

Polygons with Prescribed Turning Angles in Three Dimensions: A Counterexample to a Conjecture of Efrat, Fulek, Kobourov and Tóth, and the Corrected Criterion

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Abstract

A closed polygon in $\mathbb{R}^3$ has at each vertex a turning angle in $(0, \pi)$, and a sequence $A = (\alpha_0, \dots, \alpha_{n-1}) \in (0, \pi)^n$ is called realizable if it is the sequence of turning angles of such a polygon. Efrat, Fulek, Kobourov and Tóth conjectured (Oberwolfach Rep. 17 (2020); J. Graph Algorithms Appl. 26 (2022)) that a realizable sequence has a realization without self-intersections if and only if n is even or $\sum_i (\pi - \alpha_i) \neq \pi$, and they proved the ‘only if’ part. We show that the conjecture holds for $n = 4$ and $n = 5$ and fails for every $n \geq 6$. For $n = 6$ the sequence $(\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6})$ is realized by an explicit planar hexagon, while every polygon with these turning angles lies in a plane and has total signed turning $\pm 4\pi$, so that none is simple. The corrected criterion is the following. For a set S of indices put $e_S(A) = \sum_{i \in S} (\pi - \alpha_i) + \sum_{i \notin S} \alpha_i - \pi$. A realizable sequence has a realization without self-intersections if and only if $e_S(A) \neq 0$ for every set S of odd cardinality at least 5; the conjecture accounts only for the set of all indices. If moreover $\sum_i \alpha_i > 2\pi$ and $e_S(A) \neq 0$ for all sets S of odd cardinality, there is such a realization which is not contained in a plane. The ingredients on the sphere are known and are not claimed as new: the inequalities $e_S(A) \geq 0$ for closed spherical polygons with prescribed side lengths, and their equality case (all vertices on a great circle), belong to the theory of spherical polygonal linkages (Galitzer, Biswas, Kapovich and Millson, Buckman and Schmitt, Mondello and Panov); the realizability criterion and the ‘only if’ part are due to the authors of the conjecture, who also assert, without proof, that crossings can be avoided whenever some realization is not contained in a plane. What this note adds is the observation that the equality case for a set of at least five indices other than the set of all indices is again an obstruction to simple realizations, the counterexamples, a proof of the assertion just mentioned (by an induction which inserts one vertex at a time), and the resulting characterization. The explicit hexagon is verified in exact arithmetic; all other computations are tests and are not used in the proofs. This is an unrefereed note.

1 Introduction

1.1 Polygons with prescribed turning angles

We use the conventions of Efrat, Fulek, Kobourov and Tóth [EFKT22]. A (closed, oriented) *polygon* $P = (v_0, \dots, v_{n-1})$ in $\mathbb{R}^3$, $n \geq 3$, is a cyclic sequence of points with $v_i \neq v_{i+1}$ for all i ; indices are taken modulo n throughout, and $\mathbb{Z}_n = \{0, \dots, n-1\}$ denotes the index set. The *edges* of P are the segments $e_i = [v_i, v_{i+1}]$, with *lengths* $\ell_i = |v_{i+1} - v_i| > 0$ and unit *directions* $u_i = (v_{i+1} - v_i)/\ell_i$, so that

$$\sum_{i \in \mathbb{Z}_n} \ell_i u_i = 0. \tag{1}$$

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Conversely, unit vectors u_i and numbers $\ell_i > 0$ with (1) are the directions and lengths of a polygon, unique up to translation. We write $d(x, y) \in [0, \pi]$ for the angle between two nonzero vectors, which for unit vectors is their distance on the unit sphere $\mathbb{S}^2$. The *turning angle* of P at v_i is the angle between $v_i - v_{i-1}$ and $v_{i+1} - v_i$, that is, $d(u_{i-1}, u_i)$; it is required to lie in the open interval $(0, \pi)$ [EFKT22, p. 365]. Thus two consecutive edges are never collinear, and the angle enclosed by the two edges at v_i is π minus the turning angle.

An *angle sequence* is a point $A = (\alpha_0, \dots, \alpha_{n-1}) \in (0, \pi)^n$. A polygon P *realizes* A if its turning angle at v_i equals α_i for every $i \in \mathbb{Z}_n$, and A is *realizable* if some polygon in $\mathbb{R}^3$ realizes it. Nothing else is required of a realization: vertices may coincide, non-adjacent edges may overlap, and a polygon contained in a plane is a polygon in $\mathbb{R}^3$ (in [EFKT22] the polygons in $\mathbb{R}^3$ need not be generic, and Theorem 5 there is about sequences all of whose realizations lie in a plane). A polygon is *simple*, or *self-intersection free*, if the closed polygonal curve has no multiple point, that is, if any two non-adjacent edges are disjoint; adjacent edges meet only in their common vertex because the turning angles are smaller than π . This is the meaning of the word in [EFKT22], where a self-intersection may or may not be a proper crossing [EFKT22, p. 364].

For a polygon in an oriented plane, the *signed turning angle* at v_i is the angle $\tau_i \in (-\pi, \pi)$ of the rotation which takes u_{i-1} to u_i ; it is positive for a left turn, and $|\tau_i|$ is the turning angle. The sum $\sum_i \tau_i$ is the *total signed turning*; it is an integer multiple of 2π .

1.2 The conjecture

By [EFKT22, Theorem 2], an angle sequence A is realizable if and only if $\sum_i \alpha_i \geq 2\pi$ and there is a *spherical polygon realizing* A , that is, there are points $u_0, \dots, u_{n-1} \in \mathbb{S}^2$ with $d(u_{i-1}, u_i) = \alpha_i$ for all $i \in \mathbb{Z}_n$. The authors then observe that a realizable sequence need not have a simple realization: by [EFKT22, Theorem 5], if $n \geq 5$ is odd and $\sum_i (\pi - \alpha_i) = \pi$, then A is realizable, but every realization lies in a plane and has a self-intersection. They conjecture that this is the only obstruction.

Conjecture 1.1 ([EFKT22, Conjecture 1], [Tó20, Conjecture 5]). *Let $n \geq 4$ and let $A \in (0, \pi)^n$ be a realizable angle sequence. Then A has a simple realization in $\mathbb{R}^3$ if and only if n is even or $\sum_{i \in \mathbb{Z}_n} (\pi - \alpha_i) \neq \pi$.*

The ‘only if’ part is [EFKT22, Theorem 5]; the conjecture is introduced in [EFKT22, §5] as its converse. The statement was printed four times, in three forms (Table 1).

version	date	range of n	condition
arXiv v1, Conjecture 1	24 August 2020	$n \geq 3$	n odd or $\Sigma \neq \pi$
Oberwolfach abstract [Tó20], Conjecture 5	workshop, September 2020	$n \geq 3$	n even or $\Sigma \neq \pi$
arXiv v2, Conjecture 1	1 November 2020	$n \geq 4$	n even or $\Sigma \neq \pi$
journal version [EFKT22], Conjecture 1	June 2022	$n \geq 4$	n even or $\Sigma \neq \pi$

Table 1: The printed versions of the conjecture: a realizable angle sequence has a simple realization in $\mathbb{R}^3$ if and only if the condition in the last column holds, where $\Sigma = \sum_i (\pi - \alpha_i)$; v1 and v2 are the versions of arXiv:2008.10192.

All versions are about turning angles in $(0, \pi)$. In the first arXiv version the word ‘odd’ stands where the later versions have ‘even’; the sequences of [EFKT22, Theorem 5] have odd n , so we take this for a misprint. The first two versions allow $n = 3$, and for $n = 3$ the ‘only if’ part of the second form fails for a trivial reason: every realization is a triangle, which is simple, while n is odd and $\sum_i (\pi - \alpha_i) = \pi$. The Oberwolfach abstract, on which the record OWR-1703876-012 of the UnsolvedMath collection [ula26] is based, presents the statement as a conjecture “whose ‘only if’ part

we can prove” [Tó20, p. 1514]; the collection lists it as open. The two later versions have $n \geq 4$, and Conjecture 1.1 is this statement. It is the ‘if’ part that we refute: the sequences A_n of Corollary 1.4 contradict it for every $n \geq 6$, in each of the three printed forms (Remark 3.3).

Both [Tó20] and [EFKT22] add that the conjecture is equivalent to the claim that every realization of A has a self-intersection if and only if A can be realized in the plane as a thrackle, that is, as a polygon in which every two non-adjacent edges cross. This form fails as well (Remark 7.1).

1.3 Results

For a subset S of $\mathbb{Z}_n$ and $A \in (0, \pi)^n$ we put

$$e_S(A) = \sum_{i \in S} (\pi - \alpha_i) + \sum_{i \notin S} \alpha_i - \pi = (|S| - 1)\pi - \sum_{i \in S} \alpha_i + \sum_{i \notin S} \alpha_i. \quad (2)$$

A set $S \subseteq \mathbb{Z}_n$ is *odd* if its cardinality is odd, and an odd set S is *tight* for A if $e_S(A) = 0$. If n is odd, then $\mathbb{Z}_n$ is an odd set and $e_{\mathbb{Z}_n}(A) = \sum_i (\pi - \alpha_i) - \pi$, so the exceptional sequences of Conjecture 1.1 are those for which $\mathbb{Z}_n$ is tight.

Theorem 1.2 (obstruction). *Let $n \geq 3$, let $A \in (0, \pi)^n$, and let $S \subseteq \mathbb{Z}_n$ be an odd set with $|S| \geq 3$ and $e_S(A) = 0$.*

- (a) *A is realizable.*
- (b) *Every polygon realizing A lies in a plane. For one of the two orientations of this plane its signed turning angle at v_i is $+\alpha_i$ for $i \in S$ and $-\alpha_i$ for $i \notin S$; in particular its total signed turning is $(|S| - 1)\pi$. The unit directions of its edges are determined by A up to an isometry of $\mathbb{R}^3$.*
- (c) *If $|S| \geq 5$, no polygon realizing A is simple.*

For $S = \mathbb{Z}_n$, n odd, this is [EFKT22, Theorem 5], proved there with Fenchel’s theorem for an auxiliary polygon. The statement on the sphere behind (b), namely that the unit vectors u_i lie on a great circle in a prescribed order, is known from the theory of spherical polygonal linkages; see Section 2.

Theorem 1.3 (characterization). *Let $n \geq 3$ and $A \in (0, \pi)^n$.*

- (a) (Known: [EFKT22, Theorem 2] with the criterion for spherical polygons, see Section 2.) *A is realizable if and only if $\sum_i \alpha_i \geq 2\pi$ and $e_S(A) \geq 0$ for every odd set $S \subseteq \mathbb{Z}_n$.*
- (b) *Let A be realizable. Then A has a simple realization in $\mathbb{R}^3$ if and only if $e_S(A) > 0$ for every odd set $S \subseteq \mathbb{Z}_n$ with $|S| \geq 5$.*
- (c) *If $\sum_i \alpha_i > 2\pi$ and $e_S(A) > 0$ for every odd set $S \subseteq \mathbb{Z}_n$, then A has a simple realization which is not contained in a plane.*

Corollary 1.4 (the conjecture). (a) *For $n = 4$, every realizable angle sequence has a simple realization. For $n = 5$, a realizable angle sequence A has a simple realization if and only if $\sum_i (\pi - \alpha_i) \neq \pi$. So Conjecture 1.1 holds for $n = 4$ and for $n = 5$.*

(b) *For an odd set $S \subseteq \mathbb{Z}_n$ let $\Delta_S = \{A \in (0, \pi)^n : e_S(A) = 0\}$; this is an open simplex of dimension $n - 1$. Let $n \geq 6$. The realizable sequences $A \in (0, \pi)^n$ for which the equivalence of Conjecture 1.1 fails are exactly the points of the sets Δ_S , S odd, with $5 \leq |S| \leq n - 1$. For each of them the ‘if’ part fails: A is realizable, n is even or $\sum_i (\pi - \alpha_i) \neq \pi$, and A has no simple realization.*

(c) In particular, for every $n \geq 6$ the sequence

$$A_n = \left(\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \underbrace{b, \dots, b}_{n-5} \right), \quad b = \frac{\pi}{6(n-5)},$$

is realizable and has no simple realization, although $\sum_i (\pi - \alpha_i) = (n-5)\pi + \frac{2\pi}{3} \neq \pi$.

