

Combinatorial Spheres inside Spheres on the Same Vertex Set: Partial Results on a Question of Santos

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September 30, 2026

Abstract

F. Santos asked in an Oberwolfach problem session (2019) whether every combinatorial d -sphere S with $n \geq d+3$ vertices is a subcomplex of a combinatorial $(d+1)$ -sphere with the same n vertices. We give partial results. If S has an edge vw with $\text{lk}(v) \cap \text{lk}(w) = \text{lk}(vw)$, equivalently an edge that lies in no missing face with at least three vertices, then the union of the cones $v * \text{ast}(v)$ and $w * \text{ast}(w)$ over the two antistars is such a sphere. This extends Datta's construction for flag spheres and recovers his results for joins and for spheres with a vertex of degree $d+1$. If S has no missing face of dimension d and some vertex link is a stacked sphere, the answer is also positive. We add reformulations in terms of completable antistars, one-point suspensions and balls, closure under connected sums and stellar subdivisions, and restrictions on a smallest counterexample in dimension 3. With computer certificates that are checked by an independent program, every combinatorial 3-sphere with $6 \leq n \leq 10$ vertices lies in a 4-sphere on the same vertex set: 1320 of the 247,882 ten-vertex 3-spheres have no edge of the above kind, and each of them has an explicit extension. All 337 combinatorial 4-spheres with 9 vertices that we found have such an edge; the completeness of this list is a solver result without proof certificates, and we know no published count to compare it with. A $\mathbb{Z}_{11}$ -invariant neighborly 3-sphere without such an edge and two $\mathbb{Z}_{13}$ -invariant ones have extensions, but none in which every facet meets a fixed pair of vertices. Of nine $\mathbb{Z}_{14}$ -invariant neighborly 3-spheres without such an edge, six extend and three are undecided. These spheres are known from the enumerations of Kühnel and Lassmann and of Köhler and Lutz. The general question remains open. This is an unrefereed note.

1 Introduction

All simplicial complexes are finite. A *combinatorial d -sphere* (*d -ball*) is a simplicial complex whose geometric realisation is PL homeomorphic to the boundary of a $(d+1)$ -simplex (to a d -simplex). In the open-problem session of the 2019 Oberwolfach workshop *Geometric, Algebraic, and Topological Combinatorics*, Francisco Santos asked the following question [KNSW19, Question 12, pp. 2466–2467].

Problem 1.1 (Santos). *Let S be a combinatorial d -sphere with $n \geq d+3$ vertices. Is there always a combinatorial $(d+1)$ -sphere S' with the same vertex set and with $S \subseteq S'$?*

We call such an S' an *extension* of S . The one-point suspension of S contains S and has $n+1$ vertices [JL05], so the question is whether the additional vertex can be avoided. The bound $n \geq d+3$ is needed, because every $(d+1)$ -sphere has at least $d+3$ vertices. Santos remarks in [KNSW19] that a positive answer would follow if some antistar $S \setminus \star(v)$ could always be completed to a d -sphere

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without new vertices, but that this fails for Altshuler’s “peculiar” 10-vertex triangulation of the 3-sphere [Alt76], which is vertex-transitive and neighborly; he adds that this sphere is nevertheless contained in a 10-vertex 4-sphere. The question is record OWR-17135-036 of the UnsolvedMath dataset [ula26].

Known cases. Criado and Santos observed that the answer is positive for polytopal spheres [CS22, proof of Thm. 3.5]; see also [Dat20, Thm. 1]. Datta [Dat20] proved it for joins of spheres, for spheres with a vertex of degree $d + 1$, for flag spheres [Dat20, Thm. 4], and for stacked spheres, with a stacked extension [Dat20, Thm. 5], and he treated the two non-polytopal 3-spheres with 8 vertices [Dat20, Sect. 4]. (Datta states the question for $(d - 1)$ -spheres; we shift the dimension.) By Mani’s theorem [Man72], combinatorial d -spheres with at most $d + 4$ vertices are polytopal, so the open cases have $n \geq d + 5$. Datta also asked whether every combinatorial d -ball with $m \geq d + 2$ vertices lies in a combinatorial d -sphere with the same m vertices [Dat20, Question 2]. Santos’ remark on Altshuler’s sphere already answers this in the negative; see Remark 4.6.

Results. If some edge vw of S satisfies $\text{lk}(v) \cap \text{lk}(w) = \text{lk}(vw)$, equivalently lies in no missing face with three or more vertices (Lemma 3.2), then $(v * \text{ast}_S(v)) \cup (w * \text{ast}_S(w))$ is an extension (Theorem 3.1). In the flag case this is exactly Datta’s sphere [Dat20, proof of Thm. 4(c) and Remark 14]; his argument needs flagness only through the link condition for one edge (Remark 3.3). Corollary 3.4 recovers flag spheres, joins, spheres with a vertex of degree $d + 1$ and 2-spheres. If S has no missing face of dimension d and a stacked vertex link, then S has an extension (Lemma 4.2). Sections 4 and 5 give reformulations in terms of completable antistars, one-point suspensions and balls, closure under connected sums and stellar subdivisions, and restrictions on a smallest counterexample in dimension 3 (Corollary 5.3). Section 6 treats all 3-spheres with $6 \leq n \leq 10$ vertices (Theorem 6.1), the 4-spheres with $7 \leq n \leq 9$ vertices (Proposition 6.3, conditional on a solver result), and the cyclically invariant neighborly 3-spheres with 10, 11, 13 and 14 vertices that have no LC edge (Theorem 6.4); these spheres are known from [KL85, KL05].

Scope. The general question is open. The results of Sections 3 to 5 are proved in the text, without computer assistance. The statements of Section 6 are computer-assisted: every positive statement comes with an explicit certificate, and every non-existence statement is verified either by exhaustive direct computation (absence of LC edges and of stacked vertex links, primeness, non-isomorphism, vertex-transitivity) or by a DRUP refutation, except where a statement is explicitly labelled as a solver result. The certificates and computations are checked by a short independent program included with this note. The completeness of our lists of 3-spheres is derived from the published numbers of Lutz and of Sulanke and Lutz [Lut08, SL09]. For the 9-vertex 4-spheres we know no published number, and the completeness of our list is a solver result, so Proposition 6.3 is conditional.

2 Preliminaries

A simplicial complex K is a family of finite sets closed under taking subsets; we include the empty face. Its vertex set is $V(K)$. For a face σ let $\text{lk}_K(\sigma) = \{\tau \in K : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in K\}$ and, for a vertex v , let $\text{ast}_K(v) = \{\tau \in K : v \notin \tau\}$ be the *antistar*. For $U \subseteq V(K)$ the induced subcomplex is $K[U] = \{\tau \in K : \tau \subseteq U\}$. For $x \notin V(K)$ the cone is $x * K = K \cup \{\tau \cup \{x\} : \tau \in K\}$, and $K * L$ denotes the join of complexes on disjoint vertex sets. We write $\bar{\sigma}$ for the complex of all subsets of σ and $\partial \bar{\sigma}$ for its boundary. A *missing face* of K is a set $F \subseteq V(K)$ with $F \notin K$ all of whose proper subsets are in K ; its dimension is $|F| - 1$. A pure complex is often identified with its set of facets,

and for pure complexes $M \supseteq B$ of the same dimension, $M - B$ denotes the complex generated by the facets of M that are not in B .

