

A Proof of Paták's $kb + 1$ Conjecture for Constrained Stars, with Improved Bounds for Complete Graphs

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Abstract

In an Oberwolfach report from 2020, Paták considered closure operators on topological spaces whose closures have at most b path-connected components. He noted that $\binom{b+1}{2}(k-1) + b + 1$ points suffice for a constrained drawing of the star $K_{1,k}$, conjectured that $kb + 1$ points suffice, and asked two further questions: whether the bounds for complete graphs K_n can be improved, and whether the method extends to higher homology or homotopy. In the journal version (J. Graph Theory, 2025) the conjecture is stated for b -iatlon graphs and proved there for $b \leq 2$. We prove the conjecture for all k and b . Every b -iatlon graph with $kb + 1$ vertices contains a constrained copy of $K_{1,k}$. For every closure operator as above, every set of $kb + 1$ points admits a constrained drawing of $K_{1,k}$. The bound $kb + 1$ is sharp in both settings. The proof rests on a colouring lemma for families of graphs indexed by the subsets of a finite set and growing with the subset; it extends the bound $\chi \leq \Delta + 1$. As a consequence, $1 + b + \dots + b^{n-1}$ points force a constrained copy of K_n , improving the previous bound $O(b^{2n-3})$. Combined with Paták's topological arguments, this lowers his bounds on Radon and Helly numbers in $\mathbb{R}^d$ from $O(b^{2d+3})$ to $O(b^{d+2})$, and his bound on Radon numbers on a fixed closed surface from $O(b^6)$ to $O(b^3)$. In the b -iatlon setting, Ramsey numbers give lower bounds, which show that the exponent $n - 1$ is optimal for $n = 3, 4$. Exact values for complete graphs remain open, and the question on higher homology and homotopy is not addressed. This is an unrefereed note.

1 Introduction

A *closure operator* on a set X is a map $\text{cl} : 2^X \rightarrow 2^X$ such that $A \subseteq \text{cl } A$, $\text{cl}(\text{cl } A) = \text{cl } A$, and $\text{cl } A \subseteq \text{cl } B$ whenever $A \subseteq B$. Convex hull in $\mathbb{R}^d$ is the basic example, and every family $\mathcal{F}$ of subsets of X gives the closure operator $\text{cl}_{\mathcal{F}}(A) = \bigcap \{F \in \mathcal{F} : A \subseteq F\}$ (which is X if no member contains A). Radon and Helly numbers of such operators on topological spaces have been bounded under topological assumptions by Matoušek [Mat97], Goaoc, Paták, Patáková, Tancer and Wagner [GPP⁺17], Patáková [Pat24] and Paták [Pat25]; see the survey [PP26]. These proofs construct constrained drawings of graphs, or more generally constrained maps of simplicial complexes.

Following Paták [Pat20], let cl be a closure operator on a topological space X , let $S \subseteq X$, and let G be a graph. A *constrained drawing* of G in S places the vertices of G at distinct points of S (we identify each vertex with its point) and assigns to every edge e a set $S_e \subseteq S$ and a path that joins the two endpoints of e inside $\text{cl}(S_e \cup e)$, such that

- (i) $S_e \cap V(G) = \emptyset$ for every edge e , and
- (ii) $S_e \cap S_f = \emptyset$ for any two disjoint edges e and f .

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If the paths of two disjoint edges e, f meet, then $S_e \cup e$ and $S_f \cup f$ are disjoint subsets of S whose closures intersect. Non-embeddability theorems force such crossings, and this bounds the Radon number.

In the report [Pat20], Paták assumes that every closure has at most b path-connected components. He notes that $\binom{b+1}{2}(k-1) + b + 1$ points of S suffice for a constrained drawing of the star $K_{1,k}$, and that constrained drawings of complete graphs can be built inductively from stars. He then asks three questions:

- (1) “How many points are actually needed to find $K_{1,k}$? (Conjecture: $kb + 1$)”;
- (2) whether better bounds can be obtained for complete graphs K_n ;
- (3) whether the approach can be adapted to non-trivial homology or homotopy in higher dimensions.

The three questions together form record OWR-1703876-006 of the UnsolvedMath dataset [ula26].

In the journal version [Pat25], Paták formulates the combinatorial core for b -iatlon graphs (Definition 3.1). Numbered results of [Pat25] are cited as they are numbered in arXiv:1909.08489v4, since we did not see the typeset journal version. Let $p(G, b)$ be the least number of vertices that forces a constrained copy of G in every b -iatlon graph. He proves that $p(K_n, b) \leq b \sum_{j=0}^{n-2} \binom{b+1}{2}^j + 1 = O(b^{2n-3})$ [Pat25, Theorem 1], and uses this bound to prove polynomial bounds on Radon and Helly numbers. He also shows $p(K_{1,k}, b) \geq kb + 1$ [Pat25, Lemma 1] and $p(K_{1,k}, 2) = 2k + 1$ [Pat25, Lemma 2]. He reports that computer experiments confirmed further small cases, and he states the following conjecture [Pat25, Conjecture 1].

Conjecture 1.1 (Paták). *For all $n \geq 0$ and $b \geq 1$, every b -iatlon graph with at least $bn + 1$ vertices contains a constrained copy of $K_{1,n}$.*

Results

Theorem 1.2. *Let $k \geq 0$ and $b \geq 1$ be integers.*

- (a) *Every b -iatlon graph with at least $kb + 1$ vertices contains a constrained copy of $K_{1,k}$. Hence $p(K_{1,k}, b) = kb + 1$, and Conjecture 1.1 holds.*
- (b) *Let cl be a closure operator on a topological space X such that every set $Y \subseteq X$ of $b + 1$ points contains two points in the same path-connected component of $\text{cl}Y$. (This holds, for example, if $\text{cl}Y$ has at most b path-connected components.) Then every set $S \subseteq X$ with $|S| \geq kb + 1$ admits a constrained drawing of $K_{1,k}$.*
- (c) *For all $d \geq 1$ and $k \geq 1$ there is a closure operator on $\mathbb{R}^d$ whose closures all have at most b path-connected components, and a set of kb points that admits no constrained drawing of $K_{1,k}$.*

So the answer to question (1) is exactly $kb + 1$, in every dimension. The main tool is a colouring lemma, Theorem 2.2, for families of graphs $(G_T)_{T \subseteq V}$ that grow with T . When G_T is the induced subgraph $G[T]$ of a fixed graph G , it reduces to the bound $\chi(G) \leq \Delta(G) + 1$.