For $n = 6$ the sequence of (c) is $A^* = A_6 = (\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6})$. In Section 3 we give a planar hexagon with coordinates in $\mathbb{Q}(\sqrt{3})$ which realizes A^* , and a self-contained proof that every polygon with these turning angles is planar with total signed turning $\pm 4\pi$. Everything stated in Theorems 1.2 and 1.3 and Corollary 1.4 is proved in Sections 2 to 6; the only quoted facts are the angle sum of a simple planar polygon and the separation theorem for convex sets in the plane.

1.4 Idea of the proofs

The obstruction. The unit edge directions of a polygon realizing A form a closed chain $u_0, \dots, u_{n-1}$ on the sphere with steps $d(u_{i-1}, u_i) = \alpha_i$. Replacing u_j by $-u_j$ turns the two steps at u_j from α into $\pi - \alpha$. After an even number of such changes, placed at the indices of $S \setminus \{i_0\}$, the triangle inequality on the sphere reads $e_S(A) \geq 0$, and in the case of equality all points lie on a great arc in a prescribed order. The polygon is then planar, it turns to one side exactly at the vertices in S , and its total signed turning is $(|S| - 1)\pi$, while a simple polygon has $\pm 2\pi$.

The constructions. Let G_n be the set of all $A \in (0, \pi)^n$ with $\sum_i \alpha_i > 2\pi$ and $e_S(A) > 0$ for all odd S . We prove by induction on n that every $A \in G_n$ is realized by a simple polygon whose edge directions span $\mathbb{R}^3$. For $n = 4$ a spherical quadrilateral inscribed in a small circle yields four edges of a tetrahedron. For $n \geq 5$ two consecutive angles with the smallest sum are merged into one angle γ such that the shorter sequence lies in G_{n-1} (Lemma 4.4); in a simple polygon for the shorter sequence the corner carrying γ is cut off by a very short edge, and the polygon is closed again by a small linear correction of the edge lengths, which is possible because the directions span $\mathbb{R}^3$ (Lemma 4.3). On the boundary of the region of Theorem 1.3(a) there are simple realizations in two further cases only: convex polygons for $\sum_i \alpha_i = 2\pi$, and pseudo-triangles for a tight set of three elements (Section 5).

1.5 What is known and what is new

The inequalities $e_S(A) \geq 0$ for closed spherical polygons with side lengths α_i , the fact that they characterize the possible side lengths, and their equality case are known; references are given before Lemma 2.3. Theorem 1.3(a) combines [EFKT22, Theorem 2] with this criterion, the ‘only if’ part of Conjecture 1.1 is [EFKT22, Theorem 5], and Fenchel’s inequality is classical. We prove these statements again only to keep the note self-contained.

One more assertion of the source has to be credited. In [EFKT22, §5], just before the conjecture, and already in the first arXiv version, it is said to be “not difficult to see” that crossings can be avoided unless every realization of A in $\mathbb{R}^3$ lies in a plane, and it is concluded that the problem of finding the minimum number of crossings reduces, by [EFKT22, Theorem 1], to a test whether A has a realization which is not contained in a plane. No argument is given. The first assertion is the implication

$$A \text{ has a realization not contained in a plane} \implies A \text{ has a simple realization.} \quad (3)$$

Proposition 4.1 and Remark 7.4 prove it, and we do not know a shorter proof (Remark 7.6). Granted (3), Theorem 1.3(b) follows from Theorem 1 and Lemma 1 of [EFKT22] and from the equality cases of the inequalities $e_S \geq 0$ and of Fenchel’s inequality as soon as one knows which sequences have a

realization not contained in a plane. These are the sequences of G_n (Remark 7.4); the source leaves this test open.

To our knowledge the following is not in print: the observation that tight sets with at least five elements other than $\mathbb{Z}_n$ occur and are obstructions to simple realizations (Theorem 1.2(c) for $S \neq \mathbb{Z}_n$), so that Conjecture 1.1 fails; the counterexamples A^* and A_n ; a proof of (3); the answer to the test just mentioned, that is, the ‘if’ part of Remark 7.4, which together with (3) is Theorem 1.3(c); and the resulting characterization, Theorem 1.3(b) and Corollary 1.4, with all boundary cases. The searches, their limits and the sources read are described in the paragraph “Scope and priority” at the end; no priority is claimed.

2 Preliminaries on the sphere and in the plane

For $x, y \in \mathbb{S}^2$ we have $d(-x, y) = \pi - d(x, y)$, and d is a metric on $\mathbb{S}^2$. If $0 < d(x, z) < \pi$, the *minor arc* from x to z is the shorter arc of the great circle through x and z ; with $D = d(x, z)$ it consists of the points

$$\frac{\sin(D-t)x + \sin t z}{\sin D}, \quad 0 \leq t \leq D, \quad (4)$$

and the point with parameter t is the point of the arc at distance t from x .

Lemma 2.1 (equality in the triangle inequality). *Let $m \geq 1$ and let $x_0, \dots, x_m \in \mathbb{S}^2$ be such that $d(x_{j-1}, x_j) > 0$ for $j = 1, \dots, m$ and*

$$\sum_{j=1}^m d(x_{j-1}, x_j) = d(x_0, x_m) =: D < \pi.$$

Then all x_j lie on the minor arc from x_0 to x_m , and x_j is the point of this arc at distance $t_j = \sum_{i \leq j} d(x_{i-1}, x_i)$ from x_0 .

Proof. For $j = 0$ and $j = m$ there is nothing to prove. Let $0 < j < m$. The triangle inequality gives $D \leq d(x_0, x_j) + d(x_j, x_m) \leq t_j + (D - t_j) = D$, hence $d(x_0, x_j) = t_j$ and $d(x_j, x_m) = D - t_j$, where $0 < t_j < D$ because all steps are positive. So it suffices to treat three points x, y, z with $d(x, y) = t$, $d(y, z) = D - t$ and $d(x, z) = D$, where $0 < t < D < \pi$. Write

$$x = \cos t y + \sin t x', \quad z = \cos(D-t) y + \sin(D-t) z'$$

with unit vectors x', z' orthogonal to y . Then

$$\cos D = \langle x, z \rangle = \cos t \cos(D-t) + \sin t \sin(D-t) \langle x', z' \rangle \geq \cos t \cos(D-t) - \sin t \sin(D-t) = \cos D,$$

and since $\sin t \sin(D-t) > 0$, equality forces $\langle x', z' \rangle = -1$, that is, $z' = -x'$. Consequently $\sin(D-t)x + \sin t z = (\sin(D-t)\cos t + \sin t \cos(D-t))y = \sin D y$, which by (4) says that y is the point of the minor arc from x to z at distance t from x . $\square$

Lemma 2.2 (Fenchel’s inequality for polygons). *If a polygon in $\mathbb{R}^3$ has turning angles $\alpha_0, \dots, \alpha_{n-1}$, then $\sum_i \alpha_i \geq 2\pi$.*

This is classical [Fen29]; for polygons see [Sul08, Theorem 2.4], the reference used in [EFKT22]. We include a proof for completeness; it is the argument usually attributed to Horn [Hor71].

Proof. Let Γ be the closed curve on $\mathbb{S}^2$ formed by the minor arcs from u_{i-1} to u_i , $i \in \mathbb{Z}_n$; its length is $L = \sum_i \alpha_i$. Suppose that $L < 2\pi$. Choose points p, q on Γ which cut it into two arcs of length $L/2$. Then $d(p, q) \leq L/2 < \pi$. Let m be the midpoint of the minor arc from p to q (and $m = p$ if $p = q$), so that $\langle p, m \rangle = \langle q, m \rangle > 0$ and $p + q = 2\langle p, m \rangle m$. We claim that $\langle x, m \rangle > 0$ for all $x \in \Gamma$. Otherwise, by continuity, $\langle x, m \rangle = 0$ for some $x \in \Gamma$. The half-turn $\varrho(y) = 2\langle y, m \rangle m - y$ about m is an isometry of $\mathbb{S}^2$ with $\varrho(p) = q$ and $\varrho(x) = -x$, so that $d(x, q) = d(\varrho(x), \varrho(q)) = d(-x, p) = \pi - d(x, p)$, that is, $d(p, x) + d(x, q) = \pi$. But x lies on one of the two arcs of Γ from p to q , hence $d(p, x) + d(x, q) \leq L/2 < \pi$, a contradiction. So $\langle u_i, m \rangle > 0$ for all i , and $\langle \sum_i \ell_i u_i, m \rangle > 0$ contradicts the closing condition (1). $\square$

The next lemma is the heart of the obstruction. It is known. Part (a) is the necessity half of the criterion for the existence of a closed spherical polygon with prescribed side lengths: the possible side lengths $(r_1, \dots, r_n)$ of closed n -gons on $\mathbb{S}^2$ form the polytope given by $0 \leq r_i \leq \pi$ and $\sum_{i \in I} r_i \leq \sum_{i \notin I} r_i + (|I| - 1)\pi$ for all sets I of odd cardinality. This is a theorem of Galitzer [Gal97], quoted as [KM99, Theorem 3.1]; see also Biswas [Bis98]; Buckman and Schmitt [BS02, Theorem 2.5], the reference given in [EFKT22]; Mondello and Panov [MP16, Theorem 2.22], where the inequalities are stated for the 3-sphere literally in the form $e_S \geq 0$ and said to be essentially contained in [Bis98]; and the appendix by Mamaev in [MP24], with an elementary proof. Part (b) is the equality case. By [MP16, Lemma 2.24(i)], if equality holds in one of the inequalities, every closed polygon with these side lengths lies on a great circle, and by the appendix of [MP24] it is then unique up to isometry. In [KM99] the polygons contained in a great circle are called degenerate; they are the critical points of the side-length map [KM99, Theorem 2.9], and [KM99, §§1, 3] describe the hyperplanes $\sum_{i \in I} r_i - \sum_{i \notin I} r_i = 2\pi w$ through the set I of the edges of one direction and the winding number w of a degenerate polygon; the facets of the polytope lie in those with $|I| = 2w + 1$. This is the sign pattern and the winding number $(|S| - 1)/2$ of part (b). We give the short proof because we need the statement in exactly this form. For $|S| = 1$, part (a) is the polygon inequality $\alpha_i \leq \sum_{j \neq i} \alpha_j$.

