

A Negative Answer to the Kadu–van Leeuwen Conjecture on Dual Binary Tomography

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Abstract

Kadu and van Leeuwen proposed to reconstruct a binary image from line sums by solving the Lagrange dual of the least-squares problem and reading off the sign of its numerically computed solution, where zeros mark undetermined pixels. From exhaustive experiments with images of up to 4×4 pixels and two to four directions they conjectured, for noiseless data and every image size, that this retrieves the unique binary solution if there is one and the intersection of all binary solutions otherwise. We show that the conjecture is false. In the noiseless case the exact dual optimum is zero, so the conjecture is a statement about numerically computed solutions, and the only pixels such a solution can certify are those that take the same value on the whole solution set Q of the box relaxation. A 5×5 image with six ones has a unique binary solution for rows, columns and diagonals, but Q is a segment with a half-integral endpoint and only 14 of the 25 pixels are fixed on it. Padding gives such images for every $n \geq 5$; with four directions the first failure is at 6×6 . The second statement already fails inside the tested range, for exactly 448 of the 65536 images of size 4×4 with rows, columns and diagonals. In computations through CVXPY (not the authors' MATLAB and CVX stack), the interior-point solvers Clarabel, ECOS and PIQP return the pattern of the fixed pixels, with zeros elsewhere, or a full sign pattern with a wrong sign; of 61 solver runs (eight solvers) that returned a solution on the 5×5 image, none returns the image. Two directions give a totally unimodular system, and the fixed-pixel form of the conjecture holds for them at every size. That uniqueness among binary solutions does not imply uniqueness among fractional solutions is known from work of Fishburn, Schwander, Shepp and Vanderbei; what is new here is the explicit refutation of this conjecture. This is an unrefereed note.

1 Introduction

Binary tomography asks for a binary image from its line sums along a few directions. For three or more directions the problem is NP-hard (see [GGP99]), and convex relaxations are a natural tool. Kadu and van Leeuwen [KvL20, Kad19] proposed the following one. Write the image as $x \in \{-1, 1\}^N$ and the data as $y = Ax$, where A is the 0/1 line-sum matrix. The Lagrange dual of the least-squares problem $\min_{\mu} \frac{1}{2} \|Ax - y\|^2$ subject to $x = \text{sign}(\varphi)$ is, after elimination of the primal variables, the problem

$$\min_{\mu} \frac{1}{2} \|AA^{\dagger}(\mu - y)\|^2 + \|A^{\top}\mu\|_1, \quad (1)$$

which is problem (13) of [KvL20]. We call the use of (1) together with the recovery rule $\hat{x} = \text{sign}(A^{\top}\hat{\mu}) \in \{-1, 0, 1\}^N$ the *dual approach*; a zero marks an undetermined pixel. The authors solve (1) with CVX, set entries of the numerically computed dual solution below 10^{-9} to zero, and report

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for all binary images of size $n \times n$, $n = 2, 3, 4$, and $m = 2, 3, 4$ directions (rows and columns; + diagonal; + anti-diagonal) that a unique binary solution is retrieved, and for several solutions their intersection (their Table I; the exceptions, marked by stars, are blamed on CVX). This is *Conjecture 1* of the report [Kad19], p. 258: “If the problem has a unique solution, then the dual approach retrieves it.” The second half asks for the intersection of all solutions when there are several. Both halves are stated for noiseless data, and [KvL20] conjectures that the pattern persists for larger n and m . The statement is the corpus record [ula26] OWR-16766-004 (duplicates -002, -003; -005 adds a question on strong duality).

Results. *Scoping.* In the noiseless case the exact optimum of (1) satisfies $A^\top \mu^* = 0$ for every instance (Lemma 2.1), so its sign carries no information and the conjecture is a statement about the *numerically computed* solution of a solver, which is how it is tested in [KvL20] (Section 2). The exact part is combinatorial. Let $Q(y) = \{z \in [-1, 1]^N : Az = y\}$ be the optimal set of the box relaxation, and call a pixel *fixed* if all $z \in Q(y)$ agree there. Every $z \in Q(y)$ is a certificate for the optimality of $\mu^* = 0$ (Proposition 2.2), so a computed solution can certify a pixel only if it is fixed; interior-point codes follow the central path approximately (Section 3; the facts about the central path used there are cited from the literature and tested, not re-proved).

Counterexamples. The directions are rows, columns, diagonals $j - i = \text{const}$ (direction $(1, 1)$) and, for four directions, anti-diagonals $i + j = \text{const}$.

- A 5×5 image with six ones (Theorem 4.2) has a unique binary solution for rows, columns and diagonals. The set Q is a segment whose far endpoint has entries in $\{-1, 0, 1\}$ (half-integral in the 0/1 coding), and only 14 of the 25 pixels are fixed. The example extends to every $n \geq 5$ (Corollary 4.3); with four directions the first failure is at 6×6 (Theorem 4.5).
- Inside the range tested in [KvL20] the second half fails: for $n = 4$ with rows, columns and diagonals, exactly 112 projection classes, containing 448 of the 65536 images, have a pixel common to all binary solutions that is not fixed (Theorem 4.6). The star at $(n, m) = (4, 3)$ in Table I of [KvL20] counts 451 failures; the difference of 3, and the 93 failures at $(4, 2)$, where Q is integral, are numerical.
- For any two directions Q is the convex hull of the binary solutions, so the conjecture holds at every size (Theorem 4.1); it also holds in the fixed-pixel sense if all circuits of $\ker A$ are ternary (Theorem 4.7), which covers 5×5 with four directions.

