \leq d_k(a, b) \leq 2^{-k+1} \leq e(a) + e(b). \quad (*)$$

Let x, y, z be distinct; we show $d(x, z) \leq d(x, y) + d(y, z)$. If all three lie in one gadget, d_k is a metric. If x, z lie in one block and y in another, then the block is a gadget (if the block is $\{p\}$ then $x = z$), and $d(x, z) \leq e(x) + e(z) \leq d(x, y) + d(y, z)$ by (*). If x, y lie in one block and z in another, then $h(x) = h(y)$ and $e(x) \leq d(x, y) + e(y)$ by (*), so $d(x, z) \leq d(x, y) + e(y) + |h(y) - h(z)| + e(z) = d(x, y) + d(y, z)$; the case y, z in one block is symmetric. If the three blocks are different, the claim follows from $|h(x) - h(z)| \leq |h(x) - h(y)| + |h(y) - h(z)|$ and $e(y) \geq 0$. A cross distance is at least $e(a) + e(b) > 0$, or $e(a) + h(a) > 0$ if the other point is p . Finally $d(p, a) < 3 \cdot 2^{-k}$ for $a \in G_k$, and each gadget is finite; so every point other than p is isolated, and every sequence has a convergent subsequence.

(b) Every distance has the form $q + \sum_a n_a e(a)$ with $q \in \mathbb{Q}$ and integers n_a : inside a gadget it is 2^{-k} or 2^{-k+1} ; for a cross pair it is $e(a) + e(b) + |h(a) - h(b)|$, with $e(p) = 0$. By the linear independence of $\{1\} \cup \{e(a)\}$, two such expressions are equal only if the rational parts and all coefficients n_a agree. The e -part of a cross pair is non-zero and is the sum of e over the gadget points of the pair, which determines the pair. So no cross distance equals another distance, and the distances inside gadgets are 2^{-j} exactly on E_j , $j \geq 0$. $\square$

Theorem B. For every $N \geq 2$, X_N is a countable compact metric space that satisfies $H(N)$ and is not loosely embeddable in $\mathbb{R}^N$.

Proof. X_N is not LE in $\mathbb{R}^N$. Let φ be a loose embedding of X_N in $\mathbb{R}^N$. By (4.1) all edges of G_k have one image length S_k , positive by injectivity, and all radii of G_k one image length R_k . The points $\varphi(v_{k,i})$ are pairwise at distance S_k and $\varphi(c_k)$ is at distance R_k from all of them, so $R_k = cS_k$ by Lemma 2.2(b). Edges of G_k and radii of G_{k+1} lie in E_k , so $S_k = R_{k+1} = cS_{k+1}$. Hence $S_k = c^{-(k-1)}S_1 \rightarrow \infty$, while $S_k \leq \text{diam } \varphi(X_N) < \infty$ because $\varphi(X_N)$ is compact.

X_N satisfies $H(N)$. Fix $K \geq 1$ and let $Y_K = \{p\} \cup G_1 \cup \dots \cup G_K$. Fix a regular simplex $P_0, \dots, P_N$ in $\mathbb{R}^N$ with edge 1 and centroid 0; then $|P_i| = c$ by Lemma 2.2(b). Put $\hat{v}_{k,i} = P_i$ and $\hat{c}_k = 0$. The parameter space $\Pi = \text{SO}(N)^K \times (\mathbb{R}^N)^{K+1}$, with points $\pi = (U_1, \dots, U_K, t_p, t_1, \dots, t_K)$, is a connected real-analytic manifold. Let $\beta(a) \in \{p, 1, \dots, K\}$ be the block of a , put $w_p = 0$ and $w_a = c^{-k}U_k \hat{a}$ for $a \in G_k$, and define $\varphi_\pi(a) = t_{\beta(a)} + w_a$.

Inside G_k , $|\varphi_\pi v_{k,i} - \varphi_\pi v_{k,j}| = c^{-k}$ and $|\varphi_\pi v_{k,i} - \varphi_\pi c_k| = c^{-k}|P_i| = c^{-k+1}$. So for every π the pairs of E_j in Y_K have image length c^{-j} ($0 \leq j \leq K$), and these values are distinct because $0 < c < 1$. Call π *good* if (i) no cross pair of Y_K has image length in $\{0\} \cup \{c^{-j} : 0 \leq j \leq K\}$, and (ii) distinct cross pairs of Y_K have distinct image lengths. If π is good, φ_π is injective; two pairs inside blocks have equal image lengths exactly when they lie in the same class E_j , and by (i) and (ii) no cross pair shares its image length with another pair. By Lemma 4.1(b) these are exactly the equalities of distances in Y_K , so φ_π is a loose embedding of Y_K .