Lemma 2.3 (odd-subset inequalities and their equality case; known). *Let $u_0, \dots, u_{n-1} \in \mathbb{S}^2$ with $d(u_{i-1}, u_i) = \alpha_i \in (0, \pi)$ for all $i \in \mathbb{Z}_n$, let $A = (\alpha_0, \dots, \alpha_{n-1})$, and let $S \subseteq \mathbb{Z}_n$ be an odd set.*

- (a) $e_S(A) \geq 0$.
- (b) *If $e_S(A) = 0$, then all u_i lie on one great circle C , and for one of the two orientations of C the signed angle in $(-\pi, \pi)$ from u_{i-1} to u_i is $-\alpha_i$ for $i \in S$ and $+\alpha_i$ for $i \notin S$. The configuration $(u_0, \dots, u_{n-1})$ is determined by A and S up to an isometry of $\mathbb{S}^2$.*

Proof. Fix $i_0 \in S$ and let $F = S \setminus \{i_0\}$, a set of even cardinality. Define signs $\varepsilon_j \in \{1, -1\}$ for $j = i_0, i_0 + 1, \dots, i_0 + n - 1$ by $\varepsilon_{i_0} = 1$ and

$$\varepsilon_j = -\varepsilon_{j-1} \text{ if } j \in F, \quad \varepsilon_j = \varepsilon_{j-1} \text{ if } j \notin F.$$

The indices $i_0 + 1, \dots, i_0 + n - 1$ run through $\mathbb{Z}_n \setminus \{i_0\}$, so the sign changes $|F|$ times, and $\varepsilon_{i_0+n-1} = 1$ because $|F|$ is even. Put $w_j = \varepsilon_j u_j$. For $j = i_0 + 1, \dots, i_0 + n - 1$ we get $d(w_{j-1}, w_j) = \pi - \alpha_j$ if $j \in F$ and $d(w_{j-1}, w_j) = \alpha_j$ if $j \notin S$; all these steps are positive. Moreover $d(w_{i_0+n-1}, w_{i_0}) = d(u_{i_0-1}, u_{i_0}) = \alpha_{i_0}$, since both signs are $+1$. The triangle inequality along the chain $w_{i_0}, w_{i_0+1}, \dots, w_{i_0+n-1}$ gives

$$\alpha_{i_0} \leq \Lambda := \sum_{j \in F} (\pi - \alpha_j) + \sum_{j \notin S} \alpha_j, \quad (5)$$

and $\Lambda - \alpha_{i_0} = e_S(A)$ by (2). This proves (a).

Let $e_S(A) = 0$. Then the chain has total length $\Lambda = \alpha_{i_0} = d(w_{i_0}, w_{i_0+n-1}) < \pi$, and by Lemma 2.1 all w_j lie on the minor arc from w_{i_0} to w_{i_0+n-1} , the point w_j at distance p_j from w_{i_0} , where $p_{i_0} = 0$ and $p_j = p_{j-1} + d(w_{j-1}, w_j)$; in particular $p_{i_0+n-1} = \alpha_{i_0}$. Let C be the great circle containing this arc,

oriented from w_{i_0} towards w_{i_0+n-1} , with the angular coordinate that vanishes at w_{i_0} . Then $u_j = \varepsilon_j w_j$ has the angular coordinate $\varphi_j = p_j$ if $\varepsilon_j = 1$ and $\varphi_j = p_j + \pi$ if $\varepsilon_j = -1$, and the signed angle from u_{j-1} to u_j is $\varphi_j - \varphi_{j-1}$ modulo 2π . For $j \notin S$ the signs agree and $\varphi_j - \varphi_{j-1} = p_j - p_{j-1} = \alpha_j$. For $j \in F$ the signs differ and $\varphi_j - \varphi_{j-1} = (\pi - \alpha_j) \pm \pi \equiv -\alpha_j$. For $j = i_0$ the angle goes from $\varphi_{i_0+n-1} = \alpha_{i_0}$ to $\varphi_{i_0} = 0$, which is $-\alpha_{i_0}$. As $\alpha_j \in (0, \pi)$, these are the representatives in $(-\pi, \pi)$. Finally, the numbers p_j and ε_j depend only on A , S and i_0 , and the configuration is determined by them and by the point w_{i_0} together with the unit tangent vector of C at this point; any two such pairs of a point and a unit tangent vector are related by an isometry of $\mathbb{S}^2$. $\square$

Lemma 2.4 (the tight set is unique). *Let $n \geq 3$ and $A \in (0, \pi)^n$. If $S \neq S'$ are odd subsets of $\mathbb{Z}_n$, then $e_S(A) + e_{S'}(A) > 0$. In particular, at most one odd set is tight for A .*

Proof. By (2), $e_S(A) + \pi$ is the sum over $i \in \mathbb{Z}_n$ of $\pi - \alpha_i$ for $i \in S$ and of α_i for $i \notin S$. An index in the symmetric difference $S \Delta S'$ therefore contributes $(\pi - \alpha_i) + \alpha_i = \pi$ to $e_S(A) + e_{S'}(A) + 2\pi$, and

$$e_S(A) + e_{S'}(A) = 2 \sum_{i \in S \cap S'} (\pi - \alpha_i) + 2 \sum_{i \notin S \cup S'} \alpha_i + (|S \Delta S'| - 2)\pi.$$

The number $|S \Delta S'|$ is even and positive, so the last term is non-negative, and the right-hand side vanishes only if $S \cap S' = \emptyset$, $S \cup S' = \mathbb{Z}_n$ and $|S \Delta S'| = 2$, that is, only if $n = 2$. $\square$

We turn to the plane. The next lemma is stated in [EFKT22, §1] for generic polygons, with a reference to Garg [Gar98].

Lemma 2.5 (planar polygons with prescribed signed turning angles). *Let $\tau_0, \dots, \tau_{n-1}$ be nonzero numbers in $(-\pi, \pi)$ with $\sum_i \tau_i = 2\pi k$ for an integer $k \neq 0$. Then there is a polygon in the oriented plane $\mathbb{R}^2$ whose signed turning angle at v_i is τ_i for every $i \in \mathbb{Z}_n$.*

Proof. Let $\beta_{-1} = 0$, $\beta_i = \tau_0 + \dots + \tau_i$ and $d_i = (\cos \beta_i, \sin \beta_i)$ for $i = 0, \dots, n-1$. Since $\beta_{n-1} = 2\pi k$, the vector d_{n-1} has the polar angle β_{-1} as well, and for every $i \in \mathbb{Z}_n$ the rotation by τ_i takes d_{i-1} to d_i .

We claim that the vectors d_i are not contained in a closed half-plane $\{x : \langle x, c \rangle \geq 0\}$ with $c \neq 0$. Suppose they are, and let γ be the polar angle of c . Then each of the numbers $\beta_{-1}, \beta_0, \dots, \beta_{n-1}$ lies in one of the intervals $[\gamma - \frac{\pi}{2} + 2\pi j, \gamma + \frac{\pi}{2} + 2\pi j]$, $j \in \mathbb{Z}$, any two of which are separated by a gap of length π . Since $|\beta_i - \beta_{i-1}| = |\tau_i| < \pi$, all these numbers lie in the same interval, of length π . This contradicts $|\beta_{n-1} - \beta_{-1}| = 2\pi|k| \geq 2\pi$.

By the separation theorem for convex sets, the origin is therefore an interior point of $K = \text{conv}\{d_0, \dots, d_{n-1}\}$. Choose $\delta > 0$ so small that $-\delta \sum_i d_i \in K$, and write $-\delta \sum_i d_i = \sum_i \mu_i d_i$ with $\mu_i \geq 0$. Then $\lambda_i = \delta + \mu_i > 0$ and $\sum_i \lambda_i d_i = 0$. The polygon with edge vectors $\lambda_i d_i$ has the directions d_i , and its signed turning angle at v_i is the signed angle from d_{i-1} to d_i , which is τ_i . $\square$

Lemma 2.6 (total turning of a simple polygon). *Let P be a simple polygon in an oriented plane. Then its total signed turning is 2π or -2π .*

Proof. Reversing the direction of traversal replaces the pair of directions (u_{i-1}, u_i) at v_i by $(-u_i, -u_{i-1})$ and hence changes the sign of every signed turning angle. So we may assume that the bounded region enclosed by P lies to the left of P . Then the interior angle of P at v_i is $\pi - \tau_i$. The interior angles of a simple n -gon sum to $(n-2)\pi$, because it can be cut into $n-2$ triangles by diagonals (see, for instance, [dBCvKO08, Theorem 3.1]; the proof of [EFKT22, Lemma 1] uses the same fact). Hence $\sum_i (\pi - \tau_i) = (n-2)\pi$, that is, $\sum_i \tau_i = 2\pi$. $\square$

3 The obstruction and the counterexample

3.1 Proof of Theorem 1.2

Let $S \subseteq \mathbb{Z}_n$ be an odd set with $|S| \geq 3$ and $e_S(A) = 0$. By (2) this means

$$\sum_{i \in S} \alpha_i - \sum_{i \notin S} \alpha_i = (|S| - 1)\pi = 2\pi k, \quad k = \frac{|S| - 1}{2} \geq 1. \quad (6)$$

(a) Put $\tau_i = \alpha_i$ for $i \in S$ and $\tau_i = -\alpha_i$ for $i \notin S$. By (6) and Lemma 2.5 there is a polygon in the plane with signed turning angles τ_i ; its turning angles are the α_i .

(b) Let P be a polygon realizing A . Its unit edge directions satisfy $d(u_{i-1}, u_i) = \alpha_i$, so Lemma 2.3(b) applies to them: they lie on a great circle C , that is, in a linear subspace Π of dimension two, and P lies in the plane $v_0 + \Pi$. Orient this plane by the orientation of C opposite to the one of Lemma 2.3(b). The signed turning angle of P at v_i is the signed angle from u_{i-1} to u_i , which is now $+\alpha_i$ for $i \in S$ and $-\alpha_i$ for $i \notin S$; the total is $(|S| - 1)\pi$ by (6). By Lemma 2.3(b) the vectors u_i are determined by A and S up to an isometry of $\mathbb{S}^2$, which is the restriction of a linear isometry of $\mathbb{R}^3$, and S is determined by A (Lemma 2.4).

(c) If P is simple, then Lemma 2.6 and (b) give $(|S| - 1)\pi = 2\pi$, that is, $|S| = 3$. $\square$

Only the directions of the edges were used in (b); so the statement covers every polygon with the turning angles A , whatever its edge lengths are and whether or not vertices coincide or edges overlap. A set $S = \{i\}$ cannot be tight for a realizable sequence: then $\alpha_i = \sum_{j \neq i} \alpha_j$, hence $\sum_j \alpha_j = 2\alpha_i < 2\pi$, against Lemma 2.2. For $|S| = 3$ the total signed turning is 2π , and there are simple realizations (Lemma 5.2).