For question (2) we obtain the following. Here $R(n, m)$ is the Ramsey number, the least N such that every graph on N vertices contains n pairwise adjacent or m pairwise non-adjacent vertices.

Theorem 1.3. *For all integers $n \geq 1$ and $b \geq 1$,*

$$R(n, b + 1) \leq p(K_n, b) \leq 1 + b + b^2 + \cdots + b^{n-1}.$$

Under the hypothesis of Theorem 1.2(b), every set of at least $1 + b + \cdots + b^{n-1}$ points admits a constrained drawing of K_n .

The upper bound improves $O(b^{2n-3})$ to $O(b^{n-1})$. By known bounds on Ramsey numbers, the exponent $n - 1$ is optimal for $n = 3$ and $n = 4$ (Corollary 4.5). Moreover $p(K_3, 2) = 6 = R(3, 3)$, where the upper bound comes from an exhaustive computer search (Example 4.4). In particular, the lower bound $p(K_n, b) \geq (n - 1)b + 1$ that follows from stars is not tight.

Combined with Paták's topological arguments, Theorem 1.3 and a bound for disjoint copies of $K_{3,3}$ improve the Radon and Helly bounds of [Pat25] as follows (Section 5).

Corollary 1.4. (a) *Under the hypotheses of [Pat25, Theorem 2], a closure operator on $\mathbb{R}^d$ has Radon number at most $\sum_{j=0}^{d+2} b^j$. Under the hypotheses of [Pat25, Theorem 3], a family of sets in $\mathbb{R}^d$ has Helly number at most $\sum_{j=1}^{d+2} b^j$.*

(b) *Under the hypotheses of [Pat25, Theorem 4], a closure operator on a closed surface of genus g (orientable or not) has Radon number at most $3b^3 + b^2 + b + 1 + g(9b - 3)$.*

The previous bounds were $O(b^{2d+3})$ and, for a fixed surface, $O(b^6)$ [Pat25].

Scope. Question (1) is answered completely, both in the closure setting of the report and in the b -iatlon setting of [Pat25]. For question (2) we give improved bounds; the exact values of $p(K_n, b)$ for $n \geq 3$ remain open, and in the closure setting we know no lower bound beyond $(n - 1)b + 1$. Question (3) is not addressed.

Section 2 proves the colouring lemma. Section 3 proves Theorem 1.2, Section 4 treats complete graphs and disjoint unions, and Section 5 derives the Radon and Helly bounds.

2 A colouring lemma

Definition 2.1. Let V be a finite set. A *monotone graph system* on V is a family $(G_T)_{T \subseteq V}$ of graphs such that $V(G_T) = T$ for every T , and $E(G_T) \subseteq E(G_{T'})$ whenever $T \subseteq T' \subseteq V$. A set $I \subseteq V$ is *independent* if G_I has no edge. By monotonicity, subsets of independent sets are independent; α denotes the largest size of an independent set. For $k \geq 1$, a *good k -star* is a pair (c, L) with $c \in V$, $L \subseteq V \setminus \{c\}$ and $|L| = k$, such that

$$cx \in E(G_{V \setminus (L \setminus \{x\})}) \quad \text{for every } x \in L.$$

In words, c is joined to each leaf x in the graph of the set that is obtained from V by removing the other $k - 1$ leaves.

Theorem 2.2. *Let $k \geq 1$, and let $(G_T)_{T \subseteq V}$ be a monotone graph system on a finite set V without a good k -star. Then V is the union of k pairwise disjoint independent sets, some of which may be empty. In particular, $|V| \leq k\alpha$.*

Since subsets of independent sets are independent, every cover of V by k independent sets can be refined to k pairwise disjoint ones, so the theorem could equally be stated for covers. The proof uses the disjoint form in the induction step.

Proof. We use induction on $|V|$. If $V = \emptyset$, take k empty sets. Let $V \neq \emptyset$, fix $p \in V$ and put $V' = V \setminus \{p\}$. For $T \subseteq V'$ let G'_T be the subgraph of $G_{T \cup \{p\}}$ induced by T . So in (G'_T) every set is considered together with p ; we call it the *link* of p .

The link is a monotone graph system on V' : if $T \subseteq T' \subseteq V'$, then $E(G_{T \cup \{p\}}) \subseteq E(G_{T' \cup \{p\}})$, and hence $E(G'_T) \subseteq E(G'_{T'})$.

The link has no good k -star. Indeed, let $c \in V'$, $L \subseteq V' \setminus \{c\}$ with $|L| = k$, and $x \in L$. Since $p \notin L$, we have $(V' \setminus (L \setminus \{x\})) \cup \{p\} = V \setminus (L \setminus \{x\})$. So $G'_{V' \setminus (L \setminus \{x\})}$ is an induced subgraph of $G_{V \setminus (L \setminus \{x\})}$

that contains c and x . Hence, if (c, L) were a good k -star of the link, it would be a good k -star of (G_T) .

By the induction hypothesis, applied to the link, V' is the union of pairwise disjoint sets $B_1, \dots, B_k$ that are independent in the link. This means that $G_{B_i \cup \{p\}}$ has no edge inside B_i . So every edge of $G_{B_i \cup \{p\}}$ joins p to a vertex of B_i .

Suppose first that for some i the graph $G_{B_i \cup \{p\}}$ has no edge. Then $B_i \cup \{p\}$ is independent. For $j \neq i$, every edge of G_{B_j} is an edge of $G_{B_j \cup \{p\}}$ inside B_j , by monotonicity, and there is no such edge; so B_j is independent. Replacing B_i by $B_i \cup \{p\}$ gives k pairwise disjoint independent sets with union V , since $p \notin V'$.