Each failure of (i) or (ii) is the vanishing of one of finitely many functions $h: \Pi \rightarrow \mathbb{R}$, each a polynomial in the matrix entries and the coordinates, hence real-analytic. We show that no h vanishes identically; then each zero set is closed with empty interior, by the identity theorem, so is their finite union, and a good π exists. Let e be a unit vector and $s \in \mathbb{R}$.

- (1) A cross pair $\{a, b\}$ with blocks $k \neq l$ and squared image length $C \in \{0\} \cup \{c^{-2j} : 0 \leq j \leq K\}$: $h = |t_k - t_l + w_a - w_b|^2 - C$. With U and the other translations fixed and $t_k = se$, $h \rightarrow \infty$ as $s \rightarrow \infty$.
- (2) Cross pairs $\{a, b\}, \{a', b'\}$ with different sets of blocks $\{k, l\} \neq \{k', l'\}$: $h = |t_k - t_l + w_a - w_b|^2 - |t_{k'} - t_{l'} + w_{a'} - w_{b'}|^2$. One of the two sets contains a block index not in the other, say $k \notin \{k', l'\}$; with $t_k = se$ and the other translations 0, the first term grows like s^2 and the second is constant.
- (3) Distinct cross pairs with the same blocks, labelled so that a, a' lie in block k and b, b' in block l . With $w = w_a - w_b$ and $w' = w_{a'} - w_{b'}$,

$$h = |t_k - t_l + w|^2 - |t_k - t_l + w'|^2 = 2(t_k - t_l) \cdot (w - w') + |w|^2 - |w'|^2,$$

a non-constant affine function of t_k as soon as $w \neq w'$ for the chosen rotations; and $w - w' = (w_a - w_{a'}) - (w_b - w_{b'})$. If $a = a'$, then $b \neq b'$, so l is a gadget, and $w - w' = w_{b'} - w_b \neq 0$ because $0, P_0, \dots, P_N$ are distinct and $c^{-l}U_l$ is invertible; the case $b = b'$ is symmetric. If $a \neq a'$ and $b \neq b'$, both blocks have two points, so both are gadgets, and $w_a - w_{a'} = U_k\alpha$, $w_b - w_{b'} = U_l\alpha'$ with non-zero vectors $\alpha = c^{-k}(\hat{a} - \hat{a}')$, $\alpha' = c^{-l}(\hat{b} - \hat{b}')$. Take $U_l = I$, and $U_k = I$ if $\alpha \neq \alpha'$; otherwise let U_k be the rotation by the angle π in a plane containing α , which exists because $N \geq 2$, so that $U_k\alpha = -\alpha \neq \alpha'$. Then $w \neq w'$.

So Y_K is LE in $\mathbb{R}^N$ for every K . Every finite subset of X_N lies in some Y_K , and X_N satisfies $H(N)$ by Lemma 2.1(b). $\square$

This half of the proof is a genericity argument: it shows that almost every parameter π gives a loose embedding of Y_K , but it does not exhibit one. Section 9 lists exact checks of explicit parameters for $N = 2, 3$.

Remark 4.2. For $N = 1$ the construction fails, because the rotation step in (3) is not available: $X'_1 = \{p\} \cup \bigcup_k G_k$ built in the same way with $N = 1$ does not satisfy $H(1)$. Indeed, by Lemma 2.2(b) for $N = 1$ and the classes E_k , a loose embedding of $G_k \cup G_{k+1}$ in $\mathbb{R}$ maps G_k onto $\{u, u + \frac{S}{2}, u + S\}$ and G_{k+1} onto $\{u', u' + S, u' + 2S\}$ for some u, u' and $S > 0$; the differences $(u' + iS) - (u + \frac{i}{2}S)$ coincide for $(i, j) = (0, 0), (1, 2)$, so two distinct cross pairs have equal image length. This is why Theorem A uses a different space.

5 What Theorems A and B do not answer

Theorems A and B answer Question 1.2 in the reading “loosely embeddable in the same $\mathbb{R}^N$ ”. They do not answer [Chi22, Question 2.1], which under $H(N)$ asks only for a loose embedding in some finite-dimensional Euclidean space.

Proposition 5.1. X_1 is loosely embeddable in $\mathbb{R}^2$.

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$. We choose $g_1, g_2, \dots \in \mathbb{C}$ and put $\varphi(p) = 0$, $\varphi(x_k) = g_k$. Let $Y_k = \{p, x_1, \dots, x_k\}$ and let (I_k) be the statement that $\varphi|_{Y_k}$ is a loose embedding of Y_k . Put $g_1 = 1$; (I_1) holds. Suppose that $g_1, \dots, g_k$ satisfy (I_k) . The circle $C_k = \{z : |z - g_k| = |g_k|\}$ passes through 0 and $2g_k$, and along each of its arcs from 0 to $2g_k$ the modulus $|z|$ runs continuously from 0 to $2|g_k|$; so the open arc set $A_k = \{z \in C_k : \frac{1}{8}|g_k| < |z| < \frac{1}{4}|g_k|\}$ is infinite.