3.2 The hexagon

Let $A^* = (\alpha_0, \dots, \alpha_5) = (\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6})$, that is, $(150^\circ, 150^\circ, 150^\circ, 150^\circ, 150^\circ, 30^\circ)$. Let H be the hexagon in the plane $\mathbb{R}^2 \subset \mathbb{R}^3$ with the vertices

$$\begin{aligned} v_0 &= (0, 0), & v_1 &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), & v_2 &= \left(\frac{\sqrt{3}-1}{2}, \frac{1-\sqrt{3}}{2}\right), \\ v_3 &= \left(\frac{\sqrt{3}-1}{2}, \frac{3\sqrt{3}-1}{2}\right), & v_4 &= \left(\frac{\sqrt{3}+2}{2}, -\frac{1}{2}\right), & v_5 &= (1, 0). \end{aligned}$$

Its edge vectors are $v_{i+1} - v_i = \ell_i(\cos \varphi_i, \sin \varphi_i)$ with

$$(\varphi_0, \dots, \varphi_5) = (30^\circ, 240^\circ, 90^\circ, 300^\circ, 150^\circ, 180^\circ), \quad (\ell_0, \dots, \ell_5) = (1, 1, 2\sqrt{3} - 1, 3, 1, 1). \quad (7)$$

Indeed, the sum of the six vectors $\ell_i(\cos \varphi_i, \sin \varphi_i)$ has the coordinates $\frac{\sqrt{3}}{2} - \frac{1}{2} + 0 + \frac{3}{2} - \frac{\sqrt{3}}{2} - 1 = 0$ and $\frac{1}{2} - \frac{\sqrt{3}}{2} + (2\sqrt{3} - 1) - \frac{3\sqrt{3}}{2} + \frac{1}{2} + 0 = 0$, and the partial sums are the points v_i . The signed turning angle at v_i is $\varphi_i - \varphi_{i-1}$ reduced to $(-180^\circ, 180^\circ)$:

$$(\tau_0, \dots, \tau_5) = (-150^\circ, -150^\circ, -150^\circ, -150^\circ, -150^\circ, +30^\circ), \quad \sum_i \tau_i = -720^\circ.$$

So H realizes A^* , and A^* is realizable. The hexagon H has exactly three self-intersections, all of them proper crossings:

$$e_0 \cap e_2 = \left\{ \left(\frac{\sqrt{3}-1}{2}, \frac{1}{2} - \frac{\sqrt{3}}{6} \right) \right\}, \quad e_1 \cap e_5 = \left\{ \left(\frac{\sqrt{3}}{3}, 0 \right) \right\}, \quad e_2 \cap e_5 = \left\{ \left(\frac{\sqrt{3}-1}{2}, 0 \right) \right\};$$

the other six pairs of non-adjacent edges are disjoint, the six vertices are distinct, and no vertex lies on an edge not incident with it. With the same directions and the lengths $(2, 1, 3\sqrt{3} - 2, 5, 2, 2)$

one obtains a second hexagon H_1 realizing A^* , with the vertices $(0,0)$, $(\sqrt{3},1)$, $(\sqrt{3} - \frac{1}{2}, 1 - \frac{\sqrt{3}}{2})$, $(\sqrt{3} - \frac{1}{2}, \frac{5\sqrt{3}}{2} - 1)$, $(\sqrt{3} + 2, -1)$, $(2,0)$ and with exactly one self-intersection, the proper crossing $e_0 \cap e_2 = \{(\sqrt{3} - \frac{1}{2}, 1 - \frac{\sqrt{3}}{6})\}$; see Figure 1. All these statements can be checked from the coordinates; they are also verified in exact arithmetic by five programs (Section 8).

The sequence A^* does not belong to the exceptional family of Conjecture 1.1: $n = 6$ is even. (Also $\sum_i (\pi - \alpha_i) = \frac{5\pi}{3} \neq \pi$.) On the other hand $e_S(A^*) = 0$ for $S = \{0, 1, 2, 3, 4\}$, because $\alpha_0 + \dots + \alpha_4 - \alpha_5 = 4\pi$. By Theorem 1.2, every realization of A^* is planar with total signed turning $\pm 4\pi$, and none is simple. Since this case alone refutes Conjecture 1.1, we write the argument out for it.

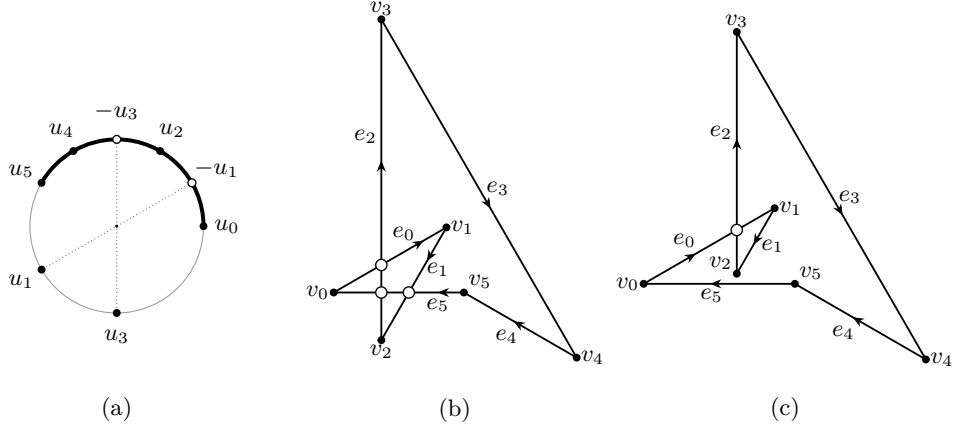


Figure 1: (a) The unit edge directions of every polygon realizing A^* , up to an isometry: $u_0, -u_1, u_2, -u_3, u_4$ are equally spaced on an arc of 120° , and u_5 lies 30° beyond u_4 . (b) The hexagon H of (7), with three crossings. (c) The hexagon H_1 , with one crossing. Both turn right by 150° at $v_0, \dots, v_4$ and left by 30° at v_5 .

Proposition 3.1. *Every polygon in $\mathbb{R}^3$ with the turning angles A^* lies in a plane and has total signed turning 4π or -4π . In particular it is not simple.*

Proof. Let $u_0, \dots, u_5$ be the unit edge directions of such a polygon, and measure angles in degrees. Then $d(u_5, u_0) = d(u_0, u_1) = d(u_1, u_2) = d(u_2, u_3) = d(u_3, u_4) = 150$ and $d(u_4, u_5) = 30$. The points

$$w_0 = u_0, \quad w_1 = -u_1, \quad w_2 = u_2, \quad w_3 = -u_3, \quad w_4 = u_4$$

satisfy $d(w_{j-1}, w_j) = 180 - 150 = 30$ for $j = 1, \dots, 4$, so that $d(u_0, u_4) \leq 120$. On the other hand $d(u_0, u_4) \geq d(u_0, u_5) - d(u_5, u_4) = 150 - 30 = 120$. Hence $d(u_0, u_4) = 120$, and both triangle inequalities hold with equality. By Lemma 2.1, applied to the chain $w_0, \dots, w_4$, these five points lie on the minor arc from u_0 to u_4 at the distances $0, 30, 60, 90, 120$ from u_0 . By Lemma 2.1, applied to the chain u_0, u_4, u_5 of length $120 + 30 = 150 = d(u_0, u_5)$, the point u_4 lies on the minor arc from u_0 to u_5 ; so u_5 lies on the great circle C through u_0 and u_4 , 30 beyond u_4 . In the angular coordinate of C which is 0 at u_0 and 120 at u_4 we get

$$(u_0, u_1, u_2, u_3, u_4, u_5) = (0, 210, 60, 270, 120, 150)$$

(Figure 1(a)). Thus the polygon lies in a plane, and with the orientation given by this coordinate its signed turning angles are the differences $0 - 150, 210 - 0, 60 - 210, 270 - 60, 120 - 270, 150 - 120$, reduced to $(-180, 180)$, that is, -150 five times and $+30$ once. Their sum is -720 ; with the other orientation it is $+720$. A simple polygon in a plane has total signed turning ± 360 by Lemma 2.6. $\square$

The directions (7) of H are the forced directions, rotated by 30° . So the realizations of A^* are, up to isometries, exactly the hexagons with these six directions and positive lengths satisfying (1); they form a cone of dimension four.

Remark 3.2 (a boundary phenomenon). If $S \subseteq \mathbb{Z}_6$ contains a of the indices $0, \dots, 4$ and $c \in \{0, 1\}$ of the index 5, then $e_S(A^*) = \frac{\pi}{6}(20 - 4a + 4c)$. So $S_0 = \{0, \dots, 4\}$ is tight and $e_S(A^*) \geq \frac{4\pi}{3}$ for every other odd set. Hence for $0 < \delta < \frac{5\pi}{6}$ the sequence $(\frac{5\pi}{6}, \dots, \frac{5\pi}{6}, \frac{\pi}{6} + \delta)$ satisfies the hypothesis of Theorem 1.3(c) and has a simple realization which is not planar, while for $-\frac{\pi}{6} < \delta < 0$ it is not realizable, since $e_{S_0} = \delta < 0$. So the obstruction disappears under a perturbation of the last angle: what matters is that a tight set forces *all* realizations into a plane.

3.3 The sequences A_n

Let $n \geq 6$ and $A_n = (\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, b, \dots, b)$ with $n - 5$ entries equal to $b = \frac{\pi}{6(n-5)}$, so that $0 < b \leq \frac{\pi}{6}$ and $A_6 = A^*$. For $S_0 = \{0, 1, 2, 3, 4\}$,

$$e_{S_0}(A_n) = 5 \cdot \frac{\pi}{6} + (n - 5)b - \pi = 0.$$

By Theorem 1.2, A_n is realizable, every realization is planar with total signed turning $\pm 4\pi$, and none is simple. Explicitly, in the frame of Proposition 3.1 the edge directions of a realization are

$$0^\circ, 210^\circ, 60^\circ, 270^\circ, 120^\circ, 120^\circ + c, 120^\circ + 2c, \dots, 150^\circ, \quad c = \frac{30^\circ}{n - 5};$$

the $n - 6$ new directions are positive combinations of the directions 120° and 150° , so giving every new edge a small length and shortening the two edges of H with these two directions accordingly, one obtains a realization of A_n . Finally

$$\sum_i (\pi - \alpha_i) = 5 \cdot \frac{\pi}{6} + (n - 5)(\pi - b) = (n - 5)\pi + \frac{2\pi}{3} \neq \pi. \quad (8)$$

Remark 3.3 (all printed forms). By (8), the right-hand side of the equivalence is true for A_n in each form of Table 1: with ‘ n even or’, with ‘ n odd or’, and for either parity of n . So every printed form of the conjecture predicts a simple realization of A_n for every $n \geq 6$, and none exists.

4 The open region: simple polygons that are not planar

Let

$$G_n = \left\{ A \in (0, \pi)^n : \sum_i \alpha_i > 2\pi \text{ and } e_S(A) > 0 \text{ for every odd set } S \subseteq \mathbb{Z}_n \right\}.$$

The set G_3 is empty, because $e_{\mathbb{Z}_3}(A) = 2\pi - \sum_i \alpha_i$.

Proposition 4.1. *Let $n \geq 4$. Every $A \in G_n$ is realized by a simple polygon whose edge directions span $\mathbb{R}^3$.*

Such a polygon is not contained in a plane, so this is Theorem 1.3(c). The proof is by induction on n : Lemma 4.2 is the case $n = 4$, and Lemmas 4.3 and 4.4 give the induction step.

Lemma 4.2 ($n = 4$). *Every $A \in G_4$ is realized by a quadrilateral whose four vertices are affinely independent. Such a quadrilateral is simple, and its edge directions span $\mathbb{R}^3$.*

Proof. Put $\theta_i = \pi - \alpha_i \in (0, \pi)$. By (2), $e_{\mathbb{Z}_4 \setminus \{l\}}(A) = \sum_{i \neq l} \theta_i - \theta_l$, so that $A \in G_4$ implies

$$\theta_0 + \theta_1 + \theta_2 + \theta_3 < 2\pi \quad \text{and} \quad \theta_l < \sum_{i \neq l} \theta_i \quad (l \in \mathbb{Z}_4). \quad (9)$$

Step 1: points on a small circle. Let θ_m be a largest of the θ_i and $r_0 = \theta_m/2 \in (0, \frac{\pi}{2})$. For $r \in [r_0, \frac{\pi}{2}]$ let $\phi_i(r) = 2 \arcsin(\sin \frac{\theta_i}{2} / \sin r) \in (0, \pi]$. These functions are continuous, $\phi_i(\frac{\pi}{2}) = \theta_i$ and $\phi_m(r_0) = \pi$. Let $F = \sum_i \phi_i$. If $F(r_0) \geq 2\pi$, then, as $F(\frac{\pi}{2}) = \sum_i \theta_i < 2\pi$, there is $r \in [r_0, \frac{\pi}{2})$ with $F(r) = 2\pi$, and we put $\Delta_i = \phi_i(r)$ for all i . If $F(r_0) < 2\pi$, consider $h = \phi_m - \sum_{i \neq m} \phi_i$. Then $h(r_0) = 2\pi - F(r_0) > 0$ and $h(\frac{\pi}{2}) = \theta_m - \sum_{i \neq m} \theta_i < 0$, so $h(r) = 0$ for some $r \in (r_0, \frac{\pi}{2})$, and we put $\Delta_i = \phi_i(r)$ for $i \neq m$ and $\Delta_m = 2\pi - \phi_m(r)$. In both cases

$$0 < r < \frac{\pi}{2}, \quad \Delta_i > 0, \quad \sum_i \Delta_i = 2\pi, \quad \sin \frac{\Delta_i}{2} = \frac{\sin(\theta_i/2)}{\sin r}.$$

Let $\psi_0 = 0$ and $\psi_j = \Delta_1 + \dots + \Delta_j$ for $j = 1, 2, 3$, so that $0 = \psi_0 < \psi_1 < \psi_2 < \psi_3 < 2\pi$ and $2\pi - \psi_3 = \Delta_0$, and let $w_j = (\sin r \cos \psi_j, \sin r \sin \psi_j, \cos r) \in \mathbb{S}^2$. For $j \in \mathbb{Z}_4$ the points w_{j-1} and w_j differ by the angle Δ_j or $2\pi - \Delta_j$ on the circle, hence $\langle w_{j-1}, w_j \rangle = \cos^2 r + \sin^2 r \cos \Delta_j = 1 - 2 \sin^2 r \sin^2 \frac{\Delta_j}{2} = \cos \theta_j$, that is, $d(w_{j-1}, w_j) = \theta_j$.