Otherwise, for every i choose a neighbour $b_i \in B_i$ of p in $G_{B_i \cup \{p\}}$, and let $L = \{b_1, \dots, b_k\}$. The b_i are distinct because the B_i are disjoint, and $p \notin L$. For each i , the set $L \setminus \{b_i\}$ is contained in $\bigcup_{j \neq i} B_j$, so $B_i \cup \{p\} \subseteq V \setminus (L \setminus \{b_i\})$. By monotonicity, $pb_i \in E(G_{V \setminus (L \setminus \{b_i\})})$. So (p, L) is a good k -star, which is excluded. Hence the first case occurs, and the induction is complete. Since each of the k sets has at most α elements, $|V| \leq k\alpha$. $\square$

Remark 2.3. Let G be a graph on V and $G_T = G[T]$. Then $cx \in E(G_{V \setminus (L \setminus \{x\})})$ if and only if $cx \in E(G)$, so a good k -star is a vertex together with k of its neighbours. Thus Theorem 2.2 says that $\Delta(G) < k$ implies $\chi(G) \leq k$, that is, $\chi(G) \leq \Delta(G) + 1$. In this case the link of p is $(G[T])_{T \subseteq V'}$ again, and the proof is the usual greedy argument. In general, adding p to a set T may create edges inside T . Therefore a colouring of $V \setminus \{p\}$ that is independent in the system $(G_T)_{T \subseteq V \setminus \{p\}}$ need not stay independent when p is added to one of its classes. Passing to the link, where p is present in every set, avoids this problem. The proof is constructive: it finds either k independent sets or a good k -star.

3 Constrained stars

3.1 b -iatlon graphs

Definition 3.1 (Paták [Pat25, Definitions 3, 4 and 7]). A *labelled graph* is a graph H in which every edge e carries a family L_e of subsets of $V(H) \setminus e$, called *labels*. Let $b \geq 1$.

- (a) H is *b -iatlon* if every set $Y \subseteq V(H)$ of $b + 1$ vertices contains the ends of an edge uv with $Y \setminus \{u, v\} \in L_{uv}$.
- (b) A *constrained copy* of a graph G in H is a subgraph G_0 of H isomorphic to G , together with a label $l(e) \in L_e$ for every $e \in E(G_0)$, such that $l(e) \cap V(G_0) = \emptyset$ for every e , and $l(e) \cap l(f) = \emptyset$ for any two disjoint edges e, f of G_0 .
- (c) $p(G, b)$ is the least N such that every b -iatlon graph on N vertices contains a constrained copy of G .

Paták shows that $p(G, b)$ is finite for every graph G [Pat25]. We use three simple facts.

Lemma 3.2. *Let H be a b -iatlon graph and let G be a graph.*

- (a) *For $X \subseteq V(H)$, let $H[X]$ be the induced subgraph on X in which every edge keeps only its labels contained in X [Pat25, Definition 6]. Then $H[X]$ is b -iatlon, and every constrained copy of G in $H[X]$ is a constrained copy of G in H . In particular, if $|V(H)| \geq p(G, b)$, then H contains a constrained copy of G .*
- (b) *If G' is a subgraph of G , then $p(G', b) \leq p(G, b)$ [Pat25, Proposition 1(7)].*

- (c) For every set Y of $b+1$ vertices choose one edge uv as in Definition 3.1(a). Keep only the labels $Y \setminus \{u, v\}$ obtained in this way, and only the edges that receive a label. The result is a b -iatlon graph in which every label has $b-1$ elements, and each of its constrained copies is a constrained copy in H .

Proof. (a) If $Y \subseteq X$ and $|Y| = b+1$, the label $Y \setminus \{u, v\}$ given by Definition 3.1(a) is contained in X . The edges and labels of $H[X]$ are edges and labels of H . For the last claim, apply Definition 3.1(c) to $H[X]$ with $|X| = p(G, b)$. (b) A constrained copy of G contains a constrained copy of G' with the same labels. (c) This is clear from the definitions. $\square$

By Lemma 3.2(c), for upper bounds on $p(G, b)$ it suffices to consider b -iatlon graphs given by a choice of one pair in every $(b+1)$ -set of vertices.

Theorem 3.3. *Let $k \geq 0$ and $b \geq 1$. Every b -iatlon graph H with $|V(H)| \geq kb + 1$ contains a constrained copy of $K_{1,k}$.*

Proof. For $k = 0$ a single vertex suffices, so let $k \geq 1$ and $V = V(H)$. For $T \subseteq V$ let G_T be the graph on T whose edges are the edges uv of H with $u, v \in T$ that have a label $l \in L_{uv}$ with $l \subseteq T$. This is a monotone graph system. Every set Y of $b+1$ vertices is dependent: by Definition 3.1(a) it contains an edge uv with the label $Y \setminus \{u, v\} \subseteq Y$, so $uv \in E(G_Y)$. Hence $\alpha \leq b$, and $|V| \geq kb + 1 > k\alpha$. By Theorem 2.2 there is a good k -star (c, L) .

For every $x \in L$ we have $cx \in E(G_{V \setminus (L \setminus \{x\})})$. So cx is an edge of H with a label $l_x \in L_{cx}$ such that $l_x \subseteq V \setminus (L \setminus \{x\})$. Labels of cx avoid c and x , so $l_x \cap (L \cup \{c\}) = \emptyset$. Therefore the star with centre c , leaves L and labels l_x is a constrained copy of $K_{1,k}$. Any two edges of a star share the centre, so the condition on disjoint edges is void. $\square$

Lemma 3.4 (Paták [Pat25, Lemma 1]). *Let $k, b \geq 1$. The disjoint union of b complete graphs K_k , in which every edge e carries all subsets of $V \setminus e$ as labels, is a b -iatlon graph on kb vertices without a constrained copy of $K_{1,k}$.*

Proof. Any $b+1$ vertices include two in the same K_k . A copy of $K_{1,k}$ is connected and has $k+1$ vertices, so it would lie in one K_k . $\square$

Proof of Theorem 1.2(a). Combine Theorem 3.3 with Lemma 3.4; for $k = 0$ we have $p(K_1, b) = 1$. $\square$

Remark 3.5. Let $r(G, K_m)$ be the least N such that every graph on N vertices contains a copy of G or m pairwise non-adjacent vertices. If every edge carries all subsets of $V \setminus e$ as labels, then constrained copies are just copies, and b -iatlon means that there are no $b+1$ pairwise non-adjacent vertices. Hence $p(G, b) \geq r(G, K_{b+1})$ for every G (Proposition 4.3). By Chvátal's theorem [Chv77], $r(K_{1,k}, K_{b+1}) = kb + 1$, and Lemma 3.4 is the extremal example. So Theorem 1.2(a) says that $p(K_{1,k}, b) = r(K_{1,k}, K_{b+1})$: labels do not help to avoid stars. For graphs with all labels, Theorem 3.3 reduces to the classical argument: a graph without $K_{1,k}$ has $\Delta < k$, hence $\chi \leq k$ and at most $k\alpha$ vertices.

