

High-Density Canonical Hard Disks Percolate

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Abstract

Using Georgii’s equivalence of ensembles and Magazinov’s high-activity theorem, we prove high-density percolation for uniform canonical hard disks. For every $\varepsilon > 0$, every weak limit of uniform square-torus packings sufficiently close to the maximum number density $1/(2\sqrt{3})$ has an infinite component when centers at distance at most $2 + \varepsilon$ are joined. The parameter conversion uses the explicit bound

$$\rho \leq \frac{1}{2\sqrt{3}} - \frac{1}{9}e^{-4\pi\lambda}$$

for stationary Gibbs packings of activity λ . We give the ensemble argument for arbitrary square sizes and treat the maximum-density endpoint separately. The conclusion addresses the canonical percolation question of Bowen, Lyons, Radin and Winkler; it is not a crystallization theorem. Separated triangular patches show why the uniform sampling hypothesis cannot be replaced by high density alone.

1 The canonical question

A unit-disk packing is a locally finite set $X \subset \mathbb{R}^2$ with $|x - y| \geq 2$ for distinct centers. Its connection graph $G_\varepsilon(X)$ joins centers whose distance is at most $2 + \varepsilon$. Write

$$\rho_{\max} = \frac{1}{2\sqrt{3}}$$

for the maximum number density. The occupied area fraction is $\pi\rho$; the activity of a Gibbs law will be denoted by λ , not by ρ .

Let $Q_L = [-L/2, L/2)^2$, with opposite sides identified, and define

$$a_L(m) = \frac{1}{m!} \int_{Q_L^m} \mathbf{1}_{\{\text{periodic hard core}\}} dx_1 \cdots dx_m. \quad (1)$$

When $a_L(m) > 0$, let $C_{L,m}$ be the normalized uniform law on these unordered configurations, periodically continued to the plane. This law is translation invariant. Weak convergence below is the usual weak convergence of laws of locally finite point measures.

Bowen, Lyons, Radin and Winkler [3, Section 7] ask whether such limits percolate at every sufficiently high fixed submaximal density. Their finite-volume measure is defined using configuration-space volume [3, pp. 1078–1079]. At a fixed density below $\rho_{\max}$, strict packings with small independently movable centers exist for all sufficiently large square sizes, so the configuration space has full dimension and their measure agrees with (1). The density question is also discussed in Magazinov’s Oberwolfach report [6, pp. 1097–1099].

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Theorem 1. *For every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ with the following property. Suppose $L_j \rightarrow \infty$, $a_{L_j}(m_j) > 0$, and*

$$\liminf_{j \rightarrow \infty} \frac{m_j}{L_j^2} > \rho_{\max} - \delta(\varepsilon).$$

Every weak limit of C_{L_j, m_j} assigns probability one to the event that G_ε has an infinite connected component. If $\lambda_0(\varepsilon) > 0$ is a threshold in Magazinov's high-activity theorem, one may take

$$\delta(\varepsilon) = \frac{1}{9} e^{-4\pi\lambda_0(\varepsilon)}. \quad (2)$$

The main percolation input is Magazinov's theorem [5, Theorem 7.4 in the arXiv version] for *every* hard-disk Gibbs measure above a sufficiently large activity. Aristoff [1] previously obtained the result with a restriction on connection distance. Our task is to transfer that input to fixed density using Georgii's ensemble theorem [4]. No uniqueness of the Gibbs measure or absence of phase coexistence is assumed. Formula (2) does not assert a numerical value of λ_0 or an optimal density threshold.

2 A uniform density gap at bounded activity

A stationary packing law P has intensity $\rho(P)$ when $\mathbb{E}_P N_X(A) = \rho(P)|A|$ for bounded Borel sets A . A hard-disk Gibbs law of activity λ satisfies the usual Dobrushin–Lanford–Ruelle (DLR) equations: conditional on the exterior of a bounded region, its interior is a Poisson process of activity λ conditioned to obey the hard core.