Step 2: orientation. For $i < j < k$ put $a = \psi_j - \psi_i$ and $b = \psi_k - \psi_j$; then $a, b > 0$ and $a + b < 2\pi$, and a direct computation gives

$$D_{ijk} := \det(w_i, w_j, w_k) = \sin^2 r \cos r (\sin a + \sin b - \sin(a+b)) = 4 \sin^2 r \cos r \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{a+b}{2} > 0.$$

Step 3: the quadrilateral. Let $u_j = (-1)^j w_j$. Then $d(u_{j-1}, u_j) = \pi - \theta_j = \alpha_j$ for all $j \in \mathbb{Z}_4$; for the pair u_3, u_0 this uses that 4 is even. By Cramer's rule, any four vectors of $\mathbb{R}^3$ satisfy $D_{123}w_0 - D_{023}w_1 + D_{013}w_2 = D_{012}w_3$, which reads

$$D_{123}u_0 + D_{023}u_1 + D_{013}u_2 + D_{012}u_3 = 0$$

with positive coefficients. So the polygon with the directions u_j and the lengths $D_{123}, D_{023}, D_{013}, D_{012}$ closes, and it realizes A . Since $\det(u_0, u_1, u_2) = -D_{012} \neq 0$, its vertices are affinely independent. Its edges are then four edges of a tetrahedron, and its two pairs of non-adjacent edges are pairs of opposite edges of the tetrahedron, which are disjoint. $\square$

Lemma 4.3 (inserting a vertex). *Let $m \geq 4$, and let P' be a simple polygon with m vertices, with unit edge directions $u_0, \dots, u_{m-1}$ which span $\mathbb{R}^3$ and with turning angles $\alpha_i = d(u_{i-1}, u_i)$. Let $s \in \mathbb{Z}_m$ and $\beta, \beta' \in (0, \pi)$ with*

$$|\beta - \beta'| < a_s < \min(\beta + \beta', 2\pi - \beta - \beta'). \quad (10)$$

Then there is a simple polygon with $m + 1$ vertices, with edge directions spanning $\mathbb{R}^3$ and with the turning angles $(a_0, \dots, a_{s-1}, \beta, \beta', a_{s+1}, \dots, a_{m-1})$.

Proof. The new direction. Let $f = u_{s-1}$, $g = u_s$ and $\gamma = a_s = d(f, g)$. Let f' be the unit vector in $\text{span}(f, g)$ orthogonal to f with $g = \cos \gamma f + \sin \gamma f'$, and let $\nu = f \times f'$. The points $x(\psi) = \cos \beta f + \sin \beta (\cos \psi f' + \sin \psi \nu)$ satisfy $d(f, x(\psi)) = \beta$, and $\langle g, x(\psi) \rangle = \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos \psi$ decreases strictly from $\cos(\gamma - \beta)$ to $\cos(\gamma + \beta)$ as ψ goes from 0 to π . By (10), $\cos(\gamma + \beta) < \cos \beta' < \cos(\gamma - \beta)$: indeed $\beta' > |\gamma - \beta|$, and $\beta' < \gamma + \beta$ if $\gamma + \beta \leq \pi$, while $\beta' < 2\pi - \gamma - \beta$ if $\gamma + \beta > \pi$. So there is $\psi \in (0, \pi)$ with $d(x(\psi), g) = \beta'$. Let $z = x(\psi)$. Its component $\sin \beta \sin \psi$ along ν is not zero, so $z \notin \text{span}(u_{s-1}, u_s)$.

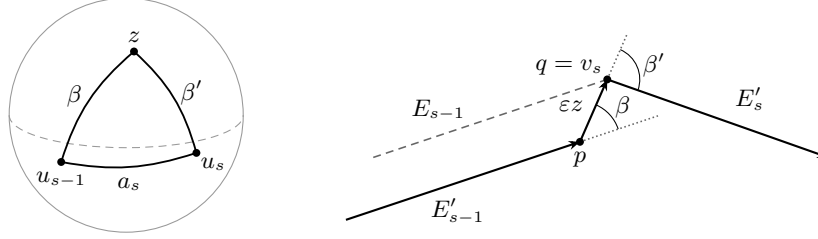


Figure 2: The insertion step of Lemma 4.3. Left: the new direction z is the third vertex of a spherical triangle over the arc from u_{s-1} to u_s , with sides β and β' . Right (schematic): the edge E_{s-1} of P' (dashed) is replaced by the parallel edge E'_{s-1} , ending at $p = v_s - \varepsilon z$, and by the short edge from p to $q = v_s$.

The new polygon. Let ℓ_i be the edge lengths and v_i the vertices of P' . As the u_i span $\mathbb{R}^3$, there is $c \in \mathbb{R}^m$ with $\sum_i c_i u_i = z$. For small $\varepsilon > 0$ the numbers $\ell'_i = \ell_i - \varepsilon c_i$ are positive, and $\sum_i \ell'_i u_i + \varepsilon z = 0$. Let P_ε be the polygon which starts at $q = v_s$, runs through the edges $\ell'_s u_s, \ell'_{s+1} u_{s+1}, \dots, \ell'_{s-1} u_{s-1}$, thereby arriving at $p = v_s - \varepsilon z$, and returns to q by the edge εz (Figure 2). We denote by E_i the edges of P' and by E'_i the edge of P_ε with direction u_i . The directions of P_ε span $\mathbb{R}^3$. Its turning angles are a_i at the common end point of E'_{i-1} and E'_i for $i \neq s$, further $d(u_{s-1}, z) = \beta$ at p and $d(z, u_s) = \beta'$ at q ; in cyclic order this is the required sequence. With $\kappa = 1 + \sum_i |c_i|$, every end point of E'_i is within $\kappa\varepsilon$ of the corresponding end point of E_i , so every point of E'_i is within $\kappa\varepsilon$ of E_i .

P_ε is simple for small ε . There are five kinds of pairs of edges. (1) E'_i, E'_j with E_i, E_j non-adjacent in P' : the disjoint compact sets E_i, E_j have a positive distance δ_{ij} , and E'_i, E'_j are disjoint as soon as $2\kappa\varepsilon < \delta_{ij}$. (2) E'_{i-1}, E'_i with $i \neq s$: they have a common end point and are not collinear, since $a_i \in (0, \pi)$. (3) E'_{s-1}, E'_s : they lie on the lines $p + \mathbb{R}u_{s-1}$ and $p + \varepsilon z + \mathbb{R}u_s$, which are disjoint because $z \notin \text{span}(u_{s-1}, u_s)$. (4) The new edge $[p, q]$ meets E'_{s-1} only in p and E'_s only in q , since $\beta, \beta' \in (0, \pi)$. (5) $[p, q]$ and E'_j with $j \notin \{s-1, s\}$: the vertex v_s does not lie on E_j because P' is simple, so it has a positive distance δ_j from E_j ; the segment $[p, q]$ is within ε of v_s , and it is disjoint from E'_j as soon as $(\kappa + 1)\varepsilon < \delta_j$. These are finitely many conditions on ε . $\square$

Lemma 4.4 (merging two consecutive angles). *Let $n \geq 5$ and $A \in G_n$, and let $s \in \mathbb{Z}_n$ be an index for which $\alpha_s + \alpha_{s+1}$ is smallest. Then there is a number γ with*

$$|\alpha_s - \alpha_{s+1}| < \gamma < \min(\alpha_s + \alpha_{s+1}, 2\pi - \alpha_s - \alpha_{s+1}) \quad (11)$$

such that the sequence obtained from A by replacing the two consecutive entries α_s, α_{s+1} by the single entry γ lies in G_{n-1} .

Proof. Write $a = \alpha_s, b = \alpha_{s+1}$ and $R = \mathbb{Z}_n \setminus \{s, s+1\}$, so that $|R| = n-2 \geq 3$, and put $\Sigma_R = \sum_{i \in R} \alpha_i$ and, for $Q \subseteq R$, $\sigma(Q) = \sum_{i \in Q} \alpha_i - \sum_{i \in R \setminus Q} \alpha_i$. The index set of the new sequence A' is $R \cup \{*\}$, where $*$ carries γ . Its odd subsets are the sets $Q \cup \{*\}$ with $|Q|$ even and the sets Q with $|Q|$ odd, $Q \subseteq R$, and by (2)

$$e_{Q \cup \{*\}}(A') = |Q|\pi - \sigma(Q) - \gamma, \quad e_Q(A') = \gamma - \sigma(Q) + (|Q| - 1)\pi.$$

Every γ with (11) lies in $(0, \pi)$. Hence $A' \in G_{n-1}$ and (11) hold if and only if $L < \gamma < U$, where L is the largest of the numbers

$$L_1 = |a - b|, \quad L_2 = 2\pi - \Sigma_R, \quad L_3(Q) = \sigma(Q) - (|Q| - 1)\pi \quad (Q \subseteq R, |Q| \text{ odd}),$$

and U is the smallest of the numbers

$$U_1 = a + b, \quad U_2 = 2\pi - a - b, \quad U_3(Q') = |Q'|\pi - \sigma(Q') \quad (Q' \subseteq R, |Q'| \text{ even});$$

here $\gamma > L_2$ is the condition that the sum of A' exceeds 2π . We show that each lower bound is smaller than each upper bound.

- (1) $L_1 < U_1$ and $L_1 < U_2$, because $0 < a, b < \pi$.
- (2) $L_1 < U_3(Q')$: the numbers $e_{Q' \cup \{s\}}(A) = U_3(Q') - (a - b)$ and $e_{Q' \cup \{s+1\}}(A) = U_3(Q') - (b - a)$ are positive.
- (3) $L_3(Q) < U_1$, because $e_Q(A) = U_1 - L_3(Q) > 0$.
- (4) $L_3(Q) < U_2$, because $e_{Q \cup \{s, s+1\}}(A) = U_2 - L_3(Q) > 0$.
- (5) $L_3(Q) < U_3(Q')$: we have $U_3(Q') - L_3(Q) = (|Q| + |Q'| - 1)\pi - \sigma(Q) - \sigma(Q')$ and

$$\sigma(Q) + \sigma(Q') = 2 \sum_{i \in Q \cap Q'} \alpha_i - 2 \sum_{i \in R \setminus (Q \cup Q')} \alpha_i \leq 2|Q \cap Q'| \pi \leq (|Q| + |Q'| - 1)\pi,$$

where the last step holds because $|Q| + |Q'| - 1 - 2|Q \cap Q'| = |Q \Delta Q'| - 1$ and $|Q \Delta Q'|$ is odd. The first inequality is an equality only if $Q \cap Q' = \emptyset$ and $Q \cup Q' = R$; then $|Q \Delta Q'| = |R| \geq 3$ and the second one is strict.