3.2 Closure operators

Let cl be a closure operator on a topological space X and $b \geq 1$, and assume

- every $Y \subseteq X$ with $|Y| = b+1$ contains two points in the same path-connected component of $\text{cl } Y$. (1)

Since $Y \subseteq \text{cl } Y$, (1) holds if every such $\text{cl } Y$ has at most b path-connected components. For a finite set $P \subseteq X$, Paták [Pat25, proofs of Lemma 4 and Theorem 4] defines a labelled graph H_P on P as

follows: for every $Y \subseteq P$ with $|Y| = b + 1$, choose two points $u, v \in Y$ in the same path-connected component of $\text{cl } Y$, make uv an edge, and add $Y \setminus \{u, v\}$ to L_{uv} . Then H_P is b -iatlon, and each of its labels has $b - 1$ elements.

Lemma 3.6. *If H_P contains a constrained copy of a graph G , then P admits a constrained drawing of G in which $|S_e| = b - 1$ for every edge e .*

Proof. Let $e = uv$ be an edge of the copy, with label $l(e)$. Then $l(e) = Y \setminus \{u, v\}$ for a set Y for which u and v were chosen, so u and v are joined by a path in $\text{cl } Y = \text{cl}(l(e) \cup e)$. Put $S_e = l(e)$ and draw e along this path. Conditions (i) and (ii) of a constrained drawing are the conditions on the labels in Definition 3.1(b). $\square$

Proof of Theorem 1.2(b). Let $|S| \geq kb + 1$; the set S may be infinite. Choose $P \subseteq S$ with $|P| = kb + 1$. By Theorem 3.3, H_P contains a constrained copy of $K_{1,k}$, and Lemma 3.6 gives a constrained drawing of $K_{1,k}$ in $P \subseteq S$. $\square$

One can also apply Theorem 2.2 directly to the graphs G_T on P in which two points are adjacent if they lie in the same path-connected component of $\text{cl } T$. For a good k -star (c, L) this gives a constrained drawing with $S_e = P \setminus (L \cup \{c\})$ for every edge.

Proposition 3.7. *Let $d, b \geq 1$. Let $R_1 = \{x \in \mathbb{R}^d : x_1 < 1\}$, $R_i = \{x \in \mathbb{R}^d : i - 1 \leq x_1 < i\}$ for $1 < i < b$, and $R_b = \{x \in \mathbb{R}^d : x_1 \geq b - 1\}$ (for $b = 1$ let $R_1 = \mathbb{R}^d$). Then*

$$\text{cl}(Y) = \bigcup_{i=1}^b \text{conv}(Y \cap R_i)$$

is a closure operator on $\mathbb{R}^d$, and $\text{cl}(Y)$ has at most b path-connected components for every $Y \subseteq \mathbb{R}^d$. For every $k \geq 1$, a set S with exactly k points in each R_i admits no constrained drawing of $K_{1,k}$.

Proof. The sets R_i are convex, pairwise disjoint, and cover $\mathbb{R}^d$. Clearly $Y \subseteq \text{cl } Y$, and cl is monotone. Since $\text{conv}(Y \cap R_i) \subseteq R_i$, we have $\text{cl}(Y) \cap R_i = \text{conv}(Y \cap R_i)$, and therefore $\text{cl}(\text{cl } Y) = \text{cl } Y$. Each $\text{cl } Y$ is a union of at most b convex sets, so it has at most b path-connected components.

Suppose that S has a constrained drawing of $K_{1,k}$ with centre $c \in R_i$. Let x be a leaf and $A = S_{cx} \cup \{c, x\}$. The path from c to x lies in $\text{cl } A$, the union of the sets $\text{conv}(A \cap R_j)$. These sets are compact, because A is finite, and pairwise disjoint. Hence each of them is open and closed in $\text{cl } A$, and the path stays in $\text{conv}(A \cap R_i) \subseteq R_i$. So $x \in R_i$. Then c and its k leaves are $k + 1$ points of S in R_i , a contradiction. $\square$

Proposition 3.7 gives Theorem 1.2(c). It is the closure-operator version of Lemma 3.4. The same argument shows that b slabs with $n - 1$ points each carry no constrained drawing of K_n , since K_n is connected.

4 Complete graphs and disjoint unions

For a graph G , let $K_1 + G$ be the graph obtained from G by adding one vertex adjacent to all vertices of G . Paták builds complete graphs from stars by the following argument [Pat25, Proposition 1(6)], and he observed that Conjecture 1.1 would give the inequality below [Pat25, Section 5]. We include the short proof.

Lemma 4.1. *For every graph G and every $b \geq 1$, $p(K_1 + G, b) \leq bp(G, b) + 1$.*

Proof. Let $N = p(G, b)$, and let H be a b -iatlon graph with at least $bN + 1$ vertices. By Theorem 3.3, H contains a constrained copy of $K_{1,N}$, with centre v , leaf set X and labels l_x ($x \in X$), where $l_x \cap (X \cup \{v\}) = \emptyset$. By Lemma 3.2(a), $H[X]$ is a b -iatlon graph on N vertices, so it contains a constrained copy G_1 of G . Its labels are contained in $X \setminus V(G_1)$.

Let F consist of G_1 together with v and the edges vy , $y \in V(G_1)$, where vy gets the label l_y . Then F is isomorphic to $K_1 + G$, and all labels are disjoint from $V(F) \subseteq X \cup \{v\}$. Two edges vy and vy' are not disjoint. An edge vy and an edge e of G_1 have disjoint labels, since $l_y \cap X = \emptyset$ and $l(e) \subseteq X$. Disjoint edges of G_1 have disjoint labels. So F is a constrained copy of $K_1 + G$. $\square$

Corollary 4.2. *For all $n, m, b \geq 1$,*

$$p(K_n, b) \leq \sum_{j=0}^{n-1} b^j \quad \text{and} \quad p(K_{m,n}, b) \leq nb^m + \sum_{j=0}^{m-1} b^j.$$

In particular, $p(K_3, b) \leq b^2 + b + 1$ and $p(K_{3,3}, b) \leq 3b^3 + b^2 + b + 1$.