Definition 2.1. An edge vw of K satisfies the *link condition* (LC) if $\text{lk}_K(v) \cap \text{lk}_K(w) = \text{lk}_K(vw)$. (The inclusion $\supseteq$ always holds.)

For a combinatorial d -sphere S and a vertex v we put $C_v = v * \text{ast}_S(v)$, where v serves as the cone point; note that $v \notin V(\text{ast}_S(v))$. For a combinatorial d -sphere X , a vertex u of X and a new vertex v , the *one-point suspension* is $\Sigma_{u,v}(X) = (v * X) \cup (u * \text{ast}_X(u))$ [JL05]. Every facet of $\Sigma_{u,v}(X)$ contains u or v . A *stacked d -ball* is a pure d -complex obtained from a d -simplex by repeatedly adding a d -simplex $\tau \cup \{x\}$ along a boundary ridge τ , where x is a new vertex; a *stacked sphere* is the boundary of a stacked ball. The *dual graph* of a pure d -complex has the facets as vertices, two facets being adjacent if they share a ridge.

Lemma 2.2. *Let $d \geq 1$.*

- (a) *If B is a combinatorial d -ball contained in a combinatorial d -sphere M , then $M - B$ is a combinatorial d -ball and $B \cap (M - B) = \partial B = \partial(M - B)$.*
- (b) *If B_1, B_2 are combinatorial d -balls and $B_1 \cap B_2$ is a combinatorial $(d - 1)$ -ball contained in $\partial B_1 \cap \partial B_2$, then $B_1 \cup B_2$ is a combinatorial d -ball.*
- (c) *If B_1, B_2 are combinatorial d -balls with $B_1 \cap B_2 = \partial B_1 = \partial B_2$, then $B_1 \cup B_2$ is a combinatorial d -sphere.*
- (d) *If v is a vertex of a combinatorial d -sphere S , then $\text{ast}_S(v)$ is a combinatorial d -ball with boundary $\text{lk}_S(v)$ and vertex set $V(S) \setminus \{v\}$, and C_v is a combinatorial $(d + 1)$ -ball with $\partial C_v = S$.*
- (e) *$\Sigma_{u,v}(X)$ is a combinatorial $(d + 1)$ -sphere with vertex set $V(X) \cup \{v\}$ for every combinatorial d -sphere X and $u \in V(X)$.*
- (f) *Let M be a closed pseudomanifold of dimension $d \geq 3$ whose vertex links are combinatorial $(d - 1)$ -spheres. If M minus one facet is collapsible, then M is a combinatorial d -sphere.*
- (g) *If v is a boundary vertex of a combinatorial $(d + 1)$ -ball B , then $\text{lk}_B(v)$ is a combinatorial d -ball with boundary $\text{lk}_{\partial B}(v)$.*

Proof. (a) and (b) are [RS72, Cor. 3.13 and 3.16], see [Dat20, Prop. 9 and Lemma 10]; (c) and (d) are [Dat20, Lemma 10] (every vertex $w \neq v$ lies in a facet avoiding v , since $\text{lk}_S(w)$ is not a cone). (e) The cones $v * X$ and $u * \text{ast}_X(u)$ are $(d + 1)$ -balls with boundary X (by (d) for the second), and their intersection is X ; apply (c). (f) Deleting a facet σ leaves a combinatorial manifold with boundary $\partial \bar{\sigma}$; a collapsible combinatorial manifold is a ball by Whitehead's theorem [Whi39] (see also [RS72, Ch. 3]); now apply (c) with the ball $\bar{\sigma}$. (g) is standard [RS72]. $\square$

Lemma 2.3. (a) *Let $d \geq 2$. A stacked d -ball is a combinatorial d -ball, and all its faces of dimension at most $d - 2$, in particular all its vertices, lie in its boundary.*

- (b) *A pure d -complex whose dual graph is a tree and which has $f_d + d$ vertices, where f_d is its number of facets, is a stacked d -ball. This is due to Datta and Singh [DS13, Lemma 2.1] (see [Dat20, Lemma 13]); we include the short proof.*
- (c) *A combinatorial 3-ball L all of whose vertices and edges lie in ∂L is stacked; in particular ∂L is a stacked 2-sphere.*

- (d) A 2-sphere Σ with at least 5 vertices is stacked if and only if it has a vertex x of degree 3 such that the 2-sphere Σ_x obtained by deleting x and adding the triangle spanned by its neighbours is stacked; and then this holds for every vertex x of degree 3.

Proof. (a) By induction on the number of facets, using Lemma 2.2(b). When $\tau \cup \{x\}$ is added along the boundary ridge τ , the only faces that become interior are τ and $\tau \cup \{x\}$: the ridges containing x lie in one facet, and every proper face of τ lies in such a ridge.

(b) In a pure d -complex with connected dual graph one can order the facets so that each facet shares a ridge with an earlier one; each facet then adds at most one vertex, so there are at most $f_d + d$ vertices. Now induct on f_d . Let σ be a leaf of the dual tree, attached to σ' along the ridge τ , and let X' be the complex generated by the other facets. Its dual graph is a tree, so X' has at most $f_d - 1 + d$ vertices; hence $\sigma = \tau \cup \{x\}$ with $x \notin V(X')$, and X' has exactly $f_d(X') + d$ vertices. The ridge τ lies in no facet of X' other than σ' , as σ is a leaf. So X' is stacked by induction, and X arises from it by one stacking step.

(c) Let m be the number of vertices of L . The 2-sphere ∂L has $3m - 6$ edges and $2m - 4$ triangles, and by assumption L has m vertices and $3m - 6$ edges. Let f_2, f_3 be the numbers of triangles and tetrahedra of L . The Euler characteristic gives $m - (3m - 6) + f_2 - f_3 = 1$, and counting incidences gives $4f_3 = 2(f_2 - 2m + 4) + (2m - 4)$. Hence $f_3 = m - 3$, and L has $f_2 - (2m - 4) = m - 4 = f_3 - 1$ interior triangles. The dual graph of L is connected, so it is a tree, and L has $f_3 + 3$ vertices. Apply (b).