- (6) $L_2 < U_1$, because $\sum_i \alpha_i > 2\pi$.
- (7) $L_2 < U_3(Q')$ for $Q' \neq \emptyset$: $U_3(Q') - L_2 = (|Q'| - 2)\pi + 2 \sum_{i \in R \setminus Q'} \alpha_i > 0$, since $|Q'| \geq 4$ or $R \setminus Q' \neq \emptyset$.
- (8) $L_2 < U_3(\emptyset)$ means $\Sigma_R > \pi$.
- (9) $L_2 < U_2$ means $a + b < \Sigma_R$.

Only (8) and (9) depend on the choice of s . The indices $s+2, s+3, s+4$ are three different elements of R , as $n \geq 5$, and $\alpha_{s+2} + \alpha_{s+3} \geq a + b$ by the choice of s ; hence $\Sigma_R > a + b$, which is (9). If $a + b > \pi$, this gives (8); and if $a + b \leq \pi$, then $\Sigma_R = \sum_i \alpha_i - (a + b) > 2\pi - \pi$. So $L < U$, and any $\gamma \in (L, U)$ has the required properties. $\square$

The choice of s matters. The sequence $A = (\frac{9}{10}\pi, \frac{9}{10}\pi, \frac{1}{10}\pi, \frac{1}{10}\pi, \frac{1}{10}\pi)$ lies in G_5 : its sum is $\frac{21}{10}\pi$, and the smallest value of $e_S(A)$ over the odd sets is $\frac{3}{10}\pi$. Merging the two large angles is impossible, since it would require $\gamma < U_2 = \frac{1}{5}\pi$ and $\gamma > L_2 = \frac{17}{10}\pi$. For $s = 2$ the lemma gives the interval $(L, U) = (\frac{1}{10}\pi, \frac{1}{5}\pi)$.

Proof of Proposition 4.1. For $n = 4$ this is Lemma 4.2. Let $n \geq 5$, $A \in G_n$, and choose s and γ as in Lemma 4.4. The set G_n and the assertion are invariant under cyclic shifts of the indices, so we may assume $s \leq n - 2$. Then $A' = (\alpha_0, \dots, \alpha_{s-1}, \gamma, \alpha_{s+2}, \dots, \alpha_{n-1}) \in G_{n-1}$ has the entry γ in position s . By induction A' is realized by a simple polygon P' whose directions span $\mathbb{R}^3$; its turning angle at the vertex number s is γ . By (11), Lemma 4.3 applies at this vertex with $\beta = \alpha_s$ and $\beta' = \alpha_{s+1}$, and yields a simple polygon with spanning directions and with the turning angles $(\alpha_0, \dots, \alpha_{s-1}, \alpha_s, \alpha_{s+1}, \alpha_{s+2}, \dots, \alpha_{n-1}) = A$. $\square$

5 The two planar cases

Lemma 5.1 ($\sum_i \alpha_i = 2\pi$: convex polygons). *Let $n \geq 3$ and $A \in (0, \pi)^n$ with $\sum_i \alpha_i = 2\pi$. Then A is realized by a convex polygon in a plane, which is simple.*

Proof. Lemma 2.5 with $\tau_i = \alpha_i$ gives a polygon P in the plane with signed turning angles $\alpha_i > 0$. Fix i , let d_i be the direction of the edge e_i , and let $h(x)$ be the signed distance of x from the line through e_i , positive on the left of d_i . The edges $e_{i+1}, \dots, e_{i+n-1}$ lead from v_{i+1} back to v_i , and the edge e_{i+j} makes the counterclockwise angle $\omega_j = \alpha_{i+1} + \dots + \alpha_{i+j}$ with d_i . These angles increase from $\omega_1 = \alpha_{i+1} \in (0, \pi)$ to $\omega_{n-1} = 2\pi - \alpha_i \in (\pi, 2\pi)$, and along e_{i+j} the function h changes by the length

of the edge times $\sin \omega_j$. So along the path from v_{i+1} to v_i the function h first increases strictly, then is constant on at most one edge, then decreases strictly, and it vanishes at both ends. Hence $h > 0$ at every vertex other than v_i and v_{i+1} , and every edge not adjacent to e_i lies strictly on the left of the line through e_i . Thus P is convex and simple. $\square$

Lemma 5.2 (a tight set of three elements: pseudo-triangles). *Let $n \geq 3$, let $S \subseteq \mathbb{Z}_n$ with $|S| = 3$, and let $A \in (0, \pi)^n$ with $e_S(A) = 0$. Then A is realized by a simple polygon in a plane; it turns left at the three vertices v_x , $x \in S$, and right at all other vertices.*

The existence of such a polygon also follows from [EFKT22, Theorem 1], which gives a generic polygon without crossings for every consistent signed sequence with sum 2π . We give a direct construction (Figure 3).

Proof. For $x \in S$ let x^+ and x^- be the successor and the predecessor of x in the cyclic order of S , let $j_1, \dots, j_r$ ($r \geq 0$) be the indices strictly between x and x^+ , and put $\rho_x = \alpha_{j_1} + \dots + \alpha_{j_r}$ and $\theta_x = \pi - \alpha_x$. By (6), $\sum_{x \in S} (\theta_x + \rho_x) = 3\pi - \sum_{i \in S} \alpha_i + \sum_{i \notin S} \alpha_i = \pi$. Hence the numbers

$$\Theta_x = \theta_x + \frac{1}{2}(\rho_{x^-} + \rho_x) \quad (x \in S)$$

are positive with sum π , and $\rho_x < \pi$. Let T be a triangle with vertices p_x , $x \in S$, in counterclockwise order corresponding to the cyclic order of S , and with the interior angle Θ_x at p_x .

The chains. Fix $x \in S$, write $\rho = \rho_x$, and let g be the unit vector from p_x to p_{x^+} . For $t = 0, \dots, r$ let f_t be g rotated counterclockwise by $\frac{\rho}{2} - (\alpha_{j_1} + \dots + \alpha_{j_t})$; so $f_0, \dots, f_r$ turn clockwise from the angle $\frac{\rho}{2}$ to the angle $-\frac{\rho}{2}$ with g . If $r = 0$, let K_x be the segment from p_x to p_{x^+} . If $r \geq 1$, the vectors f_0 and f_r are linearly independent, and g and the vectors f_t with $0 < t < r$ lie in the open cone spanned by f_0 and f_r . So for small $\mu > 0$ the vector $p_{x^+} - p_x - \mu \sum_{0 < t < r} f_t$ lies in this cone, that is, it equals $\lambda_0 f_0 + \lambda_r f_r$ with $\lambda_0, \lambda_r > 0$. With $\lambda_t = \mu$ for $0 < t < r$, let K_x be the chain from p_x to p_{x^+} with the edge vectors $\lambda_0 f_0, \dots, \lambda_r f_r$. At its t -th inner vertex it turns right by α_{j_t} .

K_x is a simple arc in a triangle. Let D_x be the triangle with the base $[p_x, p_{x^+}]$, lying on the left of the base, with both base angles equal to $\frac{\rho}{2}$ (and $D_x = K_x$ if $r = 0$). The chain K_x is a simple arc because $\langle f_t, g \rangle > 0$ for all t . It lies in D_x , which is the intersection of three half-planes. First, every f_t is obtained from f_0 by a clockwise rotation by an angle in $[0, \rho]$, and $\rho < \pi$; so K_x , starting at p_x , stays on the right of or on the line through p_x with direction f_0 . Second, every f_t is obtained from f_r by a counterclockwise rotation by an angle in $[0, \rho]$; so K_x , which ends at p_{x^+} , stays on the right of or on the line through p_{x^+} with direction f_r . Third, the components of $f_0, \dots, f_r$ orthogonal to g are the sines of angles decreasing from $\frac{\rho}{2}$ to $-\frac{\rho}{2}$; so the height of K_x over the base first increases and then decreases, from 0 to 0, and K_x stays on the left of or on the base line.

The polygon. The interior of the counterclockwise triangle T is on the left of each side, and $\frac{1}{2}\rho_x < \Theta_x, \Theta_{x^+}$ because $\theta_x, \theta_{x^+} > 0$; hence $D_x \subseteq T$. Seen from the corner p_x , the points of D_x make an angle of at most $\frac{1}{2}\rho_x$ with the side towards p_{x^+} , and the points of D_{x^-} make an angle of at most $\frac{1}{2}\rho_{x^-}$ with the side towards p_{x^-} , hence an angle of at least $\Theta_x - \frac{1}{2}\rho_{x^-} = \theta_x + \frac{1}{2}\rho_x > \frac{1}{2}\rho_x$ with the side towards p_{x^+} . So $D_{x^-} \cap D_x = \{p_x\}$. Consequently the three chains meet only in the points p_x , and together they form a simple polygon P with n edges. At p_x the polygon turns from the last edge of K_{x^-} , whose direction is that of $p_x - p_{x^-}$ rotated clockwise by $\frac{1}{2}\rho_{x^-}$, to the first edge of K_x , whose direction is that of $p_{x^+} - p_x$ rotated counterclockwise by $\frac{1}{2}\rho_x$; since T has the exterior angle $\pi - \Theta_x$ at p_x , this is a left turn by $\frac{1}{2}\rho_{x^-} + (\pi - \Theta_x) + \frac{1}{2}\rho_x = \pi - \theta_x = \alpha_x$. Numbering the vertices of P so that $v_x = p_x$ for $x \in S$, the turning angle of P at v_i is α_i for every $i \in \mathbb{Z}_n$. $\square$

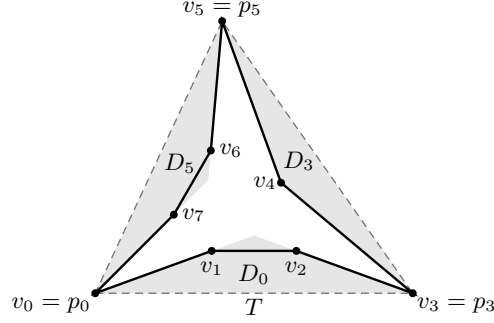


Figure 3: The construction of Lemma 5.2 for $n = 8$, $S = \{0, 3, 5\}$ and the angles $155^\circ, 20^\circ, 20^\circ, 160^\circ, 30^\circ, 155^\circ, 25^\circ, 15^\circ$. The triangle T (dashed) has the angles $65^\circ, 55^\circ, 60^\circ$; the three chains lie in the shaded triangles D_x , which meet only in the corners of T .

6 Proof of the characterization

Proof of Theorem 1.3. Part (c) is Proposition 4.1.

(a) If A is realizable, then $\sum_i \alpha_i \geq 2\pi$ by Lemma 2.2, and $e_S(A) \geq 0$ for all odd S by Lemma 2.3(a), applied to the edge directions of a realization. Conversely, let $\sum_i \alpha_i \geq 2\pi$ and $e_S(A) \geq 0$ for all odd S . If $\sum_i \alpha_i = 2\pi$, Lemma 5.1 gives a realization. If some odd set S is tight, then $|S| \neq 1$, because a tight set $\{i\}$ would give $\sum_j \alpha_j = 2\alpha_i < 2\pi$; so $|S| \geq 3$, and Theorem 1.2(a) gives a realization. In the remaining case $A \in G_n$, and Proposition 4.1 applies.