Proof. We have $p(K_1, b) = 1$ and $K_{n+1} = K_1 + K_n$, so the first bound follows from Lemma 4.1 by induction. Let $K_{0,n}$ be the graph with n vertices and no edges, so $p(K_{0,n}, b) = n$. Since $K_{m,n}$ is a subgraph of $K_1 + K_{m-1,n}$, Lemma 3.2(b) and Lemma 4.1 give $p(K_{m,n}, b) \leq bp(K_{m-1,n}, b) + 1$, and the second bound follows by induction on m . $\square$

For comparison, [Pat25, Theorem 1] gives $p(K_3, b) \leq b^{\binom{b+1}{2}} + b + 1$, which is 9 for $b = 2$, while Corollary 4.2 gives 7. For $K_{3,3}$ and $b = 3$, the bound used in [Pat25, Proposition 2] is 562, and Corollary 4.2 gives 94.

Proof of the upper bounds in Theorem 1.3. The bound on $p(K_n, b)$ is Corollary 4.2. For the second statement, let $|S| \geq \sum_{j < n} b^j$ and choose a finite $P \subseteq S$ of this size. By Lemma 3.2(a) and Corollary 4.2, H_P contains a constrained copy of K_n , and Lemma 3.6 gives the drawing. $\square$

Proposition 4.3. *For every graph G and every $b \geq 1$, $p(G, b) \geq r(G, K_{b+1})$, with $r(G, K_{b+1})$ as in Remark 3.5. In particular $p(K_n, b) \geq R(n, b+1) \geq (n-1)b + 1$.*

Proof. Let F be a graph on $r(G, K_{b+1}) - 1$ vertices that contains no copy of G and no $b+1$ pairwise non-adjacent vertices. Let every edge e of F carry all subsets of $V(F) \setminus e$ as labels. Every set of $b+1$ vertices contains an edge, which has the required label, so F is b -iatlon. It has no constrained copy of G , since it has no copy of G . The last inequality holds because b disjoint copies of K_{n-1} contain no K_n and no $b+1$ pairwise non-adjacent vertices. $\square$

Example 4.4. Let H be the 5-cycle on $\mathbb{Z}_5$ with edges $\{i, i+1\}$ and labels $L_{\{i, i+1\}} = \{\{i+2\}, \{i+3\}\}$. Every 3-subset of $\mathbb{Z}_5$ has the form $\{i, i+1, i+2\}$ or $\{i, i+1, i+3\}$, and in both cases the edge $\{i, i+1\}$ carries the remaining point as a label. So H is 2-iatlon. It has no triangle, hence no constrained copy of K_3 , and $p(K_3, 2) \geq 6$. (The 5-cycle is the classical example showing $R(3, 3) > 5$; see [GG55] for $R(3, 3) = 6$.) Conversely, by Lemma 3.2(c) it suffices to check the 3^{20} ways of choosing one pair in each 3-subset of a 6-set. An exhaustive computer search, described in the Verification paragraph, finds a constrained K_3 in each case. Hence $p(K_3, 2) = 6 = R(3, 3)$, whereas the star bound $(n-1)b + 1$ gives 5 and Corollary 4.2 gives 7.

Corollary 4.5. *As $b \rightarrow \infty$, $p(K_3, b) = \Omega(b^2 / \log b)$ and $p(K_4, b) = \Omega(b^3 / \log^4 b)$. Together with Corollary 4.2, $\log p(K_n, b) / \log b \rightarrow n - 1$ for $n \in \{3, 4\}$. For every fixed $n \geq 5$, $p(K_n, b) = \Omega((b / \log b)^{(n+1)/2})$.*

Proof. Apply Proposition 4.3 with the lower bounds $R(3, t) = \Omega(t^2/\log t)$ of Kim [Kim95], $R(4, t) = \Omega(t^3/\log^4 t)$ of Mattheus and Verstraete [MV24], and $R(n, t) = \Omega((t/\log t)^{(n+1)/2})$ for fixed n of Spencer [Spe77], with $t = b + 1$. $\square$

The next lemma is Pataák's argument for [Pat25, Proposition 1(9)], in the form we need. The bound depends on the number of labels of a copy, which is at most its number of edges.

Lemma 4.6. *Let G be a graph with v vertices and e edges, and let $t \geq 1$. Then*

$$p(t \cdot G, b) \leq p(G, b) + (t - 1)(v + (b - 1)e),$$

where $t \cdot G$ denotes the disjoint union of t copies of G .

Proof. Let H be a b -iatlon graph with at least $p(G, b) + (t - 1)(v + (b - 1)e)$ vertices. By Lemma 3.2(c) we may assume that all labels of H have $b - 1$ elements; restrictions $H[X]$ keep this property. We use induction on t ; the case $t = 1$ is Lemma 3.2(a). Let $t \geq 2$. By Lemma 3.2(a), H contains a constrained copy G_0 of G . Let Z be the union of $V(G_0)$ and the labels of G_0 , so that $|Z| \leq v + (b - 1)e$. The graph $H[V(H) \setminus Z]$ is b -iatlon and has at least $p(G, b) + (t - 2)(v + (b - 1)e)$ vertices. By induction it contains a constrained copy G_1 of $(t - 1) \cdot G$, whose labels lie in $V(H) \setminus Z$. Then $G_0 \cup G_1$ is a copy of $t \cdot G$. The labels of G_1 avoid Z , which contains $V(G_0)$ and the labels of G_0 . The labels of G_0 lie in Z , so they avoid $V(G_1)$. Hence $G_0 \cup G_1$ is a constrained copy of $t \cdot G$. $\square$

For $G = K_{3,3}$ we have $v + (b - 1)e = 9b - 3$, so Corollary 4.2 gives

$$p((g + 1) \cdot K_{3,3}, b) \leq 3b^3 + b^2 + b + 1 + g(9b - 3) \quad (g \geq 0). \quad (2)$$

For disjoint edges the bound of Lemma 4.6 is sharp, and it can be larger than the Ramsey lower bound of Proposition 4.3. Let $\nu(G)$ be the matching number of G , the largest number of pairwise disjoint edges.