(d) If Σ_x bounds a stacked ball B' , then Σ bounds $B' \cup (t \cup \{x\})$, where t is the added triangle. Conversely, let $\Sigma = \partial B$ with B stacked; the vertex added last has degree 3. Let x be any vertex of degree 3 in Σ . By (a) all edges of B lie in Σ , so the only tetrahedron of B containing x is $t \cup \{x\}$, with t the triangle of the neighbours of x . Since Σ has at least 5 vertices, $t \cup \{x\}$ is a leaf of the dual tree, and removing it leaves a complex that is stacked by (b) and has boundary Σ_x . $\square$

3 The link-condition construction

Theorem 3.1. *Let S be a combinatorial d -sphere ($d \geq 1$) with vertex set V , and let $v \neq w$ be vertices. Then $C_v \cap C_w = S$ if and only if vw is an edge of S satisfying the link condition. In that case $C_v \cup C_w$ is a combinatorial $(d + 1)$ -sphere with vertex set V that contains S .*

Proof. For $F \subseteq V$ we have $F \in C_v$ if and only if $F \setminus \{v\} \in S$. Indeed, if $v \notin F$ this says $F \in \text{ast}_S(v) \iff F \in S$, and if $v \in F$ it says $F \setminus \{v\} \in \text{ast}_S(v) \iff F \setminus \{v\} \in S$. Hence $F \in C_v \cap C_w$ if and only if $F \setminus \{v\} \in S$ and $F \setminus \{w\} \in S$. In particular $S \subseteq C_v \cap C_w$, and a set F that does not contain both v and w lies in $C_v \cap C_w$ only if $F \in S$. For $F = \sigma \cup \{v, w\}$ with $\sigma \cap \{v, w\} = \emptyset$ we get $F \in C_v \cap C_w$ if and only if $\sigma \in \text{lk}(v) \cap \text{lk}(w)$, while $F \in S$ if and only if $\sigma \in \text{lk}(vw)$. Therefore $C_v \cap C_w = S$ if and only if $\text{lk}(v) \cap \text{lk}(w) \subseteq \text{lk}(vw)$. Taking $\sigma = \emptyset$, this requires $vw \in S$; and for an edge the inclusion is the link condition.

If the condition holds, then C_v and C_w are $(d + 1)$ -balls with $\partial C_v = \partial C_w = S = C_v \cap C_w$ (Lemma 2.2(d)), so their union is a $(d + 1)$ -sphere by Lemma 2.2(c). Its vertex set is V . $\square$

Lemma 3.2. *An edge vw of a simplicial complex K violates the link condition if and only if some missing face F of K with $|F| \geq 3$ contains v and w .*

Proof. If vw violates the link condition, choose $\sigma \in \text{lk}(v) \cap \text{lk}(w)$ with $\sigma \notin \text{lk}(vw)$ minimal under inclusion; $\sigma \neq \emptyset$ because $vw \in K$. Then $F = \sigma \cup \{v, w\}$ is not a face. Every proper subset of F misses v or w , and then lies in $\sigma \cup \{w\}$ or $\sigma \cup \{v\}$, or it has the form $\sigma' \cup \{v, w\}$ with $\sigma' \subsetneq \sigma$, and then it is a face by minimality. So F is a missing face with $|F| \geq 3$. Conversely, if $F \ni v, w$ is a missing face with $|F| \geq 3$, then $\sigma = F \setminus \{v, w\}$ lies in $\text{lk}(v) \cap \text{lk}(w)$ but not in $\text{lk}(vw)$. $\square$

Remark 3.3 (Relation to Datta's construction and to edge contraction). Let S/vw be obtained from S by replacing v by w , that is, $S/vw = \text{ast}_S(v) \cup \{(\tau \setminus \{v\}) \cup \{w\} : v \in \tau \in S\}$. A direct check of the four cases according to whether F contains v and/or w shows that

$$C_v \cup C_w = \Sigma_{w,v}(S/vw)$$

for all $v \neq w$, as complexes; here v is the new vertex of the one-point suspension. So under the link condition, $S/vw = \text{lk}_{C_v \cup C_w}(v)$ is a combinatorial d -sphere, a special case of known results on edge contractions under the link condition (see [Nev07] and the references there). For a flag sphere S , a vertex v and $u \in \text{lk}_S(v)$, Datta constructs an extension from $\text{ast}_S(v)$ and $D_3 = u * \text{ast}_{\text{lk}_S(v)}(u)$ and observes that it equals $\Sigma_{u,v}(\text{ast}_S(v) \cup D_3)$ [Dat20, proof of Thm. 4(c) and Remark 14]. Since $\text{ast}_S(v) \cup D_3 = S/vu$, this is the sphere of Theorem 3.1 for the edge uv . Flagness enters his proof through the identity $\text{ast}_S(v) \cap D_3 = \text{lk}_S(v)$, and one checks that this identity is equivalent to the link condition for uv . Thus Theorem 3.1 is Datta's construction with flagness replaced by the link condition for one edge. We have given the short direct proof for completeness.

Corollary 3.4. *A combinatorial d -sphere S with n vertices has an extension in each of the following cases:*

- (a) S is flag [Dat20, Thm. 4(c)];
- (b) $S = S_1 * S_2$ is a join of two spheres [Dat20, Thm. 4(a)];
- (c) S has a vertex of degree $d+1$ and $n \geq d+3$ [Dat20, Thm. 4(b)]; this includes all stacked spheres with $n \geq d+3$ (for them [Dat20, Thm. 5] gives the stronger conclusion that the extension can be chosen stacked);
- (d) $d = 2$ and $n \geq 5$ (this also follows from [CS22], since 2-spheres are polytopal; see [Dat20, Cor. 3]).

Proof. By Theorem 3.1 and Lemma 3.2 it suffices to find an edge that lies in no missing face with at least three vertices.

(a) All missing faces of a flag complex have two vertices.

(b) A set F is a face of $S_1 * S_2$ if and only if $F \cap V(S_i) \in S_i$ for $i = 1, 2$; hence a missing face lies in $V(S_1)$ or in $V(S_2)$, and every edge vw with $v \in V(S_1)$, $w \in V(S_2)$ is as required.

(c) Let $\text{lk}(v) = \partial\tau$ with $|\tau| = d+1$. If $\tau \in S$, then S would contain the closed pseudomanifold $\partial(\tau \cup \{v\})$ of the same dimension, hence be equal to it, and $n = d+2$. So $\tau \notin S$. Let $w \in \tau$, and let $F \ni v, w$ be a missing face with $|F| \geq 3$. Every $x \in F \setminus \{v\}$ is a neighbour of v , so $F \setminus \{v\} \subseteq \tau$. As $F \notin S$, the set $F \setminus \{v\}$ is not in $\text{lk}(v)$, so $F \setminus \{v\} = \tau$, which is then a face, a contradiction. A stacked sphere with $n \geq d+3$ vertices has a vertex of degree $d+1$, namely the vertex added last.

(d) Missing faces of a 2-sphere with $n \geq 5$ vertices containing an edge and having at least three vertices are *empty triangles*, missing faces with three vertices (four vertices would give $\partial\Delta^3 \subseteq S$). If there is no empty triangle we are done. Otherwise choose an empty triangle abc and one of the two discs D into which the cycle abc divides S such that D has the smallest possible number of interior vertices. A triangulated disc whose boundary is a triangle and which has no interior vertex is that triangle, so the triangle of D on ab has a third vertex x in the interior of D . If ax lay in an empty triangle axy , then $y \in D$, since all neighbours of x lie in D , and the disc bounded by axy inside D would have fewer interior vertices than D , because x is not one of them. So ax lies in no empty triangle. This is the classical fact that every triangulated 2-sphere other than $\partial\Delta^3$ has an edge that can be contracted. $\square$

4 Cones, one-point suspensions and balls

Proposition 4.1. *Let S be a combinatorial d -sphere with vertex set V and let $v \in V$. The following are equivalent.*

- (i) *Some combinatorial $(d+1)$ -sphere S' with $V(S') = V$ contains C_v .*
- (ii) *$\text{ast}_S(v) \subseteq T$ for some combinatorial d -sphere T with $V(T) = V \setminus \{v\}$.*
- (iii) *There is a combinatorial d -ball L with $\partial L = \text{lk}_S(v)$, $V(L) \subseteq V(\text{lk}_S(v))$ and $L \cap S = \text{lk}_S(v)$.*

If they hold, $\Sigma_{u,v}(T)$ is such an S' for every $u \neq v$, and it contains $S \subseteq C_v$. We then call $\text{ast}_S(v)$ completable.