Let $z \in C_k$ and $g_{k+1} = z$. The pairs of Y_{k+1} through x_{k+1} get the image lengths $|z|$, $|z - g_k| = |g_k|$ and $|z - g_j|$ ($j < k$). By (3.1) the only equality of distances in Y_{k+1} that involves a pair through x_{k+1} is $d(x_k, x_{k+1}) = d(p, x_k)$, and it holds for the images since $|z - g_k| = |g_k - 0|$; moreover (p, x_k) is the only pair of Y_k with image length $|g_k|$, by (I_k) . Hence (I_{k+1}) holds as soon as $z \notin \{0, g_1, \dots, g_k\}$ and the numbers $|z|$, $|z - g_j|$ ($j < k$) are pairwise distinct and distinct from the finitely many image lengths of pairs of Y_k . Each of these finitely many conditions excludes at most two points of C_k : the equations $|z| = v$ and $|z - g_j| = v$ ($v > 0$, $j < k$) describe circles whose centres 0, g_j differ from the centre g_k of C_k , and $|z| = |z - g_j|$, $|z - g_i| = |z - g_j|$ ($i < j < k$) describe lines. Choose $g_{k+1} \in A_k$ outside the excluded points. Then (I_{k+1}) holds and $|g_{k+1}| < \frac{1}{4}|g_k|$.

The resulting $\varphi: X_1 \rightarrow \mathbb{C}$ is injective and satisfies the condition of Definition 1.1 for any two pairs, since any two pairs lie in some Y_k . It is continuous: the x_k are isolated, and $|g_k| \leq 4^{1-k} \rightarrow 0 = \varphi(p)$. $\square$

So X_1 satisfies $H(1)$, is not LE in $\mathbb{R}$, and is LE in $\mathbb{R}^2$: it is not a counterexample to [Chi22, Question 2.1]. For $N \geq 2$ we have not decided whether X_N is LE in some Euclidean space, and we make no claim about it. [Chi22, Question 2.1] is not answered in this note.

6 Flat tori

6.1 Elementary facts

Let $\Lambda \subset \mathbb{R}^n$ be a lattice, $n \geq 2$, and $T = \mathbb{R}^n/\Lambda$ with the flat metric; write $[x]$ for the class of $x \in \mathbb{R}^n$. Its geodesic distance is $d([x], [y]) = \min_{v \in \Lambda} |x - y - v|$; it is invariant under the translations $[x] \mapsto [x + a]$. Let λ_1 be the length of a shortest non-zero vector of Λ .

(T1) If $|z| < \lambda_1/2$, then $d([z], [0]) = |z|$, since $|z - v| \geq \lambda_1 - |z| > |z|$ for $v \in \Lambda \setminus \{0\}$. Hence points of a ball of radius $\lambda_1/4$ in $\mathbb{R}^n$ have, in T , their Euclidean mutual distances.

(P1) For $x, y \in T$ and $k \geq 1$ there are $z_0 = x, z_1, \dots, z_k = y$ with $d(z_{i-1}, z_i) = d(x, y)/k$: take lifts $\tilde{x}, \tilde{y}$ with $|\tilde{x} - \tilde{y}| = d(x, y)$ and the classes of the equally spaced points of the segment between them; then $d(z_{i-1}, z_i) \leq d(x, y)/k$ and $d(x, y) \leq \sum_i d(z_{i-1}, z_i)$, so all steps equal $d(x, y)/k$.

(P2) Let $l_0 = \lambda_1/8$. For $l \in (0, l_0]$ and $s \in (0, 2l]$ there are $a, b, c \in T$ with $d(a, b) = s$ and $d(c, a) = d(c, b) = l$: take a Euclidean isosceles triangle with apex 0, legs l and base s in a plane of $\mathbb{R}^n$, and apply (T1).

(P3) T has no isolated points.

6.2 From finite subsets to a continuous map

Lemma 6.1. Let (M, d) be a compact metric space and $l_0 > 0$ such that (P1), (P2) (with this l_0) and (P3) hold for M . If M satisfies $H(N)$, then there are a continuous non-constant map $\psi: M \rightarrow \mathbb{R}^N$ and a function $\Phi: D(M) \rightarrow [0, \infty)$ with $|\psi x - \psi y| = \Phi(d(x, y))$ for all $x, y \in M$.

Proof. Normalisation. Fix $p \in M$ and, by (P2), points a_0, b_0 with $d(a_0, b_0) = t_0 := l_0$; let $A_0 = \{p, a_0, b_0\}$ and let $\mathcal{F}$ be the set of finite subsets of M containing A_0 . For $F \in \mathcal{F}$ choose a loose embedding φ_F of F in $\mathbb{R}^N$ and put $\psi_F = (\varphi_F - \varphi_F(p))/|\varphi_F(a_0) - \varphi_F(b_0)|$, a loose embedding of F by Lemma 2.1(c). By Lemma 2.1(a) there is an injective $\Phi_F: D(F) \rightarrow [0, \infty)$ with $|\psi_F x - \psi_F y| = \Phi_F(d(x, y))$ on F , and $\psi_F(p) = 0$, $\Phi_F(t_0) = 1$.