(b) Let A be realizable. If an odd set S with $|S| \geq 5$ is tight, then no realization is simple by Theorem 1.2(c). Conversely, assume that no odd set with at least five elements is tight. By (a), one of the following holds: $\sum_i \alpha_i = 2\pi$, and Lemma 5.1 gives a simple realization; or a set of three elements is tight, and Lemma 5.2 gives one; or $A \in G_n$, and Proposition 4.1 gives one. $\square$

Table 2 lists all combinations of the boundary conditions

(F) $\sum_i \alpha_i = 2\pi$; (E_k) some odd set with k elements is tight ($k = 1, 3$); (E _{≥ 5}) some odd set with at least five elements is tight

for a sequence $A \in (0, \pi)^n$ with $\sum_i \alpha_i \geq 2\pi$ and $e_S(A) \geq 0$ for all odd S . It shows that the three cases of the proof are exhaustive and that their overlaps are harmless. The entries follow from three facts. (E₁) is impossible, as shown in the proof of (a). Two different odd sets are never both tight (Lemma 2.4). If (F) holds and an odd set S with $|S| \geq 3$ is tight, then (6) gives $2\sum_{i \notin S} \alpha_i = 2\pi - (|S| - 1)\pi$, which is possible only for $|S| = 3$ and $S = \mathbb{Z}_n$, that is, for $n = 3$.

conditions that hold	occurs	simple realization	by
none ($A \in G_n$)	for $n \geq 4$	yes, not planar	Proposition 4.1
(F) only	for $n \geq 4$	yes, convex	Lemma 5.1
(E ₃) only	for $n \geq 4$	yes, a pseudo-triangle	Lemma 5.2
(F) and (E ₃)	for $n = 3$ only	yes, a triangle	either lemma
(E _{≥ 5}) only	for $n \geq 5$	no	Theorem 1.2(c)
(E ₁), alone or with others	never		Lemma 2.2
(F) and (E _{≥ 5})	never		(6)
two different tight sets	never		Lemma 2.4

Table 2: The combinations of boundary conditions for a realizable sequence.

Proof of Corollary 1.4. (a) For $n = 4$ there is no odd set with five elements, and Theorem 1.3(b) gives a simple realization of every realizable sequence. For $n = 5$ the only such set is $\mathbb{Z}_5$, and $e_{\mathbb{Z}_5}(A) = \sum_i (\pi - \alpha_i) - \pi$; so by Theorem 1.3(b) a realizable A has a simple realization if and only if $\sum_i (\pi - \alpha_i) \neq \pi$.

(b) Let $n \geq 6$ and let S be an odd set. The map which sends A to the point x with $x_i = \pi - \alpha_i$ for $i \in S$ and $x_i = \alpha_i$ for $i \notin S$ is an affine bijection of $(0, \pi)^n$ onto itself, and $e_S(A) = \sum_i x_i - \pi$. So Δ_S corresponds to the set of all x with $x_i > 0$ and $\sum_i x_i = \pi$, an open simplex of dimension $n - 1$. If $|S| \geq 3$, every point of Δ_S is realizable by Theorem 1.2(a). Now let A be realizable. By Theorem 1.3(b), A has no simple realization if and only if $A \in \Delta_S$ for an odd set S with $|S| \geq 5$, and this set is unique (Lemma 2.4). The right-hand side of the equivalence in Conjecture 1.1 is false if and only if n is odd and $A \in \Delta_{\mathbb{Z}_n}$. Hence the equivalence fails for A exactly if $A \in \Delta_S$ for an odd set S with $5 \leq |S| \leq n - 1$; in this case the right-hand side is true and there is no simple realization.

(c) This is Section 3.3: $A_n \in \Delta_{S_0}$ with $S_0 = \{0, \dots, 4\}$, and (8). $\square$

7 Remarks

Remark 7.1 (the formulation with thrackles). The sequence A^* contradicts the formulation with thrackles quoted in Section 1.2 as well: all its realizations have a self-intersection, but none of them is a thrackle. Indeed, let P be a polygon in the plane with $n \geq 4$ vertices in which any two non-adjacent edges cross. The edges e_{i-1} and e_{i+1} are non-adjacent and have a common point x . The vertices v_{i-1} and v_{i+2} do not lie on the line g through v_i and v_{i+1} , since the turning angles are not 0 or π ; hence e_{i-1} meets g only in v_i and e_{i+1} meets g only in v_{i+1} , the point x is not on g , and v_{i-1} and v_{i+2} lie on the same side of g as x . But v_{i-1} lies on the left of the directed line g exactly if P turns left at v_i , and v_{i+2} does so exactly if P turns left at v_{i+1} . So the turns at v_i and v_{i+1} have the same sign, for every i , and all turns of P have the same sign. Every realization of A^* is planar and has turns of both signs (Proposition 3.1). The same holds for every sequence in a set Δ_S with $5 \leq |S| \leq n - 1$, by Theorem 1.2(b).

Remark 7.2 (a linear-time test; generic sequences). By (2), $e_S(A) + \pi$ is a sum of n terms, the i -th being $\pi - \alpha_i$ or α_i according as $i \in S$ or not. So the minimum $m(A)$ of $e_S(A)$ over the odd sets is found as follows: let $S_1 = \{i : \alpha_i > \frac{\pi}{2}\}$; if $|S_1|$ is odd, S_1 is a minimizer; otherwise a minimizer is obtained from S_1 by changing the membership of one index for which $|\pi - 2\alpha_i|$ is smallest. By Theorem 1.3, A is realizable if and only if $\sum_i \alpha_i \geq 2\pi$ and $m(A) \geq 0$, and it then has a simple realization unless $m(A) = 0$ and the minimizer, which is unique in this case by Lemma 2.4, has at least five elements. This takes $O(n)$ arithmetic operations and depends only on the multiset of the angles, not on their order. In particular, for every A outside the finitely many hyperplanes $e_S = 0$, hence for generic A , the criterion of Conjecture 1.1 is right: such a sequence has a simple realization as soon as it is realizable.

Remark 7.3 (the number of crossings). A generic planar realization of A^* is not simple, so it has at least one proper crossing, and H_1 has exactly one. So the minimum number of crossings of generic realizations of A^* is one. More generally, if an odd set S is tight for A , all realizations are planar and realize, up to orientation, the signed sequence of Theorem 1.2(b), whose sum is $2\pi k$ with $k = (|S| - 1)/2$; by [EFKT22, Theorem 1] the minimum number of crossings of its generic realizations is then $k - 1 = (|S| - 3)/2$. We did not prove this theorem again, and it is not used in the proofs of this note. In all other cases a realizable sequence has a simple realization. That the question of [EFKT22, §5] for the minimum number of crossings reduces in this way to a test for realizations not contained in a plane is the remark of the source quoted in Section 1.5; supplied here are the test, in linear time (Remarks 7.2 and 7.4), and the proof of (3). Crossings are counted for generic planar polygons when all realizations are planar.

Remark 7.4 (realizations which are not planar). In [EFKT22, §5] it is asked how to test whether A “admits a fully 3D realization” (see Section 1.5). The answer is: *a realizable sequence A has a realization not contained in a plane if and only if $A \in G_n$* . The ‘if’ part is Theorem 1.3(c); for closed polygons on the 3-sphere, the appendix of [MP24] shows that a polygon not contained in a great circle exists when all the inequalities $e_S \geq 0$ are strict. For the ‘only if’ part, if an odd set is tight, all realizations are planar by Theorem 1.2(b). If $\sum_i \alpha_i = 2\pi$, all realizations are planar by the equality case of Fenchel’s theorem [Fen29, Sul08], which is also noted in [EFKT22, §3]; here is a direct argument. Let Γ, p, q be as in the proof of Lemma 2.2; now the two arcs of Γ from p to q have length π , so $d(p, x) + d(x, q) \leq \pi$ for all $x \in \Gamma$. If $q \neq -p$, then $d(x, q) \leq \pi - d(p, x) = d(x, -p)$ means $\langle x, q \rangle \geq -\langle x, p \rangle$, that is, $\langle x, m \rangle \geq 0$ for the midpoint m of the minor arc from p to q . If $q = -p$, let x be a point of one of the two arcs, which cuts it into pieces of lengths t and $\pi - t$ with $0 < t < \pi$; then $t \geq d(p, x)$ and $\pi - t \geq d(x, -p) = \pi - d(p, x)$, so $d(p, x) = t$, and by Lemma 2.1, applied to p , the points u_i in the interior of the piece from p to x , and x , this piece is the minor arc from p to x . As this holds for every x , both arcs are great semicircles from p to $-p$. They leave p in the directions of two unit vectors a, b orthogonal to p , and $\langle x, m \rangle \geq 0$ on Γ for $m = (a + b)/|a + b|$ if $a \neq -b$, and for a unit vector m orthogonal to p and a if $a = -b$. In both cases (1) forces $\langle u_i, m \rangle = 0$ for all i , so the u_i lie on a great circle.

Remark 7.5 (higher dimensions). The proofs of Lemmas 2.1 to 2.3 work on the unit sphere of $\mathbb{R}^d$ for every $d \geq 3$, and the constructions take place in $\mathbb{R}^3$. So Theorems 1.2 and 1.3 and Corollary 1.4 hold for polygons in $\mathbb{R}^d$, $d \geq 3$; compare the remark on higher dimensions in [EFKT22, §1].

Remark 7.6 (general position). The source gives no argument for (3). For fixed directions in general position a generic choice of the lengths suffices. Let $u_0, \dots, u_{n-1}$ be the unit edge directions of a polygon, let $L = \{\ell \in \mathbb{R}^n : \sum_k \ell_k u_k = 0\}$, and suppose that for any two non-adjacent indices i, j each of the sets $\{u_i, u_{i+1}, \dots, u_j\}$ and $\{u_j, u_{j+1}, \dots, u_i\}$ spans $\mathbb{R}^3$. Then the polygon with the directions u_k and the lengths ℓ_k is simple for all $\ell \in L$ with positive entries outside finitely many proper linear subspaces of L . Indeed, let i, j be non-adjacent, $W = \text{span}(u_i, u_j)$, and let I be the set of the indices strictly between i and j . If e_i and e_j have a common point, then $v_j - v_{i+1} = \sum_{k \in I} \ell_k u_k \in W$. If this held for all $\ell \in L$, then, as $\ell \mapsto \sum_k \ell_k u_k$ maps $\mathbb{R}^n$ onto $\mathbb{R}^3$ with kernel L , there would be a linear map $\lambda: \mathbb{R}^3 \rightarrow \mathbb{R}^3/W$ with $\lambda(u_k) = u_k + W$ for $k \in I$ and $\lambda(u_k) = 0$ for $k \notin I$; the second set of the hypothesis gives $\lambda = 0$, so that $u_i, \dots, u_j \in W$, against the hypothesis on the first set. Without the hypothesis the conclusion can fail, although the directions span $\mathbb{R}^3$: for $u_0 = (1, 0, 0)$, $u_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, $u_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ and $u_3, u_4 = (-\frac{3}{10}, \frac{3\sqrt{3}}{10}, \pm\frac{4}{5})$ the closing lengths are $(\ell + \frac{6}{5}t, \ell, \ell + \frac{6}{5}t, t, t)$ with $\ell, t > 0$, and in every such pentagon the edges e_0 and e_2 cross in the point $v_0 + \frac{6}{5}t u_0$; its angle sequence lies in G_5 . So in general the directions have to be changed as well; Proposition 4.1 constructs suitable directions together with the lengths.

8 Computations and reproducibility

No theorem, proposition or lemma of this note depends on a computation. The properties of H and H_1 stated in Section 3.2 can be checked from the printed coordinates; they were also verified by programs in exact arithmetic. All other computations are tests. The package which accompanies this note contains the programs with which the results were first obtained (folder `original/`, O below), the programs of three independent verification runs (`verification_run_A/`, `verification_run_B/` and `independent_run_2/`; A, B and R2 below), all recorded outputs, and the program `writing_stage/make_figures.py` written with this note. Each number below is followed by the program which produced it.