Proof. (i) $\Rightarrow$ (iii). By Lemma 2.2(a),(d), $B = S' - C_v$ is a $(d+1)$ -ball with $B \cap C_v = \partial B = S$. As $v \in \partial B$, $L = \text{lk}_B(v)$ is a d -ball with $\partial L = \text{lk}_S(v)$ (Lemma 2.2(g)). If x is a vertex of L , then $\{v, x\} \in B \cap C_v = S$, so $x \in V(\text{lk}_S(v))$. If $\sigma \in L \cap S$, then $\sigma \cup \{v\} \in B$, and $\sigma \cup \{v\} \in C_v$ because $\sigma \in \text{ast}_S(v)$; so $\sigma \cup \{v\} \in S$ and $\sigma \in \text{lk}_S(v)$.

(iii) $\Rightarrow$ (ii). Put $T = \text{ast}_S(v) \cup L$. Then $\text{ast}_S(v) \cap L \subseteq S \cap L = \text{lk}_S(v) = \partial L = \partial \text{ast}_S(v)$, and $\text{lk}_S(v)$ lies in both, so T is a d -sphere by Lemma 2.2(c), with vertex set $V \setminus \{v\}$.

(ii) $\Rightarrow$ (iii). Put $L = T - \text{ast}_S(v)$. By Lemma 2.2(a), $L \cap \text{ast}_S(v) = \partial L = \text{lk}_S(v)$. No face of L contains v , so $L \cap S = L \cap \text{ast}_S(v) = \text{lk}_S(v)$. Every vertex of L is a vertex of $\text{ast}_S(v)$, hence lies in $L \cap \text{ast}_S(v) = \text{lk}_S(v)$.

(ii) $\Rightarrow$ (i). By Lemma 2.2(e), $\Sigma_{u,v}(T)$ is a $(d+1)$ -sphere with vertex set V , and it contains $v * T \supseteq C_v$. $\square$

Lemma 4.2. *Let S be a combinatorial d -sphere, $d \geq 2$, with $n \geq d+3$ vertices and without missing faces of dimension d . If $\text{lk}_S(v)$ is a stacked $(d-1)$ -sphere, then $\text{ast}_S(v)$ is completable, so S has an extension.*

Proof. Let B be a stacked d -ball with $\partial B = \text{lk}_S(v)$. By Lemma 2.3(a), all vertices and all faces of dimension at most $d-2$ of B lie in $\text{lk}_S(v)$. Let t be an interior $(d-1)$ -face of B . Then $t \notin \text{lk}_S(v)$, and every proper subset of t lies in $\text{lk}_S(v)$. If t were in S , then every proper subset of $t \cup \{v\}$ would be a face of S while $t \cup \{v\}$ is not; so $t \cup \{v\}$ would be a missing face of dimension d . Hence $t \notin S$. If B has at least two facets, every facet of B contains an interior $(d-1)$ -face, so no facet of B is in S . If B is a single simplex $\bar{\tau}$, then $\text{lk}_S(v) = \partial \bar{\tau}$ and $\tau \notin S$, as in the proof of Corollary 3.4(c). In both cases $B \cap S = \text{lk}_S(v)$, and Proposition 4.1(iii) holds with $L = B$. $\square$

Proposition 4.3. *Let S be a combinatorial 3-sphere and v a vertex such that any two vertices of $\text{lk}_S(v)$ span an edge of S ; for example, S neighborly. If $\text{ast}_S(v)$ is completable, then $\text{lk}_S(v)$ is a stacked 2-sphere. In particular, if S is neighborly and no vertex link of S is stacked, then no antistar of S is completable, and no extension of S contains a cone C_v .*

Proof. Let L be as in Proposition 4.1(iii). All vertices of L lie in $\text{lk}_S(v) = \partial L$. Every edge of L joins two vertices of $\text{lk}_S(v)$, so it lies in S , hence in $L \cap S = \text{lk}_S(v) = \partial L$. By Lemma 2.3(c), $\partial L = \text{lk}_S(v)$ is stacked. $\square$

Proposition 4.4. *Let S be a combinatorial d -sphere with vertex set V and let $u \neq v$ be vertices. The following are equivalent.*

- (i) *S has an extension S' in which every facet contains u or v .*
- (ii) *Some combinatorial d -sphere X with $V \setminus \{u, v\} \subseteq V(X) \subseteq V \setminus \{v\}$ contains $S[V \setminus \{u, v\}] \cup \text{lk}_S(v)$.*

If X is as in (ii), then $\Sigma_{u,v}(X)$ (if $u \in V(X)$), or the suspension $\{u, v\} * X$ with the 0-sphere $\{u, v\}$ (if $u \notin V(X)$), is such an S' . If uv is an edge of S , then $u \in V(X)$.

Proof. (ii) $\Rightarrow$ (i). Let $u \in V(X)$; the other case is similar. $S' = \Sigma_{u,v}(X)$ is a $(d+1)$ -sphere on V (Lemma 2.2(e)) whose facets contain u or v . A face of S avoiding u and v lies in X ; a face $\tau \cup \{v\}$ with $\tau \in \text{lk}_S(v)$ lies in $v * X$; and a face $\tau \cup \{u\}$ with $v \notin \tau$ has $\tau \in S[V \setminus \{u, v\}] \subseteq \text{ast}_X(u)$, so it lies in $u * \text{ast}_X(u)$.

(i) $\Rightarrow$ (ii). Let $X = \text{lk}_{S'}(v)$, a combinatorial d -sphere with $V(X) \subseteq V \setminus \{v\}$ and $\text{lk}_S(v) \subseteq X$. The $(d+1)$ -ball $A = \text{ast}_{S'}(v)$ has boundary X (Lemma 2.2(d)), and all its facets contain u . If a d -face ρ of A avoids u , it lies only in the facet $\rho \cup \{u\}$ of A , so $\rho \in \partial A = X$. Every face of A avoiding u lies in such a ρ , hence in X . In particular $S[V \setminus \{u, v\}] \subseteq X$, and so $V \setminus \{u, v\} \subseteq V(X)$. $\square$

The spheres of Theorem 3.1 and Proposition 4.1 are of the form $\Sigma_{u,v}(X)$, with $X = S/vw$ and $X = T$ (Remark 3.3). So for a sphere S we have the implications

LC edge $\implies$ completable antistar $\implies$ extension with a facet cover by two vertices $\implies$ extension.

Theorem 6.4 shows that the second and third implications cannot be reversed.