Uniform bound. Let $x, y \in M$, $t = d(x, y) > 0$ and $k = \lceil t/t_0 \rceil$, so $t/k \leq t_0$. Take $z_0, \dots, z_k$ for (x, y) by (P1), and by (P2), with $l = t_0$ and $s = t/k$, points a, b, c with $d(a, b) = t/k$ and $d(c, a) = d(c, b) = t_0$. Let $S(x, y)$ be the finite set formed by A_0 , x , y , the z_i and a, b, c . If $F \in \mathcal{F}$ contains $S(x, y)$, then

$$|\psi_F x - \psi_F y| \leq \sum_{i=1}^k |\psi_F z_{i-1} - \psi_F z_i| = k \Phi_F(t/k) \leq k(|\psi_F a - \psi_F c| + |\psi_F c - \psi_F b|) = 2k,$$

and $2k \leq C := 2(\text{diam } M/t_0 + 1)$.

Limit. The sets $\mathcal{F}_A = \{F \in \mathcal{F} : F \supseteq A\}$, $A \subset M$ finite, have the finite intersection property ($\mathcal{F}_A \cap \mathcal{F}_B = \mathcal{F}_{A \cup B}$); let $\mathcal{U}$ be an ultrafilter on $\mathcal{F}$ containing all of them. For $x \in M$ we have $|\psi_F(x)| = |\psi_F x - \psi_F p| \leq C$ for $F \in \mathcal{F}_{S(p, x)} \in \mathcal{U}$ (and $\psi_F(p) = 0$), so the limit $\psi(x) = \lim_{\mathcal{U}} \psi_F(x)$ exists in the closed ball of radius C . If $d(x, y) = d(x', y')$, then $|\psi_F x - \psi_F y| = |\psi_F x' - \psi_F y'|$ for all $F \in \mathcal{F}_{\{x, y, x', y'\}}$, and in the limit $|\psi x - \psi y| = |\psi x' - \psi y'|$. So $\Phi(d(x, y)) := |\psi x - \psi y|$ is well defined, and $\Phi(t_0) = \lim_{\mathcal{U}} 1 = 1$; in particular ψ is not constant.

Continuity. For $l \in (0, l_0]$ and $s \in (0, 2l]$, (P2) and the triangle inequality in $\mathbb{R}^N$ give $\Phi(s) \leq 2\Phi(l)$. Suppose that $\Phi(s_n) \geq \varepsilon > 0$ for some $s_n \in D(M)$ with $s_n \rightarrow 0$. For every $l \in (0, l_0]$ choose n with $s_n \leq 2l$; then $\Phi(l) \geq \varepsilon/2$. So any two distinct points of M at distance at most l_0 have images at least $\varepsilon/2$ apart. By (P3) a ball of radius $l_0/2$ in M is infinite; its images would be infinitely many $\varepsilon/2$ -separated points of a bounded subset of $\mathbb{R}^N$, which is impossible. Hence $\Phi(s) \rightarrow 0$ as $s \rightarrow 0$, and ψ is uniformly continuous. $\square$

We do not use Theorem 2.4 of [Chi22] (weak loose embeddability in $\mathbb{R}^N$ under $H(N)$); Lemma 6.1 proves directly the weaker statement needed here, with the conclusion “non-constant” in place of “injective”.

6.3 Maps of flat tori whose chords depend only on the distance

Lemma 6.2. *Let $T = \mathbb{R}^n/\Lambda$, $n \geq 2$, with its geodesic distance d . If $\psi: T \rightarrow \mathbb{R}^N$ is continuous and $|\psi x - \psi y|$ depends only on $d(x, y)$, then ψ is constant.*

Proof. Step 1. Translations are isometries of T , so $|\psi(x+a) - \psi(y+a)| = |\psi x - \psi y|$ for all $x, y, a \in T$.

Step 2 (this step is [Hei02, Lemma 1] in the present setting). Let $W = \text{aff } \psi(T)$ and $m = \dim W$; if $m = 0$ there is nothing to prove. Choose $x_0, \dots, x_m \in T$ such that $\psi(x_0), \dots, \psi(x_m)$ is an affine basis of W . For $a \in T$ the families $(\psi(x))_{x \in T}$ and $(\psi(x+a))_{x \in T}$ have equal mutual distances by Step 1, so some isometry A_a of $\mathbb{R}^N$ satisfies $A_a \psi(x) = \psi(x+a)$ for all x . It maps W onto $\text{aff } \psi(T+a) = W$; its restriction B_a to W is the unique affine map of W with $B_a \psi(x_i) = \psi(x_i+a)$. Two affine maps of W that agree on $\psi(T)$ are equal, hence $B_{a+b} = B_a B_b$ and $B_0 = \text{id}$; and $a \mapsto B_a$ is continuous, since in barycentric coordinates $\lambda_i(w)$ with respect to the $\psi(x_i)$ we have $B_a(w) = \sum_i \lambda_i(w) \psi(x_i+a)$.