Lemma 2. *Every stationary hard-disk Gibbs law of finite activity λ satisfies*

$$\rho(P) \leq \rho_{\max} - \frac{1}{9} e^{-4\pi\lambda}. \quad (3)$$

Proof. In a disk B of radius two, the conditional partition function, relative to unnormalized Poisson configuration measure, is bounded above by

$$\sum_{k \geq 0} \frac{\lambda^k |B|^k}{k!} = e^{4\pi\lambda}.$$

The empty configuration is always compatible with the exterior and has weight one. The DLR equations therefore give

$$P\{N_X(B) = 0\} \geq e^{-4\pi\lambda}. \quad (4)$$

Stationarity makes the intensity measure a multiple of Lebesgue measure, so a fixed boundary circle contains no centers almost surely.

Independently choose U uniformly in $[0, 3)^2$. For each $y \in U + 3\mathbb{Z}^2$ with no center of X within distance two, add y to X . The resulting set Y is a packing: new centers are separated by at least three and are compatible with every old center. Its law is stationary, and integration over U gives

$$\rho(Y) = \rho(P) + \frac{1}{9} P\{N_X(B_2(0)) = 0\}. \quad (5)$$

For clarity, the geometric bound $\rho(Y) \leq \rho_{\max}$ needs no ergodicity assumption. Restrict any packing to a half-open square of side R and repeat that finite set with period $R + 2$ in each coordinate. This is a periodic packing, so the planar disk-packing bound gives

$$N_X(Q_R) \leq \rho_{\max}(R + 2)^2.$$

Taking expectations, dividing by R^2 and letting $R \rightarrow \infty$ proves the bound for every stationary law. Apply it to Y in (5) and use (4). $\square$

In particular, a stationary Gibbs law with $\rho > \rho_{\max} - e^{-4\pi\lambda_0}/9$ cannot have activity $\lambda \leq \lambda_0$. This implication does not require monotonicity of the density in the activity.

3 Canonical limits and ensemble equivalence

We record the measure-level consequence of Georgii's results that is needed here, including the passage from his odd-integer box sizes to arbitrary side lengths.

Proposition 3. *Suppose $L_j \rightarrow \infty$ and $m_j/L_j^2 \rightarrow \rho$ with $0 < \rho < \rho_{\max}$. Every weak limit of C_{L_j, m_j} is a stationary hard-disk Gibbs measure at one finite positive activity.*

Proof. Use the pair potential

$$\phi(x) = \begin{cases} +\infty, & |x| < 2, \\ 0, & |x| \geq 2. \end{cases}$$

It satisfies the regularity and hard-core assumptions (A1)–(A2) of [4]. Attach a momentum $p \in \mathbb{R}^2$ to each center and use kinetic energy $|p|^2$. We use Georgii's specific entropy S , mean energy U , and thermodynamic entropy $s(\rho, e)$, with unordered Lebesgue configuration measure including its $1/m!$ factor.

The interior thermodynamic domain. For every $\rho < \rho_{\max}$, start with a triangular lattice of spacing strictly greater than two and density greater than ρ . Apply a uniform random translation, independent small absolutely continuous jitters in disjoint neighborhoods, and independent thinning to intensity ρ . The jitters can be chosen small enough to preserve all hard-core distances. The resulting positional law has finite specific entropy: on each lattice cell its occupation is Bernoulli and its conditional position has a bounded density in a fixed region. Random translation cannot increase the specific relative entropy. Independent Gaussian momenta of arbitrarily small variance then show that Georgii's $\varepsilon_{\min}(\rho) = 0$. The packing bound and this construction identify Georgii's maximal thermodynamic density with $\rho_{\max}$. Thus (ρ, ρ) lies in the interior domain Σ of [4, Theorem 3.2].

The positional entropy limit. A microcanonical measure on the torus with m particles and total energy at most m has positional marginal exactly $C_{L, m}$. Indeed, the volume of its momentum ball in $\mathbb{R}^{2m}$ is

$$b_m = \frac{\pi^m m^m}{m!},$$

independently of the allowed positions. Theorem 3.2 of [4] and Stirling's formula imply, for his side lengths $L = 2n + 1$,

$$\lim_{\substack{L \rightarrow \infty \\ m/L^2 \rightarrow \rho}} \frac{\log a_L(m)}{L^2} = h(\rho) := s(\rho, \rho) - \rho(1 + \log \pi). \quad (6)$$

Here the microcanonical shell width is infinite, which is among the allowed choices.