Proposition 4.5. *Let S be a combinatorial d -sphere with vertex set V and let $v \in V$. Then S has an extension if and only if there is a combinatorial $(d+1)$ -ball A with $V(A) = V \setminus \{v\}$, $\text{ast}_S(v) \subseteq A$ and $\text{lk}_S(v) \subseteq \partial A$. Consequently, Problem 1.1 in dimension d has a positive answer if and only if every combinatorial d -ball D with $m \geq d+2$ vertices is a subcomplex of a combinatorial $(d+1)$ -ball A with $V(A) = V(D)$ and $\partial D \subseteq \partial A$.*

Proof. If S' is an extension, take $A = \text{ast}_{S'}(v)$, a ball with boundary $\text{lk}_{S'}(v) \supseteq \text{lk}_S(v)$ and vertex set $V \setminus \{v\}$. Conversely, $S' = A \cup (v * \partial A) = \partial(v * A)$ is a $(d+1)$ -sphere on V containing $\text{ast}_S(v)$ and $v * \text{lk}_S(v)$, hence S . For the second statement note that every d -ball D is $\text{ast}_S(v)$ for the d -sphere $S = D \cup (v * \partial D)$, which has $m+1$ vertices, and that $\text{lk}_S(v) = \partial D$. $\square$

Remark 4.6 (Datta's Question 2). Datta asks whether every combinatorial d -ball D with $m \geq d+2$ vertices lies in a combinatorial d -sphere with the same m vertices [Dat20, Question 2]. This is equivalent to $D \subseteq \partial A$ for some $(d+1)$ -ball A with $V(A) = V(D)$, which is stronger than the condition of Proposition 4.5. By Proposition 4.1, an antistar that is not completable is a counterexample. Santos' remark in [KNSW19] that no antistar of Altshuler's sphere can be completed therefore already answers Datta's question in the negative, with $d=3$ and $m=9$. Altshuler's sphere is the sphere A_{10} of Table 1 (see Section 6.2), and for it the remark is proved by Proposition 4.3: A_{10} is neighborly and none of its vertex links is stacked.

5 Connected sums, subdivisions and a smallest counterexample

Let S_1, S_2 be combinatorial d -spheres with $V(S_1) \cap V(S_2) = \sigma$, where σ is a facet of both. Their connected sum is $S_1 \#_\sigma S_2 = (S_1 \setminus \{\sigma\}) \cup (S_2 \setminus \{\sigma\})$.

Proposition 5.1. *If each S_i is $\partial(\overline{\sigma \cup \{x_i\}})$ or has an extension, then $S = S_1 \#_\sigma S_2$ has an extension.*

Proof. If $S_1 = \partial(\overline{\sigma \cup \{x_1\}})$, then x_1 has degree $d+1$ in S and S has $|V(S_2)| + 1 \geq d+3$ vertices, so Corollary 3.4(c) applies. Otherwise let S'_i be extensions. The ridge σ of S'_i lies in two facets $\sigma \cup \{a_i\}$ and $\sigma \cup \{b_i\}$, and by Lemma 2.2(a) the complex P_i generated by the other facets is a $(d+1)$ -ball with $\partial P_i = \partial \bar{\sigma} * \{a_i, b_i\}$; it consists of the faces of S'_i that do not contain σ . Let $E_a = \partial \bar{\sigma} * \overline{\{a_1, a_2\}}$ and $E_b = \partial \bar{\sigma} * \overline{\{b_1, b_2\}}$, two $(d+1)$ -balls. Since $a_2, b_2 \notin V(S_1)$, we have $P_1 \cap E_a = \partial \bar{\sigma} * \{a_1\}$, so $Q = P_1 \cup E_a$

is a ball with $\partial Q = \partial\bar{\sigma} * \{b_1, a_2\}$ (Lemma 2.2(b)); then $Q \cap E_b = \partial\bar{\sigma} * \{b_1\}$, and $R = Q \cup E_b$ is a ball with $\partial R = \partial\bar{\sigma} * \{a_2, b_2\}$. A face of R with all vertices in $V(S_2)$ lies in $\partial\bar{\sigma} * \{a_2, b_2\}$, so $R \cap P_2 = \partial R = \partial P_2$, and $S' = R \cup P_2$ is a $(d+1)$ -sphere with vertex set $V(S_1) \cup V(S_2)$ (Lemma 2.2(c)). The only face of S_i containing σ is σ , so $S_i \setminus \{\sigma\} \subseteq P_i$, and $S \subseteq S'$. $\square$

If $d = 3$ and S has a missing face σ with 4 vertices, then $\partial\bar{\sigma}$ is a PL 2-sphere in $|S|$; by Alexander's theorem [Ale24] the closures of the two components of its complement are 3-balls A_1, A_2 , which are subcomplexes with $A_1 \cap A_2 = \partial\bar{\sigma}$. Then $S = S_1 \#_{\sigma} S_2$ with $S_i = A_i \cup \bar{\sigma}$, and each S_i has fewer vertices than S , since $\sigma \notin S$.

For a face σ of K with $|\sigma| \geq 2$ and a new vertex x , the stellar subdivision is $\text{sd}_{\sigma}(K) = \{\rho \in K : \sigma \not\subseteq \rho\} \cup (x * \partial\bar{\sigma} * \text{lk}_K(\sigma))$. It is PL homeomorphic to K .

Proposition 5.2. *If $S \subseteq S'$ is an extension and $\sigma \in S$ with $|\sigma| \geq 2$, then $\text{sd}_{\sigma}(S) \subseteq \text{sd}_{\sigma}(S')$, and $\text{sd}_{\sigma}(S')$ is an extension of $\text{sd}_{\sigma}(S)$.*

Proof. Faces of S not containing σ are faces of S' not containing σ , and $\text{lk}_S(\sigma) \subseteq \text{lk}_{S'}(\sigma)$. Both subdivisions have vertex set $V(S) \cup \{x\}$. $\square$

Corollary 5.3. *Suppose that some combinatorial 3-sphere with at least 6 vertices has no extension, and let S be one with the smallest number n of vertices. Then:*

- (a) $n \geq 11$;
- (b) S is not polytopal;
- (c) S has no missing face with 4 vertices, and it is not a stellar subdivision of another 3-sphere;
- (d) every edge of S lies in a missing triangle;
- (e) no vertex link of S is stacked; in particular every vertex has degree at least 6, and every vertex of degree 6 has the octahedron as its link;
- (f) no antistar of S is completable.

Proof. (a) is Theorem 6.1 below, and (b) is [CS22]. (c) By the remark after Proposition 5.1, a missing face with 4 vertices would write $S = S_1 \#_{\sigma} S_2$ with each S_i either $\partial\Delta^4$ (the only 3-sphere with 5 vertices) or a sphere with $6 \leq |V(S_i)| < n$ vertices, which has an extension by minimality; so Proposition 5.1 would give one for S . The same argument with Proposition 5.2 excludes stellar subdivisions, as $n - 1 \geq 6$. (d) follows from Theorem 3.1, Lemma 3.2 and (c), since a 3-sphere with $n \geq 6$ vertices has no missing face with 5 vertices. (e) follows from Lemma 4.2 and (c): the 2-spheres with at most 5 vertices are stacked, and of the two 2-spheres with 6 vertices the one other than the octahedron has a vertex of degree 3 whose removal leaves the stacked 5-vertex bipyramid (Lemma 2.3(d)). (f) is Proposition 4.1. $\square$

6 Computations

6.1 Certificates and how they are checked

All complexes are stored by their facets. The checking program described in the Verification paragraph uses only the following tests.