Proposition 4.7. *For every graph G and every $b \geq 1$, $p(G, b) \geq |V(G)| + (b - 1)\nu(G)$. In particular, for all $t, b \geq 1$,*

$$p(t \cdot K_2, b) = t(b + 1) \quad \text{and} \quad r(t \cdot K_2, K_{b+1}) = 2t + b - 1.$$

Proof. Let $N < |V(G)| + (b - 1)\nu(G)$. On an N -set, choose one pair uv in every $(b + 1)$ -set Y and give it the label $Y \setminus \{u, v\}$; for $N \leq b$ this is the empty graph. The result is a b -iatlon graph in which every label has $b - 1$ elements. Suppose that it contains a constrained copy G_0 of G . The edges of a maximum matching of G_0 are pairwise disjoint, so their $\nu(G)$ labels are pairwise disjoint, and all labels avoid $V(G_0)$. Hence $N \geq |V(G)| + (b - 1)\nu(G)$, a contradiction. This proves the lower bound.

For $t \cdot K_2$ the lower bound is $t(b + 1)$. In particular $p(K_2, b) \geq b + 1$, and Corollary 4.2 gives $p(K_2, b) \leq b + 1$. Now Lemma 4.6 with $v = 2$ and $e = 1$ gives $p(t \cdot K_2, b) \leq (b + 1) + (t - 1)(b + 1) = t(b + 1)$. For the Ramsey number, the complete graph K_{2t-1} together with $b - 1$ isolated vertices has $2t + b - 2$ vertices, no t pairwise disjoint edges and no $b + 1$ pairwise non-adjacent vertices. Conversely, let a graph on at least $2t + b - 1$ vertices have no t pairwise disjoint edges. A maximum matching has at most $t - 1$ edges, and the at least $b + 1$ vertices that it misses are pairwise non-adjacent, since otherwise the matching could be enlarged. $\square$

So $p(t \cdot K_2, b) - r(t \cdot K_2, K_{b+1}) = (t - 1)(b - 1)$, which is positive for $t, b \geq 2$. For example, $p(2 \cdot K_2, 2) = 6$ and $r(2 \cdot K_2, K_3) = 5$. Similarly, let $K_2 \cup K_1$ be an edge together with an isolated vertex. Every graph on 3 vertices contains $K_2 \cup K_1$ or 3 pairwise non-adjacent vertices, and K_2 contains neither, so $r(K_2 \cup K_1, K_3) = 3$, whereas Proposition 4.7 gives $p(K_2 \cup K_1, 2) \geq 4$.

5 Radon and Helly numbers

The *Radon number* $r(\text{cl})$ of a closure operator on X is the least r such that every $S \subseteq X$ with $|S| \geq r$ contains two disjoint sets $A, B \subseteq S$ with $\text{cl } A \cap \text{cl } B \neq \emptyset$ [Pat25, Definition 2]. The *Helly number* $h(\mathcal{F})$ of a family $\mathcal{F}$ of sets is the least h such that every finite subfamily of $\mathcal{F}$ with empty intersection contains h (not necessarily distinct) members whose intersection is empty [Pat25, Definition 5]. By Levi [Lev51], $h(\mathcal{F}) + 1 \leq r(\text{cl}_{\mathcal{F}})$.

Paták's proofs of [Pat25, Theorems 2 and 4] have the same structure. Given $S \subseteq X$ with $|S| \geq N$, choose $P \subseteq S$ with $|P| = N$ and form the b -iatlon graph H_P of Section 3. A constrained copy of a suitable graph in H_P gives, by topological arguments, two disjoint subsets of P with intersecting closures. The number N is used only to guarantee this constrained copy. So the proofs remain valid for every N that forces the copy in every b -iatlon graph with labels of $b - 1$ elements. We now check this in the two cases.

Theorem 5.1. *Let $d, b \geq 1$, and let cl be a closure operator on $\mathbb{R}^d$ such that*

- (i) *$\text{cl } X$ has at most b path-connected components for every $X \subseteq \mathbb{R}^d$ with $|X| = b + 1$, and*
- (ii) *$H_i(\text{cl } Y; \mathbb{Z}_2) = 0$ for every integer i with $1 \leq i \leq \lceil d/2 \rceil - 1$ and every $Y \subseteq \mathbb{R}^d$ with $b + 1 + i \leq |Y| \leq i + 2 + (b - 1) \binom{i+2}{2}$.*

Then $r(\text{cl}) \leq p(K_{d+3}, b) \leq \sum_{j=0}^{d+2} b^j$.

Proof. These are the hypotheses of [Pat25, Theorem 2]. Paták derives that theorem from [Pat25, Lemmas 5 and 6] and [GPP⁺17, Corollary 14], applied to the $\lceil d/2 \rceil$ -skeleton K of the $(d + 2)$ -dimensional simplex, which has $n = d + 3$ vertices. Lemma 6 of [Pat25] constructs a non-trivial chain map from the simplicial chains of K to the singular chains of $\mathbb{R}^d$ that is constrained by P . By [GPP⁺17, Corollary 14], two disjoint faces of K have intersecting supports, and Lemma 5 of [Pat25] turns them into two disjoint subsets of P with intersecting closures.