Step 3. Let $w^* = \int_T \psi(a) da$ (normalised Haar measure). Then $w^* \in W$, and $B_b w^* = \int_T \psi(a+b) da = w^*$, because affine maps commute with averages. Let $V = W - w^*$, a linear subspace of $\mathbb{R}^N$. Then $\rho(a)u = B_a(w^* + u) - w^*$ defines a continuous orthogonal representation ρ of T on V , and $\psi(a) - w^* = \rho(a)v$ with $v = \psi([0]) - w^*$.

Step 4. The complexification of ρ is a continuous unitary representation of the compact abelian group T on $V \otimes \mathbb{C}$; it is the orthogonal direct sum of subspaces V_ξ , ξ in a finite subset Ξ of the dual

lattice $\Lambda^* = \{\xi : \xi \cdot \Lambda \subset \mathbb{Z}\}$, on which $\rho([z])$ acts as multiplication by $e^{2\pi i \xi \cdot z}$. Writing $v = \sum_{\xi \in \Xi} v_\xi$ with $v_\xi \in V_\xi$, we get

$$K(z) := |\psi([z]) - \psi([0])|^2 = \sum_{\xi \in \Xi} |e^{2\pi i \xi \cdot z} - 1|^2 |v_\xi|^2 = \sum_{\xi \in \Xi} 2(1 - \cos(2\pi \xi \cdot z)) |v_\xi|^2.$$

So K is a trigonometric polynomial on $\mathbb{R}^n$: real-analytic, Λ -periodic, with $K(0) = 0$.

Step 5. By (T1), $K(z) = \Phi(|z|)^2$ for $|z| < \lambda_1/2$, where Φ is the function with $|\psi x - \psi y| = \Phi(d(x, y))$. For $Q \in \text{SO}(n)$ the real-analytic function $z \mapsto K(Qz) - K(z)$ vanishes on this ball, hence on $\mathbb{R}^n$. Since $\text{SO}(n)$ acts transitively on each sphere $\{|z| = r\}$ ($n \geq 2$), $K(z) = k(|z|)$ for a function $k: [0, \infty) \rightarrow \mathbb{R}$.

Step 6. Fix $v_1 \in \Lambda$ with $|v_1| = \lambda_1$. For $r \geq 0$ and $|z| = r$, periodicity gives $k(|z + v_1|) = K(z + v_1) = K(z) = k(r)$. As z runs over the sphere of radius r , which is connected since $n \geq 2$, $|z + v_1|$ takes every value in $J(r) = [|r - \lambda_1|, r + \lambda_1]$. Hence k is constant, equal to $k(r)$, on $J(r)$. For $r = 0$, $J(0) = \{\lambda_1\}$ and $k(\lambda_1) = k(0) = K(0) = 0$. Every $r \in [0, 2\lambda_1]$ has $\lambda_1 \in J(r)$, so $k(r) = k(\lambda_1) = 0$. If $k = 0$ on $[0, j\lambda_1]$ for some $j \geq 2$ and $r \in [j\lambda_1, (j+1)\lambda_1]$, then $J(r)$ contains $r - \lambda_1 \in [(j-1)\lambda_1, j\lambda_1]$, so $k(r) = k(r - \lambda_1) = 0$. By induction $k = 0$, hence $K = 0$ and $\psi([z]) = \psi([0])$ for all z . $\square$

6.4 Flat tori fail the hypothesis

Theorem C. *Let $T = \mathbb{R}^n/\Lambda$ be a flat torus of dimension $n \geq 2$ with its geodesic distance. For every $N \geq 1$ some finite subset of T is not loosely embeddable in $\mathbb{R}^N$. So T satisfies $H(N)$ for no N . The proof is non-constructive: for general N it gives no explicit finite subset and no bound on its size.*

Proof. T is compact and satisfies (P1)–(P3) with $l_0 = \lambda_1/8$. If T satisfied $H(N)$, Lemma 6.1 would give a continuous non-constant ψ with $|\psi x - \psi y| = \Phi(d(x, y))$, contradicting Lemma 6.2. $\square$

The square torus $\mathbb{R}^2/\mathbb{Z}^2$ is a compact connected Riemannian manifold, so Theorem C answers Question 1.3 negatively; this is also a negative answer to [Chi22, Question 2.2], which is the same question.