- *Sphere recognition.* A pure 1-complex is a 1-sphere if it is a connected cycle. A pure 2-complex is a 2-sphere if it is a closed pseudomanifold with connected dual graph, its vertex links are cycles and its Euler characteristic is 2. For $d = 3, 4$, a pure d -complex is accepted as a combinatorial d -sphere if it is a closed pseudomanifold with connected dual graph, all its vertex links are accepted as combinatorial $(d - 1)$ -spheres, and after deleting one facet a greedy sequence of elementary collapses reduces it to a single vertex. By Lemma 2.2(f), acceptance proves that the complex is a combinatorial sphere. (In dimension 3, triangulated and combinatorial spheres agree, so the published counts below refer to the same objects.)
- *Extension certificates.* A certificate for S is the list of facets of a complex S' ; it is accepted if $V(S') = V(S)$, every facet of S is a face of S' , and S' is accepted as a combinatorial $(d + 1)$ -sphere. Where a facet cover is claimed, it is also checked that every facet of S' meets the given vertex set.
- *Completion certificates* (Proposition 4.1). A certificate is a vertex v and a list of facets L ; it is accepted if $T = \text{ast}_S(v) \cup L$ is accepted as a combinatorial 3-sphere with vertex set $V(S) \setminus \{v\}$. The extension $\Sigma_{u,v}(T)$ is then checked as well.
- *Isomorphism.* Lists are shown to be pairwise non-isomorphic by an isomorphism invariant (vertex-wise multisets of edge degrees), refined inside collisions by an exact canonical form: the lexicographically least relabelling produced by breadth-first traversals of the dual graph from all admissible starting flags.
- *Unsatisfiability.* Non-existence statements are reduced to the unsatisfiability of an explicit CNF formula (Section 6.5). The formula is regenerated by the checker, and a DRUP refutation, a list of clauses each of which follows by unit propagation from the formula and the earlier clauses and ending in a conflict [GN03], is verified by the checker's own forward RUP procedure.

6.2 Three-spheres with at most ten vertices

Theorem 6.1. *Every combinatorial 3-sphere with $6 \leq n \leq 10$ vertices has an extension. More precisely:*

- (a) *all $2 + 5 + 39 + 1296 = 1342$ combinatorial 3-spheres with $6 \leq n \leq 9$ vertices, and all but 1320 of the 247,882 combinatorial 3-spheres with 10 vertices, have an edge satisfying the link condition;*
- (b) *the remaining 1320 spheres are prime (they have no missing face with 4 vertices); 1318 of them have a stacked vertex link, so Lemma 4.2 applies; one of the other two has f -vector $(10, 40, 60, 30)$ and a completable antistar; the last one is the sphere A_{10} of Table 1; all 1320 have explicit extension certificates.*

Proof (computer-assisted). Our lists of 3-spheres with $n = 5, \dots, 10$ vertices were produced by a breadth-first search over bistellar flips [Pac91] starting from $\partial\Delta^4$ and never exceeding 10 vertices. The checker verifies that every entry is a combinatorial 3-sphere on n vertices, that the entries are pairwise non-isomorphic, and that there are 1, 2, 5, 39, 1296, 247,882 of them. These are the published numbers of combinatorial 3-spheres with $5, \dots, 10$ vertices [Lut08, Table 1], [SL09, Table 9], which go back to [Bar73, AS76] for $n \leq 9$. For $n = 10$ the checker also verifies that the distribution of the number of edges agrees with [Lut08, Table 2]; in particular there are 3540 neighborly spheres. Hence the lists contain every combinatorial 3-sphere with at most 10 vertices up to isomorphism. The checker finds an LC edge in every entry with $6 \leq n \leq 9$ vertices and in all but 1320 entries with 10 vertices; Theorem 3.1 applies to them. For the 1320 others it verifies an extension certificate each,

and it verifies the classification in (b), including a completion certificate for the sphere with f -vector $(10, 40, 60, 30)$ and the isomorphism of the last sphere with A_{10} . $\square$

The statement rests on the correctness of the published numbers, which were obtained by lexicographic enumeration [Lut08, SL09]; our flip search is an independent method that reproduced them.

Corollary 6.2. *Up to isomorphism, A_{10} is the only combinatorial 3-sphere with 10 vertices in which no antistar is completable. It is also the only vertex-transitive neighborly 3-sphere with 10 vertices without an LC edge.*

Proof. A sphere with an LC edge vw has the extension $C_v \cup C_w \supseteq C_v$, so $\text{ast}(v)$ is completable by Proposition 4.1. The proof of Lemma 4.2 completes an antistar in each of the 1318 spheres of Theorem 6.1(b) with a stacked vertex link, and the sphere with f -vector $(10, 40, 60, 30)$ has a verified completion certificate. By Proposition 4.3, no antistar of A_{10} is completable. For the second statement, the checker computes the automorphism groups of the 3540 neighborly 10-vertex spheres: exactly three are vertex-transitive, two of them (with automorphism groups of order 20) have LC edges, and the third is A_{10} , with the cyclic automorphism group of order 10. $\square$

The sphere A_{10} is Altshuler’s “peculiar” sphere [Alt76]. In the table of vertex-transitive combinatorial 3-manifolds of Köhler and Lutz [KL05, Table 4], the neighborly sphere ${}^310_1^1$, with automorphism group $\mathbb{Z}_{10}$ and facet orbits 1235, 1236, 1246, 1368 on the vertices $1, \dots, 10$, is attributed to [Alt76] and to [Alt77], where it is N_{3574}^{10} ; we checked by computer that it is isomorphic to A_{10} (Section 6.4). Corollary 6.2 confirms the identification independently, given Santos’ description of Altshuler’s sphere in [KNSW19] as a vertex-transitive neighborly 3-sphere with 10 vertices none of whose antistars can be completed. We did not see [Alt76] or [Alt77] ourselves.

6.3 Four-spheres with at most nine vertices

Proposition 6.3. *Our lists of combinatorial 4-spheres with 6, 7, 8, 9 vertices have 1, 2, 8, 337 entries. The entries are pairwise non-isomorphic combinatorial 4-spheres, and every entry with $n \geq 7$ vertices has an LC edge. If the list with 9 vertices is complete, then every combinatorial 4-sphere with $7 \leq n \leq 9$ vertices has an extension.*

For $n \leq 8$ the lists are complete and the conclusion is known: such spheres are polytopal [Man72], hence boundaries of simplicial 5-polytopes with 7 and 8 vertices, of which there are 2 and 8 (see [Fir17, Table 1]), and [CS22] applies. The list with 9 vertices was produced twice: by a breadth-first search over bistellar flips, and by an exhaustive SAT-based enumeration in which a vertex of maximum degree k has as link one of the $1 + 2 + 5 + 39$ combinatorial 3-spheres with $5 \leq k \leq 8$ vertices and all completions satisfying degree and link conditions are enumerated with blocking clauses. Both gave the same 337 spheres. The final unsatisfiability answers of the enumeration carry no proof certificates, so the completeness of the list is a solver result, and we know no published count of combinatorial 4-spheres with 9 vertices to compare with. (For comparison only: there are 322 combinatorial types of simplicial 5-polytopes with 9 vertices, a count due to Fukuda, Miyata and Moriyama [FMM13]; see [Fir17, Table 1].)