The same limit holds for arbitrary L . At fixed m , $a_L(m)$ is nondecreasing in L : embed the smaller centered square into the larger one. For a coordinate difference t of two points in the smaller square, the corresponding torus distance is $\min\{|t|, L - |t|\}$, which does not decrease as L grows. Hence every allowed configuration in the smaller torus remains allowed in the larger one. Bracket L by neighboring odd integers L_-, L_+ . Then

$$\log a_{L_-}(m) \leq \log a_L(m) \leq \log a_{L_+}(m),$$

and $m/L_{\pm}^2 \rightarrow \rho$, proving (6). Positivity of the smaller partition function holds eventually by the cited theorem.

Independent blocks. In each cell of a square grid of side $T = L + 3$, put an independent sample from $C_{L, m}$ in the central square of side L . The intervening empty corridors have width three. Mark all centers independently with density

$$q(p) = \pi^{-1} e^{-|p|^2},$$

and translate the entire cell grid by an independent uniform shift. Call the resulting stationary marked packing law P_L . It obeys the hard core across cell boundaries as well as within cells, and

$$\rho(P_L) = U(P_L) = \frac{m}{T^2}. \quad (7)$$

The independent-block entropy estimate underlying [4, Proposition 6.1] gives

$$S(P_L) \geq \frac{\log a_L(m) + m(1 + \log \pi)}{T^2}. \quad (8)$$

Here is its normalization explicitly. Relative to the marked Poisson law of positional intensity π and mark density q , the relative entropy of one full cell is

$$I_{\text{cell}} = \pi T^2 - \log a_L(m) - m \log \pi.$$

Tensorization over cells, restriction to observation boxes, and convexity under spatial averaging give $I(P_L) \leq I_{\text{cell}}/T^2$. The identity $S(P_L) = -I(P_L) + U(P_L) + \pi$ yields (8). Boundary cells contribute only a boundary-order number of cell terms as the observation box grows. This argument does not require matching configurations on adjacent cell boundaries.

Identification of every limit. Equations (6)–(8) give a uniform lower bound on $S(P_L)$ and an upper bound on $U(P_L)$. By [4, Proposition 3.1], a subsequence converges in his local topology $\tau_{\mathcal{L}}$, whose test functions are local and grow at most linearly in the number of particles and their momentum norms. Denote a limit by P .

The positional local limits of P_L and $C_{L,m}$ agree. For any fixed observation window, the probability, over the uniform grid shift, that it meets a corridor or cell boundary is $O(L^{-1})$. On the complement its positional law is the corresponding restriction of the translation-invariant torus law. Hard-core exclusion bounds the number of centers in the window uniformly. The independent Gaussian marks also give uniform control of local momentum tails. The marking law is preserved under weak convergence, as follows directly from its local Laplace functionals. Consequently P has intensity ρ , independent marks of density q , and

$$U(P) = \rho.$$

In particular, no loss of kinetic energy is hidden in the local limit; the Gaussian squared-momentum tails are uniformly integrable.

Upper semicontinuity of S in $\tau_{\mathcal{L}}$ now gives

$$S(P) \geq \limsup S(P_L) \geq s(\rho, \rho).$$

The reverse inequality is the variational formula in [4, Theorem 3.2]. Thus P is a microcanonical entropy maximizer at (ρ, ρ) .

A supporting plane to s at this interior point supplies fixed parameters $z, \beta > 0$. By [4, equation (3.12) and Theorem 3.4], P is a grand canonical Gibbs measure for these parameters. Integrating out momenta gives a positional hard-disk Gibbs law of activity

$$\lambda = z \int_{\mathbb{R}^2} e^{-\beta|p|^2} dp = \frac{z\pi}{\beta} \in (0, \infty).$$

This is one activity for the entire law, even if that law is not ergodic. Finally, beginning with any prescribed weakly convergent subsequence of the canonical laws, the stronger compactness argument above produces a subsubsequence with the same positional limit. The conclusion therefore holds for every weak limit. $\square$