Corollary D. *No flat torus of dimension $n \geq 2$ is loosely embeddable in any $\mathbb{R}^N$.*

Proof. A loose embedding is continuous and injective, hence not constant, and its chord lengths depend only on the distance (Lemma 2.1(a)); Lemma 6.2 excludes this. $\square$

At the start of Section 2 of [Chi22] the author writes that he does not know whether compact Riemannian metric spaces are always LE; for flat tori of dimension at least 2, Corollary D answers this negatively. The abstract of [Chi22] presents its results as evidence that Riemannian metric spaces admit loose embeddings into finite-dimensional Euclidean spaces; Corollary D shows that flat tori of dimension at least 2 do not. For $n = 1$ Step 5 of the proof of Lemma 6.2 is empty, and indeed the circle is LE in $\mathbb{R}^2$ [Chi22, Lemma 1.2].

For $N \leq 2$ an explicit finite witness exists on one flat torus.

Proposition 6.3. *Let $\omega_1 = (1, 0)$, $\omega_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $T = \mathbb{R}^2/\Lambda$ (the hexagonal torus). The four points $[0]$, $[(\frac{1}{2}, 0)]$, $[(\frac{1}{4}, \frac{\sqrt{3}}{4})]$, $[(\frac{3}{4}, \frac{\sqrt{3}}{4})]$ are pairwise at distance $\frac{1}{2}$. Hence they form a subset of T that is not LE in $\mathbb{R}^2$, and any three of them form a subset that is not LE in $\mathbb{R}$.*

Proof. $|m\omega_1 + n\omega_2|^2 = m^2 + mn + n^2 \geq 1$ for $(m, n) \neq 0$, so $\lambda_1 = 1$. The four points are $[\omega/2]$ for $\omega \in \{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$, and the difference of two of them is $[u/2]$ for a non-zero $u \in \Lambda$ whose class modulo 2Λ is that of ω_1 , ω_2 or $\omega_1 + \omega_2$. Now $d([u/2], [0]) = \frac{1}{2} \min\{|u'| : u' \in u + 2\Lambda\}$. Every $u' \in u + 2\Lambda$ is a non-zero vector of Λ , so $|u'| \geq 1$, and the three classes contain the unit vectors ω_1 , ω_2 and $\omega_2 - \omega_1$. So all six distances equal $\frac{1}{2}$. A loose embedding would map the four points (or

three of them) to points pairwise at the same positive distance, which is impossible in $\mathbb{R}^2$ (or $\mathbb{R}$) by Lemma 2.2(a). $\square$

We know no explicit witness for the square torus, or for $N \geq 3$.

7 Manifolds that are loosely embeddable

By Lemma 2.1(b) a compact Riemannian manifold that is LE in $\mathbb{R}^N$ satisfies $H(N)$. The following manifolds with their standard metrics are LE by classical maps; in each case the chord length is a strictly increasing function of the geodesic distance.

- The round sphere S^n of radius ϱ (for $n = 1$ this is [Chi22, Lemma 1.2]): the inclusion in $\mathbb{R}^{n+1}$ has chord $2\varrho \sin(\theta/2\varrho)$ at geodesic distance $\theta \in [0, \pi\varrho]$. So S^n satisfies $H(n+1)$.
- $\mathbb{R}P^n$ with the quotient metric of the unit sphere, for which $d([x], [y]) = \theta = \arccos |x \cdot y| \in [0, \frac{\pi}{2}]$: the map $x \mapsto xx^\top$ into the real symmetric matrices with the Frobenius norm has $|xx^\top - yy^\top|^2 = 2 - 2(x \cdot y)^2 = 2 \sin^2 \theta$. So $\mathbb{R}P^n$ satisfies $H(\frac{(n+1)(n+2)}{2})$.
- $\mathbb{C}P^n$ with the Fubini–Study metric normalised to diameter $\frac{\pi}{2}$, for which $d([x], [y]) = \theta = \arccos |\langle x, y \rangle|$: the map $x \mapsto xx^*$ into the Hermitian matrices has $|xx^* - yy^*|^2 = 2 - 2|\langle x, y \rangle|^2 = 2 \sin^2 \theta$. So $\mathbb{C}P^n$ satisfies $H((n+1)^2)$.

Thus Question 1.3 has a negative answer because of flat tori, while round spheres and real and complex projective spaces satisfy the hypothesis. We did not examine quaternionic projective spaces, the Cayley plane, compact symmetric spaces of higher rank or non-homogeneous metrics.