6.4 Vertex-transitive neighborly examples

A SAT search over unions of orbits of 4-sets under the cyclic group, followed by sphere certification and isomorphism reduction, found 2, 2, 1, 3, 10 cyclically invariant neighborly 3-spheres with $n = 10, \dots, 14$ vertices; 1, 1, 0, 2, 9 of them have no LC edge, and these are listed in Table 1. The counts 2, 2, 1, 3, 10

sphere	n	facet orbit representatives under $x \mapsto x + 1 \bmod n$	in [KL05]	in [KL85]
A_{10}	10	0124, 0125, 0135, 0257*	${}^310_1^1$	1_{10}
A_{11}	11	0124, 0125, 0135, 0258	${}^311_1^1$	1_{11}
Z_{13}^0	13	0123, 0139, 0157, 0159, 017b	${}^313_3^1$	2_{13}
Z_{13}^1	13	0137, 0138, 0148, 0149, 0179	${}^313_5^1$	4_{13}
Z_{14}^0	14	0125, 0129, 0149, 0157, 0178*, 025a	${}^314_{17}^1$	10_{14}
Z_{14}^1	14	0129, 012c, 018a, 019b, 01ac, 0279*	${}^314_{14}^1$	9_{14}
Z_{14}^2	14	0126, 0127, 0135, 0137, 024b, 0279*	${}^314_{26}^1$	16_{14}
Z_{14}^3	14	012a, 012b, 019b, 0246, 0257, 0279*	${}^314_{18}^1$	11_{14}
Z_{14}^4	14	0129, 012a, 013a, 013b, 018b, 037a*	${}^314_{27}^1$	17_{14}
Z_{14}^5	14	0137, 0138, 0178*, 0268, 027a, 0369	${}^314_4^1$	1_{14}
Z_{14}^6	14	0178*, 017a, 018a, 0268, 027a, 0369	${}^314_8^1$	4_{14}
Z_{14}^7	14	0123, 0137, 017c, 0269, 0369, 037a*	${}^314_7^1$	7_{14}
Z_{14}^8	14	0123, 013a, 017a, 017c, 0369, 037a*	${}^314_{11}^1$	8_{14}

Table 1: Cyclically invariant neighborly 3-spheres without LC edges. Vertices $10, \dots, 13$ are written $a, \dots, d$; an asterisk marks an orbit of size $n/2$, all other orbits have size n . A_{10} has 35 facets, A_{11} has 44, the Z_{13}^i have 65 and the Z_{14}^i have 77. The last two columns give the names of the spheres in the table of Köhler and Lutz [KL05, Table 4] and in the enumeration of Kühnel and Lassmann [KL85], the latter as quoted in [KL05, Table 4].

agree with the enumeration of all vertex-transitive combinatorial 3-manifolds with at most 15 vertices by Köhler and Lutz [KL05, Tables 2 and 4]: for $n = 10, \dots, 14$, the neighborly 3-spheres in their Table 4 whose automorphism group contains an n -cycle are, up to isomorphism, exactly the spheres found by our search, namely the boundary of the cyclic polytope $C_4(n)$ and the spheres of Table 1. (For $n = 10$, the third neighborly vertex-transitive sphere of [KL05], ${}^310_2^4$, has an automorphism group of order 20 without a 10-cycle.) The spheres of Table 1 are therefore not new: they appear in [KL05, Table 4], and, as quoted there, in the earlier enumeration of neighborly combinatorial 3-manifolds with a vertex-transitive cyclic or dihedral group action by Kühnel and Lassmann [KL85]. We rebuilt the spheres of [KL05, Table 4] from their orbit lists and checked these statements by computer (Verification, item 1). The results below do not depend on the completeness of any of these enumerations. The checker verifies that every sphere of Table 1 is a cyclically invariant, neighborly combinatorial 3-sphere with no LC edge and no missing face with 4 vertices, and that no vertex link is stacked except in Z_{14}^5 , all of whose vertex links are stacked.

Theorem 6.4. *For the spheres of Table 1 the following hold.*

- (a) A_{10} has an extension, and no extension of A_{10} contains a cone C_v . Up to rotation, the pairs $\{u, v\}$ for which A_{10} has an extension with every facet meeting $\{u, v\}$ are exactly $\{0, 1\}$ and $\{0, 5\}$.
- (b) A_{11} has an extension, no extension contains a cone, and no extension has all facets meeting a pair of vertices. For 14 of the 15 rotation classes of 3-sets T there is an extension in which every facet meets T . Hence the least size of a vertex set meeting all facets of an extension of A_{11} is 3.
- (c) Z_{13}^0 and Z_{13}^1 have extensions (for Z_{13}^1 one in which every facet meets $\{0, 1, 11\}$), no extension contains a cone, and no extension has all facets meeting a pair of vertices.
- (d) Six of the nine spheres Z_{14}^i have extensions: Z_{14}^i for $i = 0, 2, 4, 7, 8$ by explicit certificates, and Z_{14}^5 by Lemma 4.2. For Z_{14}^1 , Z_{14}^3 and Z_{14}^6 no extension contains a cone; we do not know whether they have extensions.

Proof (computer-assisted). The existence statements are extension certificates (for A_{10} also for the pairs $\{0, 1\}$ and $\{0, 5\}$, for A_{11} for the 14 classes of 3-sets, and for Z_{14}^5 a completion certificate), all verified by the checker. The statements about cones follow from Proposition 4.3. The statements about pairs: up to rotation the pairs are $\{0, k\}$ with $1 \leq k \leq \lfloor n/2 \rfloor$, and each is an edge. For A_{10} with $k = 2, 3, 4$, for A_{11} with $k = 1, \dots, 5$ and for Z_{13}^0, Z_{13}^1 with $k = 1, \dots, 6$, the formula $\Phi(S, 0, k)$ of Section 6.5 is unsatisfiable; the checker regenerates each of these 20 formulas and verifies its DRUP refutation. By Proposition 4.4 and Lemma 6.5 no extension has all facets meeting $\{0, k\}$. In (b), the minimum is at least 3 by the pair statement and at most 3 by the certificates; no vertex meets all facets of a sphere. $\square$

For the fifteenth class of 3-sets for A_{11} , that of $\{0, 1, 9\}$, the search program of the finder reported unsatisfiability; this is a solver result without a certificate, and we do not use it. The extension of Z_{13}^0 that we found has no vertex cover of its facets with fewer than 4 vertices, so for Z_{13}^0 the least such size is 3 or 4. For Z_{14}^1, Z_{14}^3 and Z_{14}^6 neither a general search nor searches restricted to extensions whose facets meet a fixed 3-set finished within our computing budget, and no obstruction was found. By Theorem 6.4, A_{10} has an extension with a facet cover by two vertices but no completable antistar, and $A_{11}, Z_{13}^0, Z_{13}^1$ have extensions but none with a facet cover by two vertices. So a general positive answer to Problem 1.1 would need constructions that are not one-point suspensions.