The hexagons. Closure, turning angles, total signed turning, genericity and the list of intersecting

pairs of non-adjacent edges of H and H_1 were verified in $\mathbb{Q}(\sqrt{3})$ by five programs with different implementations of the arithmetic: `O verify_counterexample.py`, `A a1`, `B b1`, `make_figures.py` and `R2 r2_01`. `A a6` drew 20,000 realizations of A^* with random rational lengths $\ell_1, \dots, \ell_4$, the other two lengths being solved exactly: none is simple; all are generic, with 1, 3 or 5 crossings (16,398, 3,284 and 318 times).

The criterion of Theorem 1.3(a). `B b2a` compared, in integer arithmetic, the condition “ $e_S(A) \geq 0$ for all odd S ” with an independent decision procedure for the existence of a closed spherical polygon (a recursion for the set of possible values of $d(u_0, u_k)$). For all 69,324,911 sequences with $\alpha_i = k_i\pi/m$ on nine grids with $3 \leq n \leq 9$, and for random samples of 6,000,000 and 3,000,000 such sequences with $(n, m) = (8, 12)$ and $(10, 12)$, the two agree; no sequence has two tight sets, and the rule of Remark 7.2 finds the minimum. In floating-point arithmetic, `A a12` (199,986 sequences) and `O test_against_paper_algorithm.py` (32,000 sequences, against the recursion of [EFKT22, Theorem 3]) found no disagreement.

Simple polygons in the open region. `O test_construction.py` ran the construction of Proposition 4.1 on 6,741 sequences of G_n , $4 \leq n \leq 11$, with 50 digits: the computed polygons have the prescribed angles up to 10^{-43} , and they are simple and not planar, which is decided in rational arithmetic on the computed vertices. `A`, with an implementation of its own and 80 to 140 digits, did the same for 36,518 sequences with $4 \leq n \leq 12$ (`a3`, `a5`, `a10`), among them sequences with some e_S as small as 10^{-16} and sequences close to two faces at once. `A a5` also classified, in rational arithmetic, all 34,061 sequences with entries in six sets of special angles, $4 \leq n \leq 8$, up to rotation and reflection: the equivalence of Conjecture 1.1 fails for none with $n \leq 5$, for A^* alone among those with $n = 6$, and for 340 of those with $n = 7, 8$. `B` searched without the construction (random spherical polygons, linear programming for the lengths, Newton’s method, exact tests on the rational vertices) and found a simple non-planar polygon whose angles agree with A up to 10^{-38} for each of 11,427 sequences of G_n , $4 \leq n \leq 10$ (`b3`, `b3_v2`); for one of them its first method found nothing, and a second one was needed (`b3x`).

The faces and the planar cases. On faces with $|S| \geq 5$, the total signed turning is $(|S| - 1)\pi$ and every sampled polygon has intersecting non-adjacent edges: `O test_bad_faces_exact.py` (240 sequences with rational directions, 9,600 polygons with random rational lengths), `A a9` (500 sequences, 12,500 polygons, each with at least $(|S| - 3)/2$ crossings) and `B b4`, `b6` (3,600 rational sequences and A_n for $n \leq 30$, with the rigidity confirmed by the exact recursion). The angle sequences of random planar polygons lie on a face in 1,708 of 24,000 cases (`O test_end_to_end.py`; 48 of them with $|S| = 5$) and in 1,722 of 24,000 cases (`B b7`; 42 with $|S| \geq 5$); in all of them S is the set of the turns of one sign and the turning number is $\pm(|S| - 1)/2$. The constructions of Lemmas 5.1 and 5.2 were carried out in rational arithmetic for 2,700 and 480 cases (`O test_planar_good_cases_exact.py`) and tested by `A a9` and `B b5`. The bounds of Lemma 4.4 were compared with the definition for 5,250 pairs (A, s) (`O test_lemma43.py`) and 11,372 pairs (`A a4`).

The programs of run 2, written from the statements of this note, repeat these tests. `R2 r2_03` carried out the construction of Proposition 4.1 as it is printed, with 100 digits, for 1,010 random sequences of G_n , $4 \leq n \leq 8$, and for the 30,684 classes of G_n among the 35,253 classes, up to rotation and reflection, of sequences with entries $k\pi/M$ on fifteen grids ($4 \leq n \leq 8$, $5 \leq M \leq 12$; two of the grids were sampled): every polygon is simple and not planar, which is decided exactly on its vertices, with angles correct up to 10^{-88} . The constructions of Lemmas 5.1 and 5.2 gave simple polygons for the 862 and 285 classes concerned, and the equivalence of Conjecture 1.1 fails for 20 classes, all with $n = 6$. `R2 r2_02` (1,200 points of faces with $|S| \geq 5$: every distance $d(u_i, u_j)$ is forced; 9,600 polygons with rational directions on such faces, each with a crossing), `R2 r2_04` and `r2_04b` (4,665,369 sequences and 219,560 points of the sets Δ_S , in integer arithmetic: Theorem 1.3(a), Table 2, Corollary 1.4(b)) and `R2 r2_05` (Lemma 4.4 on 7,000 pairs (A, s) ; Figures 1 and 3; the pentagons of Remark 7.6, in

$\mathbb{Q}(\sqrt{3})$) found no disagreement.

The script `run_checks.sh` of the package runs the programs of O, A and B again, one process at a time, and compares the results with the recorded outputs; the file `README.md` lists the commands, the running times and the required software. On 10 October 2026 the script was run on a copy of the package: all 86 recorded outputs were reproduced, identically or up to the fields which record running times. Its quick part (48 outputs) was run once more by run 2, from the archive, with the same result; `independent_run_2/run_run2.sh` does the same for the programs of run 2.

Verification. The results were first obtained, with proofs and with the programs in `original/`, in a version whose statements are numbered differently; a concordance is in the package. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. Runs A and B examined the first version, and neither saw the report of the other. Run 2, which knew both reports, examined the text of this note.

Run A compared the definitions and the conjecture with the texts of [Tó20, EFKT22] and examined whether another reading would rescue the conjecture; none does. It derived every proof again, line by line, including the signs in Lemma 2.3, the nine comparisons of Lemma 4.4 and the case distinction of Table 2, and made the tests marked A. It found no gap, and required six corrections of the text (the hypotheses of Lemma 2.1, two attributions, a step in the proof of Lemma 5.2, the example after Lemma 4.4, and Remarks 7.1 and 7.4).

Run B computed the hexagons again, confirmed by an exact computation which does not use Lemma 2.3 that the spherical polygons for A^* and for A_n , $n \leq 30$, are rigid, made the tests marked B, reproduced the 16 recorded outputs of the original programs byte for byte, and read these programs (one of them only in part). It found three defects in test programs, none of which had changed a result; they are corrected in the package. It found that Lemma 2.3 is known in both parts and that the first arXiv version prints another form of the conjecture (Table 1), listed the sources which it could not read, and required the corresponding corrections of the credits and of the wording about novelty.

When the note was written, these corrections were made, the proofs were put into the present form and numbering, and the statement of Corollary 1.4(b), Remark 3.2, the case $q = -p$ of Remark 7.4 and the figures were added.

Run 2 examined the text written in this way, and these additions first. It read the abstract [Tó20], the journal version of [EFKT22] and the sources of both arXiv versions completely and compared with them every statement quoted here; read the passages of [KM99, MP16, MP24] cited here; derived all proofs again, line by line; read the three corrected programs; computed the data printed in Section 3 and the coordinates of the figures again; made the tests marked R2; and repeated the search of the literature. It found no wrong statement and no gap in a proof. Its main finding concerns credit: the text which it examined did not mention that (3) is asserted in the source, and counted Theorem 1.3(c) among the statements not in print. It required the following corrections of the note, which are made in the present text: the credit for (3) (abstract, Section 1.5, Remarks 7.3 and 7.4, “Scope and priority”); the range of δ at the end of Remark 3.2; two steps of the case $q = -p$ in Remark 7.4; the words “ S odd” in Corollary 1.4(b); two attributions (before Lemmas 2.3 and 2.5); two labels in Figure 3. Remark 7.6 was added by this run; its example is verified in exact arithmetic by R2 `r2_05`. These corrections and Remark 7.6 have not been examined by a further independent run.

Scope and priority. The note refutes the ‘if’ part of Conjecture 1.1 for every $n \geq 6$, in each printed form (Remark 3.3), proves the conjecture for $n = 4$ and $n = 5$, and characterizes the angle sequences which have a simple realization (Theorem 1.3(b) and (c)). The setting, the conjecture, its ‘only if’ part [EFKT22, Theorem 5] and the realizability criterion (Theorem 1.3(a), which is [EFKT22, Theorem 2]

with the characterization of spherical polygons cited there) are due to Efrat, Fulek, Kobourov and Tóth. So are the assertion (3), which [EFKT22, §5] states without proof and for which a proof is given here, and the reduction of the question for the minimum number of crossings to a test for realizations not contained in a plane (Section 1.5). The inequalities $e_S \geq 0$ and their equality case (Lemma 2.3) are known in the literature on spherical polygonal linkages [Gal97, KM99, Bis98, BS02, MP16, MP24]; [EFKT22] itself cites [BS02] and [KM99]. Lemma 2.2 is Fenchel’s inequality, Lemma 2.5 is stated in [EFKT22, §1] for generic polygons, with a reference to [Gar98], and the existence statement of Lemma 5.2 follows from [EFKT22, Theorem 1].

What was read: of [Tó20], the abstract on pp. 1512–1514; of [EFKT22], the journal version and the sources of both arXiv versions, completely; of [KM99], Sections 1 to 3; of [MP16], Theorem 2.22 and Lemma 2.24 with their context, and of [MP24], the statements of the appendix by Mamaev, both in the arXiv versions (the journal versions were not compared). Not read: the proceedings chapter [EFKT20], to which the first arXiv version corresponds according to its arXiv comment (whether it prints ‘odd’ was not checked); the preprint [BS02], whose address given in [EFKT22] did not answer; [Gal97] and [Gar98], known to us only through [KM99] and [EFKT22]; [Bis98] beyond its review in zbMATH; and [Fen29, Hor71, Sul08, dBCvKO08], cited for classical facts (the attribution of the proof of Lemma 2.2 to [Hor71] was not checked against that paper). The proofs of the cited theorems were not checked; [EFKT22, Theorem 1] is used only in Remark 7.3 and in the comments of Section 1.5.

The searches were made on 10 October 2026, when the results were obtained and by the three verification runs: queries to arXiv (keywords and authors; titles, abstracts and comments), OpenAlex (titles, abstracts, full texts, citing works), Semantic Scholar and OpenCitations (citing works), zbMATH, Crossref (which registers no correction to [EFKT22] or [EFKT20]), DataCite and Zenodo, and eight web searches. We found no publication that disproves, proves or modifies Conjecture 1.1. The journal version of June 2022 states it as open, and there is no arXiv version after the second. Four works citing the paper were found (M. Schaefer, *J. Graph Algorithms Appl.* 25 (2021) 195–218; C. D. Tóth, arXiv:2311.15043; R. A. Moy, J. Schmurr and J. Varlack, arXiv:2306.01633; A. Baumann et al., arXiv:2607.01423); they were read at the place of the citation, and none discusses the conjecture. In [EFKT22] itself, tight sets other than $\mathbb{Z}_n$ do not occur. Limits of the search: MathSciNet and Google Scholar were not available, DBLP did not answer automated requests, and the works citing [KM99, MP16] were not examined. A search that finds nothing is not a proof of novelty; the authors of the conjecture are active in this field and may know more than is in print. No priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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