8 Conclusion

In the reading of the problem record, both questions of [Chi19] have negative answers. For every $N \geq 1$ there is a countable compact metric space that satisfies $H(N)$ and is not loosely embeddable in the same $\mathbb{R}^N$ (Theorems A and B); and no flat torus of dimension at least 2 satisfies $H(N)$ for any N (Theorem C, by a non-constructive proof), nor is it loosely embeddable in any Euclidean space (Corollary D). The negative answer to the first question concerns only the conclusion “in the same $\mathbb{R}^N$ ”. Chirvasitu’s [Chi22, Question 2.1], which under $H(N)$ asks for a loose embedding in some finite-dimensional Euclidean space, remains open; X_1 is not a counterexample to it (Proposition 5.1). Further questions left open here: whether X_N , $N \geq 2$, is LE in some Euclidean space; explicit finite witnesses for Theorem C on the square torus or for $N \geq 3$, and bounds on their size; and which compact Riemannian manifolds satisfy $H(N)$ (Section 7). The other two questions of [Chi19] (loose embeddability of every compact metric space in ℓ^2 , and quantum isometry groups of all compact metric spaces) are not treated.

9 Computations and reproducibility

No proof in this note depends on a computation. The programs test the explicit constructions on finite instances; they are in the folder `reproducibility/` of the source archive, with their recorded outputs. All of them run in a few seconds.

The program of this note, `check_note.py` (standard library, about three seconds), uses exact arithmetic only: rationals, the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and the Gaussian rationals. It checks (A) the metric axioms of X_1 on $\{p, x_1, \dots, x_{16}\}$ and the pattern (3.1) on $\{p, x_1, \dots, x_{40}\}$; (B) that the doubling map

is a loose embedding of Y_K for $K \leq 40$, and that the map of Remark 3.2, with $M = 2^{m+3}$ and with $M = 2^{m+4}$, is a loose embedding of all 2047 non-empty subsets of $\{p, x_1, \dots, x_{10}\}$ and of 2000 random subsets of $\{p, x_1, \dots, x_{40}\}$ (with $M = 2^m$ it fails, as a control); (C) Lemma 2.2 for $N \leq 12$; (D) the triangle inequality on $\{p\} \cup G_1 \cup \dots \cup G_6$ for $N = 2, 3, 4$ with rational heights in three regimes, and the pattern (4.1) there, with the heights treated as independent symbols; (E) that the maps φ_π of the proof of Theorem B, with rational rotations (Cayley transforms) and rational translations, realise the pattern of Y_5 for $N = 2$ and of Y_4 for $N = 3$ exactly (the first random parameter worked in both cases); (F) Remark 4.2; (G) the construction of Proposition 5.1, with points in $\mathbb{Q}(i)$, on $\{p, x_1, \dots, x_{14}\}$; (H) Proposition 6.3; (I) the two matrix identities of Section 7 on 200 pairs of rational unit vectors each. All checks pass.

The programs with which the results were first obtained (original/): `chain_N1.py` (exact) checks the metric and the pattern of X_1 on $\{p, x_1, \dots, x_{14}\}$ and the map of Remark 3.2 with $M = 2^{m+4}$ on all 2047 non-empty subsets of $\{p, x_1, \dots, x_{10}\}$ and on 3000 random subsets of $\{p, x_1, \dots, x_{40}\}$; `gadget_N.py` is a numerical test at 60 digits of X_2 and X_3 with six gadgets and the explicit heights of Section 4: the triangle inequality, the pattern (4.1) and the maps of Theorem B with random rotations on 400 random finite subsets for each N and on $\{p\} \cup G_1 \cup \dots \cup G_4$.

Verification. Two independent verification runs, both AI-assisted, examined the first written version of these results. Each re-derived every proof, ran the two original programs again from copies, with identical outputs, and wrote its own programs; these are in the folders `verification_run_A/` and `verification_run_B/`. The first run checked the map of Remark 3.2 exactly on all 8191 non-empty subsets of $\{p, x_1, \dots, x_{12}\}$ and on 5000 random subsets of $\{p, x_1, \dots, x_{60}\}$, tested X_N numerically for $N = 2, 3, 4$ (with Lemma 2.2 exactly) and the construction of Proposition 5.1 at 60 digits, and compared the statements about [Chi19, Chi21, Chi22] with these texts. The second run checked the map of Remark 3.2 exactly on the same 8191 subsets and on 1500 random subsets of $\{p, x_1, \dots, x_{60}\}$, and the maps of Theorem B for $N = 2, 3$ exactly in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, with rational rotations, on $\{p\} \cup G_1 \cup \dots \cup G_4$ and on random subsets of $\{p\} \cup G_1 \cup \dots \cup G_8$; it found the witness of Proposition 6.3 and checked it exactly, checked the bibliographic data and repeated the literature search. Neither run found a mathematical error. The corrections they required concern the presentation and have been made: the scope of the answer to Question 1.2 is stated in the abstract, the introduction, the Main Theorem and the conclusion; Step 6 of the proof of Lemma 6.2 is written out; the block $\{p\}$ is treated in the proof of Lemma 4.1(a); the constant M of Remark 3.2 is stated for the text and for the programs; the bibliographic data were checked; and Proposition 6.3 was added. In writing the note, the proofs of the finite-subset halves of Theorems A and B were arranged through the truncations Y_K and Lemma 2.1(b); for Y_K the map of Remark 3.2 is the doubling map, and the genericity argument is the one checked by the runs, applied to Y_K . The exact program of this note checks the finite parts again.