4 Percolation and the close-packing endpoint

Proof of Theorem 1. First suppose $m_j/L_j^2 \rightarrow \rho < \rho_{\max}$. Proposition 3 identifies any limiting law as Gibbs at a finite activity λ . Lemma 2 and (2) imply $\lambda > \lambda_0(\varepsilon)$. A Gibbs law is locally absolutely continuous and therefore almost surely has no pair at distance exactly two. It belongs to the strict-separation space used in [5]. Magazinov's Theorem 7.4 now gives almost-sure percolation of G_ε . The infinite-component event is used only after the Gibbs identification, not passed directly through weak convergence.

For the endpoint, let P be any stationary packing law of intensity $\rho_{\max}$. Average its rotations over $SO(2)$. The resulting law $\bar{P}$ is rigid-motion invariant and maximizes the occupied area fraction. Theorem 1 and its proof in Bowen, Holton, Radin and Sadun [2] show that an ergodic maximizing invariant disk-packing law is supported on the triangular packing orbit. Their statement extends to all maximizing invariant laws by ergodic decomposition: the density of each component is at most $\rho_{\max}$, and its mean is $\rho_{\max}$. Thus $\bar{P}$ is supported on that orbit. Membership in the orbit is rotation invariant, so P itself is supported there. Every such packing percolates in G_ε for $\varepsilon > 0$.

It remains to check the stated liminf condition. For a nonnegative continuous compactly supported function f with $\int f > 0$, hard-core exclusion bounds $\sum_{x \in X} f(x)$ uniformly. Translation invariance yields

$$\mathbb{E}_{C_{L_j, m_j}} \sum_{x \in X} f(x) = \frac{m_j}{L_j^2} \int_{\mathbb{R}^2} f.$$

Along a weakly convergent subsequence these expectations converge. The intensities therefore converge to the intensity ρ of the limit, with $\rho_{\max} - \delta(\varepsilon) < \rho \leq \rho_{\max}$. The submaximal argument or the endpoint argument applies. $\square$

Remark 4. The maximum-density argument applies to any stationary packing law. It does not apply ensemble equivalence at a singular boundary point of the thermodynamic domain. It also covers limits of uniform lower-dimensional configuration-space laws if their intensities converge to $\rho_{\max}$.

5 High density without canonical sampling

For completeness, high density by itself does not imply percolation, even for stationary ergodic laws.

Proposition 5. *For every $\varepsilon > 0$ there are stationary ergodic unit-disk packing laws with intensities tending to $\rho_{\max}$ for which every component of G_ε is finite.*

Proof. Choose an integer $b \geq 2 + \varepsilon$. For integers $N > b$ with $N \geq 2/\varepsilon$, put $s = 1 + 1/N$ and take the finite patch

$$X_N^0 = \{s(2i + (j \bmod 2), \sqrt{3}j) : 0 \leq i < N - b, \quad 0 \leq j < 2N - b\}.$$

Repeat it with horizontal period $2sN$ and vertical period $2s\sqrt{3}N$. Each patch lies in a triangular lattice with nearest distance $2s > 2$. Between horizontally separated cells, the horizontal gap between patches is at least $s(2b + 1) > 2 + \varepsilon$. For vertically separated cells the vertical gap is at least $s\sqrt{3}(b + 1) > 2 + \varepsilon$. Thus no edge of G_ε joins different patches, and every component is finite. The packing's intensity is

$$\frac{(N - b)(2N - b)}{4\sqrt{3}s^2N^2} \longrightarrow \rho_{\max}.$$

Uniformly translate it over its rectangular period cell. The resulting law is stationary and ergodic under the full translation group, which acts transitively on its periodic orbit. $\square$

These laws are not the uniform canonical laws of Theorem 1. Thus a formulation that omits the sampling distribution is genuinely different. Neither the canonical percolation conclusion nor the maximum-density equality case proves crystallization at a fixed submaximal density or symmetry breaking.

This research note was developed with AI assistance and checked by the author. It is unrefereed. The principal ensemble-equivalence, high-activity percolation and optimal-packing inputs are the cited published theorems; no absolute priority claim is made for their canonical consequence.

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