6.5 The formula for one-point-suspension extensions

Let S be a combinatorial 3-sphere with vertex set V , and let uv be an edge. Put $W = V \setminus \{v\}$ and $K = S[V \setminus \{u, v\}] \cup \text{lk}_S(v)$. The formula $\Phi(S, u, v)$ has a variable x_F for every 4-subset F of W and a variable y_e for every 2-subset e of W , together with the auxiliary variables of totalizer encodings of cardinality constraints [BB03]. Its clauses express:

- (P1) x_F for every 4-set $F \in K$;
- (P2) for every 3-subset t of W , the number of true x_F with $F \supset t$ is 0 or 2, and it is 2 if $t \in K$;
- (P3) y_e holds if and only if x_F holds for some $F \supset e$, and y_e holds for every edge e of K ;
- (P4) for every $w \in W$, with E_w the number of true y_e with $e \ni w$ and T_w the number of true x_F with $F \ni w$: $E_w \geq 4$ and $T_w = 2E_w - 4$.

Lemma 6.5. *If X is a combinatorial 3-sphere as in Proposition 4.4(ii), then $\Phi(S, u, v)$ is satisfiable.*

Proof. Since $uv \in S$, X has vertex set W . Set x_F true if and only if F is a facet of X , y_e true if and only if e is an edge of X , and the auxiliary variables to the corresponding partial counts. (P1)–(P3) hold because X is a closed pseudomanifold containing K . For $w \in W$ the link $\text{lk}_X(w)$ is a 2-sphere with $E_w \geq 4$ vertices and T_w triangles, and a 2-sphere with m vertices has $2m - 4$ triangles; this gives (P4). $\square$

The formulas have between 3330 and 15,777 variables and between 37,737 and 380,386 clauses. The solver Glucose, called through PySAT [IMMS18], found the 20 formulas of Theorem 6.4 unsatisfiable within seconds; their DRUP refutations have between 24 and 8949 lemmas. For A_{10} and the pairs $\{0, 1\}$ and $\{0, 5\}$ the formula is satisfiable, in accordance with the certificates. The finder's programs and the two independent verification runs used three further encodings (one of them records the two sides of S' explicitly) and obtained the same answers for all these pairs.

7 Concluding remarks

Problem 1.1 remains open. For $d = 3$, Corollary 5.3 reduces it to prime non-polytopal spheres with at least 11 vertices in which every edge lies in a missing triangle, no vertex link is stacked and no antistar is completable. The spheres A_{11} , Z_{13}^0 , Z_{13}^1 and Z_{14}^i , $i \neq 5$, have all these properties; they are not polytopal, because all antistars of a polytopal sphere are completable [Dat20, proof of Thm. 1]. All of them for which our searches finished have extensions. The first undecided cases known to us are Z_{14}^1 , Z_{14}^3 and Z_{14}^6 . In dimension 4, random bistellar flip sequences produced only 10-vertex 4-spheres with LC edges; this is evidence only.

Verification. The computations were checked as follows.

1. A self-contained program by the author, `check_certificates.py` (standard library only, about three minutes on 16 cores), regenerates all tests of Section 6.1 with its own code.
 - It checks the lists of 3-spheres with $5 \leq n \leq 10$ vertices (sphere recognition, pairwise non-isomorphism, the published counts, the edge distribution of [Lut08, Table 2]), the LC edges, the 1320 extension certificates, the classification in Theorem 6.1(b) and Corollary 6.2.
 - It checks the 4-sphere lists of Proposition 6.3.
 - It checks all statements about Table 1: the 26 extension and completion certificates, and the 20 DRUP refutations of Theorem 6.4, with the formulas regenerated by its own encoder.
 - As negative controls, it rejects the 138 neighborly closed 3-manifolds with 9 and 10 vertices that are not spheres, corrupted certificates, and invalid proofs.

The DRUP refutations were produced by a separate script with Glucose through PySAT. A second program by the author, `check_koehler_lutz.py` (standard library only, about 20 seconds), rebuilds the neighborly 3-spheres with 10 to 14 vertices of [KL05, Table 4] from their orbit lists. With its own canonical forms and automorphism groups it confirms the names in Table 1, the isomorphism of A_{10} with ${}^310_1^1$, and the agreement of our orbit search with [KL05] stated in Section 6.4.

2. The lists, certificates and searches come from the finder's programs: flip searches for 3- and 4-spheres, the SAT enumeration of 9-vertex 4-spheres, an independent SAT enumeration of the neighborly 3-spheres with $7 \leq n \leq 10$ vertices (1, 4, 50 and 3540 spheres, the last number as in [Lut08, Table 2]), extension searches with a SAT model that records both sides of S' , and its own DRUP-checked refutations of the pair forms for A_{11} , Z_{13}^0 and Z_{13}^1 .
3. Independent verification runs, which were AI-assisted, re-derived the proofs of Sections 3 to 5 and rechecked the computations with separate code.
 - C programs checked the sphere lists (manifold test, collapsibility, their own canonical forms, LC edges and, in the second run, stackedness of the vertex links and automorphism groups).
 - Their own sphere recognition re-verified the 1320 extension and the completion certificates. Reductions by bistellar flips to the boundary of a simplex, a test independent of collapsing, certified all extension certificates of Theorems 6.1 and 6.4 and both completion certificates once more. Their own flip search reproduced the numbers 1, 2, 5, 39, 1296 of 3-spheres and 1, 2, 8, 337 of 4-spheres with identical sets.
 - Two further SAT encodings based on Proposition 4.4 confirmed all pair-form results of Theorem 6.4; the refutations of one of them were checked by a separately written RUP checker.

- The second run re-examined the candidates rejected during the SAT enumeration of 9-vertex 4-spheres (none of them is a 4-sphere) and matched the spheres of Table 1 with [KL05, Table 4].

These runs also found that Z_{14}^5 is settled by Lemma 4.2, which an earlier version of our analysis had missed, and that the spheres of Table 1, including Altshuler’s sphere, appear in [KL85, KL05].

Scope and priority. The note gives partial answers to Problem 1.1; the general question is open. Theorem 3.1 is a modest generalisation of Datta’s construction for flag spheres [Dat20, Remark 14], and the link condition is classical in the theory of edge contractions [Nev07]. Lemma 4.2 extends the idea of Datta’s case of a vertex of degree $d + 1$, whose link is the boundary of a simplex, and Lemma 2.3(b) is due to Datta and Singh [DS13]. The negative answer to Datta’s Question 2 is due to Santos’ remark in [KNSW19]. The spheres of Table 1 were found before by Kühnel and Lassmann [KL85] and by Köhler and Lutz [KL05]; our results about them are those of Theorem 6.4, and we found no earlier treatment of their extensions. The identification of A_{10} with Altshuler’s sphere rests on [KL05, Table 4], which attributes the corresponding sphere to [Alt76, Alt77], and on our computer check of the isomorphism; Corollary 6.2 confirms it, given the description in [KNSW19]. We did not see [Alt76], [Alt77], [KL85], [DS13] or [FMM13]; what we say about them is taken from [KL05], [KNSW19], [Dat20] and [Fir17]. The statements of Section 6 are computer-assisted and depend on the published counts of 3-spheres; Proposition 6.3 is conditional on the completeness of our enumeration of 9-vertex 4-spheres, which is a solver result. We read the problem session in [KNSW19], Datta’s preprint [Dat20] and the tables of [KL05], and we searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, including works citing [Dat20] and [CS22], and made two web searches. We found no other work on the question and no statement of Theorem 3.1 or Lemma 4.2; they are elementary and may be known to experts. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and the computer certificates, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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