In the proof of Lemma 6 (through the case of dimension one in the proof of Lemma 4 of [Pat25]), the assumption $|P| \geq b \sum_{j=0}^{n-2} \binom{b+1}{2}^j + 1$ is used once: to obtain, by [Pat25, Theorem 6(1)], a constrained copy of K_n in H_P . The rest of the proof uses only this copy, hypothesis (i), which makes H_P b -iatlon, and the fact that the labels of H_P have $b - 1$ elements. For a face σ with at least two vertices, let $\Phi(\sigma)$ be the union of σ and the labels of its edges; hypothesis (ii) is applied to these sets, and the last fact gives $|\sigma| + b - 1 \leq |\Phi(\sigma)| \leq |\sigma| + (b - 1) \binom{|\sigma|}{2}$. If $|P| \geq p(K_n, b)$, then H_P contains a constrained copy of K_n by Lemma 3.2(a), again with labels of $b - 1$ elements, and the rest of the proof applies unchanged. Hence $r(\text{cl}) \leq p(K_{d+3}, b)$, and Corollary 4.2 gives the second inequality. $\square$

Corollary 5.2. *Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ such that, for every finite $\mathcal{H} \subseteq \mathcal{F}$, the intersection $\bigcap \mathcal{H}$ has at most b path-connected components and satisfies $H_i(\bigcap \mathcal{H}; \mathbb{Z}_2) = 0$ for $1 \leq i \leq \lceil d/2 \rceil - 1$. Then $h(\mathcal{F}) \leq \sum_{j=1}^{d+2} b^j$.*

Proof. These are the hypotheses of [Pat25, Theorem 3], which Paták derives from [Pat25, Theorem 2] and Levi's inequality. The Helly number is determined by finite subfamilies. For a finite $\mathcal{G} \subseteq \mathcal{F}$, every set $\text{cl}_{\mathcal{G}}(Y)$ is $\mathbb{R}^d$ or the intersection of a finite subfamily, so $\text{cl}_{\mathcal{G}}$ satisfies the hypotheses of Theorem 5.1. Hence $h(\mathcal{G}) \leq r(\text{cl}_{\mathcal{G}}) - 1 \leq \sum_{j=0}^{d+2} b^j - 1$. $\square$

Theorem 5.3. *Let M be a closed surface, and let g be its genus if M is orientable and its non-orientable genus otherwise. Let cl be a closure operator on M , and let $b \geq 1$ be such that $\text{cl } X$ has at most b path-connected components for every $X \subseteq M$ with $|X| = b + 1$. Then*

$$r(\text{cl}) \leq p((g + 1) \cdot K_{3,3}, b) \leq 3b^3 + b^2 + b + 1 + g(9b - 3).$$

For a closure operator on $\mathbb{R}^2$ with the same property, $r(\text{cl}) \leq p(K_{3,3}, b) \leq 3b^3 + b^2 + b + 1$.

Proof. These are the hypotheses of [Pat25, Theorem 4]. In Paták’s proof, if H_P contains a constrained copy C_0 of $(g + 1) \cdot K_{3,3}$, each edge e of C_0 is drawn along a path in $\text{cl}(e \cup l(e))$. By the result of Schaefer and Štefankovič [SŠ13], as used in [Pat25], two disjoint edges e, f of C_0 have intersecting drawings. Then $e \cup l(e)$ and $f \cup l(f)$ are disjoint subsets of P whose closures intersect. The number of points enters only through the existence of C_0 , which Paták obtains from [Pat25, Theorem 6(4)]. By Lemma 3.2(a) it suffices that $|P| \geq p((g + 1) \cdot K_{3,3}, b)$, and (2) gives the bound.

For $\mathbb{R}^2$, Paták states and proves [Pat25, Proposition 2] only for $b = 3$, with the bound 562. His proof takes a constrained copy of $K_{3,3}$ in the graph H_P , which exists by [Pat25, Theorem 6(3)], draws it along the chosen paths, and applies the Hanani–Tutte theorem [Cho34, Tut70] for $K_{3,3}$. This argument works verbatim for every b , and the number of points enters only through the copy of $K_{3,3}$. So $|P| \geq p(K_{3,3}, b)$ suffices, and Corollary 4.2 gives the bound. $\square$

Remark 5.4. In [Pat25, Section 5], Paták lists the bounds that would follow from Conjecture 1.1. For $\mathbb{R}^d$, Theorem 5.1 gives the bound stated there. For surfaces, our term linear in g is different. The reason is that the bound for disjoint copies depends on the number of distinct labels in a copy (Lemma 4.6). The stars of [Pat25, Proposition 1(5)] use a single label, whereas the stars of Theorem 3.3 may use a different label on every edge; so we allow nine labels for each copy of $K_{3,3}$. For a fixed surface the bound is still $O(b^3)$. The same substitution can be made in the proof of [Pat25, Theorem 5] on manifolds of even dimension; we do not pursue this.

6 Concluding remarks

Theorem 1.2 settles question (1) of [Pat20] and Conjecture 1.1. For complete graphs, Theorem 1.3 leaves a gap between $R(n, b + 1)$ and $\sum_{j < n} b^j$. By Remark 3.5 and Example 4.4, $p(G, b) = r(G, K_{b+1})$ holds for all stars and for $(G, b) = (K_3, 2)$; it also holds for $b = 1$, where both sides equal $|V(G)|$ [Pat25, Proposition 1(3)]. It fails for some disconnected graphs: by Proposition 4.7 and the remarks after it, $p(K_2 \cup K_1, 2) \geq 4 > 3 = r(K_2 \cup K_1, K_3)$, and $p(t \cdot K_2, b) = t(b + 1) > 2t + b - 1 = r(t \cdot K_2, K_{b+1})$ for all $t, b \geq 2$. For a connected graph G on n vertices, the disjoint union of b copies of K_{n-1} shows that $r(G, K_{b+1}) \geq (n - 1)b + 1$, and this is at least the bound $n + (b - 1)\nu(G)$ of Proposition 4.7, since $\nu(G) \leq n - 1$. Computations of the second verification run (item 5 of the Verification paragraph below) give $p(G, 2) = r(G, K_3)$ for the connected graphs P_4 , C_4 , the paw (a triangle with a pendant edge) and $K_4 - e$, where both values are 7, and for K_3 , where both are 6. These are SAT computations without certificates; the lower bounds $p(G, 2) \geq r(G, K_3)$ are Proposition 4.3.

Problem 6.1. *Is $p(G, b) = r(G, K_{b+1})$ for every connected graph G and every $b \geq 1$? In particular, is $p(K_n, b) = R(n, b + 1)$?*

Problem 6.2. *In the closure setting of Theorem 1.2(b), how many points force a constrained drawing of K_n ? Theorem 1.3 and the slab example give the bounds $(n - 1)b + 1$ and $\sum_{j < n} b^j$. The lower bound of Proposition 4.3 uses b -iatlon graphs that need not come from a closure operator.*

Question (3) of [Pat20], on non-trivial homology or homotopy in higher dimensions, is not addressed here.