A further independent verification run, also AI-assisted, then examined the text of this note itself. It read the sources again from their PDF files, re-derived every proof of the note line by line, wrote its own programs (folder `independent_run_2/`), and ran all programs of the source archive again from an extracted copy; every output was identical to the recorded one. Its programs check, exactly, the map of Remark 3.2 with $M = 2^{m+3}$ and with $M = 2^{m+2}$ on all 8191 non-empty subsets of $\{p, x_1, \dots, x_{12}\}$ and on 2000 random subsets of $\{p, x_1, \dots, x_{60}\}$, and by brute force the integer loose maps of $\{p, x_1, \dots, x_4\}$; at 60 digits, the metric and the pattern of X_2 and X_3 with six gadgets and the explicit heights, and the maps of Theorem B with random rotations on Y_6 ($N = 2$), on Y_5 ($N = 3$) and on 300 random subsets of Y_8 for each N ; and, exactly, Proposition 6.3, the construction of Proposition 5.1 in $\mathbb{Q}(i)$ on $\{p, x_1, \dots, x_{12}\}$, Lemma 2.2 for $N \leq 12$ and the identities of Section 7. It found no mathematical error. The corrections it required concern citations and credit and have

been made: the numbers cited from [Chi21] are those of its second arXiv version; the first appearance of both questions, in arXiv:2004.09962v1, is credited; the theorem of Heitzig [Hei02], which this run found, is cited with its relation to Lemma 6.2; and the list of renumberings in the journal version of [Chi22] is completed.

Scope and priority. The note answers Question 1.2 negatively in the reading “loosely embeddable in the same $\mathbb{R}^N$ ”, which is the reading of the problem record, for every $N \geq 1$, and Question 1.3 (= [Chi22, Question 2.2]) negatively. It does not answer [Chi22, Question 2.1], and Theorem C is non-constructive. The definitions and both questions are Chirvasitu’s: in [Chi19], in the first arXiv version of [Chi21] and in [Chi22]; the known results we use or compare are listed, with their sources, in the introduction. Lemma 6.2 is the analogue for flat tori of Heitzig’s theorem [Hei02, Theorem 1] on maps between Euclidean spaces, and its proof follows his (see Section 1); it differs in that the squared chord is known to be radial only near 0, which Step 5 extends by real-analyticity, and in that Step 6 uses the periodicity. The underlying mechanism, that an orbit of a group of isometries spanning a finite-dimensional space yields a trigonometric polynomial, is classical in spirit (the screw functions of von Neumann and Schoenberg [vNS41]). The literature on helical immersions (for example [Sak82, Sak86], and later papers by Hong and by Pak and Sakamoto) concerns smooth isometric immersions whose chord length depends only on the arc length; we know it only from bibliographic data and search snippets and did not read it. Corollary D and Theorem C are not stated in [Hei02]; Corollary D may be known or folklore, although we found it nowhere.

What was read: the abstract [Chi19] (in the PDF file of the report) and the record [ula26]; the arXiv version v1 of [Chi22] completely, and its journal version (the copy in the NSF Public Access Repository) for the numbering and the questions; of [Chi21], the introduction, Sections 2 and 3 and the statements cited in arXiv:2004.09962v2, and the statements cited and the section with the questions in v1; the preprint [Hei02] completely. We did not read [Gos15, DM90, Bes40, vNS41]: what we say about [Gos15, DM90] is taken from [Chi21, Chi22], [Bes40] is used only for the optional explicit heights, and [vNS41] only for the remark above.

The searches were made on 20 October 2026: queries to the arXiv API (the author Chirvasitu with the words embedding, metric, distance or isometry; the phrases “loose embedding”, “loose embeddings” and “loosely embeddable”; and, in the later verification run, the phrases “equality of distances”, “distance patterns” and “quantum isometry” with embedding, and “flat torus” with chord, of which the first found [Hei02]), to Crossref, to zbMATH Open and to the citation lists of Semantic Scholar and OpenAlex, and a few web searches. No printed answer to either question was found in these sources. Semantic Scholar lists only [Chi22] as citing [Chi21], and no work citing [Chi22]. OpenAlex counts two works citing [Chi21] and Crossref one. OpenAlex refused the list of citing works (HTTP 429, in both searches), so the second work counted there was not identified. A zbMATH Open search for documents citing either paper gave no result. We found no citing work that addresses the questions. A search that finds nothing is not a proof of novelty, and no priority is claimed beyond the statement that we found no prior answer.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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