Verification. The proofs above are complete and do not depend on computation, except the upper bound $p(K_3, 2) \leq 6$ in Example 4.4. The values for connected graphs quoted before Problem 6.1 are computations and are used in no proof. The following checks were made.

1. A program by the author, `check_theorem.py` (standard library only, under a minute), does the following.

- It checks Theorem 2.2 by brute force for every k : on all 46,656 monotone graph systems on 4 points, and on 20,000, 3,000 and 300 random systems on 5, 6 and 7 points. A system on n points is given by an up-closed family of subsets of $V \setminus \{u, v\}$ for every pair uv .
 - It verifies Example 4.4. An exhaustive backtracking search shows that each of the 3^{20} choices of one pair in every 3-subset of a 6-set yields a constrained K_3 . On 5 points the same search finds exactly the 2,124 choices without one that a direct enumeration finds.
 - It runs the proof of Theorem 2.2 as an algorithm on random b -iatlon graphs with $kb + 1$ vertices, for (k, b) up to $(5, 2)$, $(3, 3)$ and $(2, 5)$. Each returned star is checked against Definition 3.1.
 - It checks the arithmetic of the bounds in Sections 4 and 5.
2. A second program by the author, `sat_checks.py`, uses the SAT solver CaDiCaL through PySAT. It finds no counterexample to Theorem 2.2 among all monotone graph systems with at most 8 points. Working directly from Definition 3.1, it confirms $p(K_{1,k}, b) = kb + 1$ for $(k, b) \in \{(2, 2), (3, 2), (2, 3), (4, 2), (2, 4), (3, 3)\}$ and $p(K_3, 2) = 6$.
 3. The first computations were made before the proof was found. SAT encodings of the closure setting and of Definition 3.1 reproduced the threshold $kb + 1$ for the same six cases, and a counterexample-guided search confirmed the colouring statement for monotone systems of partitions with 7 or 8 points and $k \leq 4$, and with 9 points and $k \leq 5$. Small examples showed that a naive induction, which deletes a point instead of passing to its link, fails (Remark 2.3). These scripts also run the proof as an algorithm on all systems with 4 points and on 12,000 random systems. They are included in the release.
 4. A first independent verification run, which was AI-assisted, was made before this note was written. It compared the statements with the Oberwolfach report and arXiv version 4 of [Pat25]. It checked the proofs of Theorem 2.2, of the reduction to b -iatlon graphs (Theorem 3.3), of the reduction for closure operators (in the direct form given after the proof of Theorem 1.2(b)), of the slab example (Proposition 3.7), and of Lemma 4.1 with the bound for K_n in Corollary 4.2. It re-ran scripts of item 3 and reproduced their recorded outputs; it did not run the programs of items 1 and 2, which were written later. With its own code it checked Theorem 2.2 by brute force on all systems with 4 points and on random systems with 5 and 6 points, and by a SAT search for counterexamples among all monotone graph systems with at most 8 points, and with 9 points for $2 \leq k \leq 4$. It also confirmed, by a SAT encoding of Definition 3.1, the thresholds kb and $kb + 1$ in the six cases of item 2, and $p(K_3, 2) = 6$. It did not check Proposition 4.3, Corollary 4.5, Lemma 4.6 or Section 5.
 5. A second independent verification run, also AI-assisted, checked a draft of this note. It compared the statements with the Oberwolfach report and arXiv version 4 of [Pat25]. It checked the proofs of Sections 2 to 4 step by step. For Section 5 it checked in [Pat25] where the number of points enters Paták's proofs, but it did not re-verify the topological arguments. With its own code, written before it read the programs of items 1–3, it checked Theorem 2.2 by brute force on all systems with 4 points and on random systems with 5, 6 and 7 points, and by a SAT search for counterexamples among all monotone graph systems with at most 9 points, for every k . It confirmed $p(K_{1,k}, b) = kb + 1$ by a SAT encoding of Definition 3.1 for thirteen pairs (k, b) , up to $(3, 3)$, $(5, 2)$ and $(2, 5)$, and showed that 13 vertices suffice for $(k, b) = (6, 2)$. It confirmed $p(K_3, 2) = 6$ by an exhaustive enumeration in C and by SAT, and checked Example 4.4, Lemma 3.4 and the arithmetic of Sections 4 and 5. It re-ran the programs of items 1 and 2 and several scripts of item 3, and reproduced their recorded outputs; it also ran the

counterexample-guided search of item 3 for the remaining cases with at most 8 points. It found that Problem 6.1, stated in the draft for all graphs, fails for some disconnected graphs, and it made the computations for connected graphs quoted before Problem 6.1. The corrections it asked for are made in this version; Proposition 4.7, which is new here, follows the argument given in its report. Its programs and outputs are included in the release.

Scope and priority. Theorem 1.2 answers question (1) of [Pat20] completely, both in the closure setting of the report and in the b -iatlon setting of [Pat25], where it proves Conjecture 1.1. For question (2) we give the upper bound of Theorem 1.3 and the lower bounds of Proposition 4.3; exact values are open. Question (3) is not addressed. Section 5 uses the topological parts of [Pat25, GPP⁺17, SŠ13] as published; we checked where the number of points enters Paták’s proofs, but we did not re-verify the topological arguments. We read the report [Pat20] and arXiv version 4 of [Pat25]; we did not see the typeset journal version. In September 2026 we searched arXiv, Crossref, OpenAlex, zbMATH Open and Semantic Scholar, and made two web searches; the second verification run made similar searches independently and found nothing further. Semantic Scholar lists four records citing [Pat25]. The first is the survey [PP26], which still quotes the bound $O(b^{2d+3})$ for Helly numbers [PP26, Theorem 30]. The second is the paper of Avvakumov, Bin and Goaoc [ABG26], which partially answers Paták’s Conjecture 2 on chain maps into manifolds. The other two are the preprint and the journal version of Pálvölgyi’s paper [Pál22], which cites the arXiv version in a list of related work. None of them treats Conjecture 1.1. OpenAlex listed no citing work. A zbMATH Open query for documents citing the journal version returned no results, although the same query form does return citing documents for an older paper; since zbMATH’s citation data are incomplete, this is weak evidence. We found no earlier proof of Conjecture 1.1 for $b \geq 3$. The argument is short, and an expert may know it; this negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the proofs, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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