Numerical confirmation. We solve the authors’ problem, with the protocol of their public script, on CVXPY with Clarabel, ECOS, SCS, OSQP, HiGHS, CVXOPT, PIQP and DAQP (Section 5). No run returns the 5×5 image, and no interior-point run returns the intersection on the 112 failing 4×4 classes. We did not run MATLAB, CVX, SDPT3 or Gurobi.

Relation to earlier work. That a uniquely determined set need not be determined among fractional sets, and that interior-point methods return the centre of the solution set, is in Fishburn, Schwander, Shepp and Vanderbei [FSSV97]: their Theorem 3 gives an 11-point set in a 49×70 box, unique for the directions $(1, 0), (0, 1), (1, 1)$ but not among fuzzy sets (we checked it, Remark 4.4), and Section 7 describes the centre property. They attribute the failure of the converse of additivity to [FLRS91], and non-additive sets of uniqueness are studied further in [FS99, BDP13, BDP14], which we could not read. All of these predate Conjecture 1 ([FSSV97] by about twenty years), and [FSSV97] treats linear programs, not the dual of a least-squares problem. The relaxations of [WSH03, KSP17] are of a different kind ([KSP17] does not discuss sets of uniqueness). New here are the explicit refutation of the conjecture, the small instances (six ones on 5×5 ; first failures at $n = 5$ and $n = 6$; none

for $n \leq 4$ in part (i); new as far as we could check, see the last paragraph of Section 6), the failure of part (ii) inside the tested range, and the confirmation on real solvers.

2 The dual approach and its exact solution

Setting. An image is $x \in \{-1, 1\}^N$ on an $n \times n$ grid, pixel (i, j) with row i and column j (indices from 0, row 0 on top), $N = n^2$. In pictures we write 1 for +1 and 0 for -1. The *rows*, *columns*, *diagonals* ($j - i$ constant) and *anti-diagonals* ($i + j$ constant) are the lines; $A \in \{0, 1\}^{M \times N}$ has one row per line of the chosen directions, and $y = Ax$. Reflecting the image in a vertical axis exchanges diagonals and anti-diagonals and fixes rows and columns, so all statements hold for either convention of the diagonal direction. Put $C = [-1, 1]^N$ and

$$Q(y) = \{z \in C : Az = y\}, \quad S(y) = Q(y) \cap \{-1, 1\}^N.$$

A pixel p is *fixed* if z_p is the same for all $z \in Q(y)$; it is *common* if s_p is the same for all $s \in S(y)$. Let $F(y)$ and $I(y)$ be the sets of fixed and of common pixels. If $S(y) \neq \emptyset$, then $F(y) \subseteq I(y)$, and a fixed pixel has value ± 1 . The *fixed-pixel pattern* $\phi(y) \in \{-1, 0, 1\}^N$ equals the common value on $F(y)$ and 0 elsewhere; the *intersection pattern* $\iota(y)$ is defined in the same way for $I(y)$. In this language Conjecture 1 says that the output of the dual approach is x when $S(y) = \{x\}$, and $\iota(y)$ in general.

Lemma 2.1. *Let $y = Az$ for some $z \in C$ (for example $y = Ax$ with x an image). Then the minimum value of (1) is $\frac{1}{2}\|y\|^2$, and it is attained exactly on $\ker A^\top$. In particular $A^\top \mu^* = 0$ for every minimiser, and the minimiser is $\mu^* = 0$ if A has full row rank (the form (14) of [KvL20], in which dependent rows have been removed).*

Proof. Let $P = AA^\dagger$, the orthogonal projector onto the range of A . Since $y \in \text{range } A$, we have $Py = y$, and $A^\top(I - P) = 0$ gives $A^\top \mu = A^\top w$ with $w = P\mu$. So the objective is $\frac{1}{2}\|w - y\|^2 + \|A^\top w\|_1$. By Hölder's inequality and $\|z\|_\infty \leq 1$, $\|A^\top w\|_1 \geq z^\top A^\top w = y^\top w$. Hence the objective is at least $\frac{1}{2}\|w - y\|^2 + y^\top w = \frac{1}{2}\|w\|^2 + \frac{1}{2}\|y\|^2$, with equality at $w = 0$, where the value is $\frac{1}{2}\|y\|^2$. The bound exceeds $\frac{1}{2}\|y\|^2$ unless $w = P\mu = 0$, that is $\mu \in \ker A^\top$. $\square$

So the exact dual optimum is zero and is the same for every image with the same line sums. The one-dimensional example of [KvL20] shows the same ($A = 1$, $|y| \leq 1$ gives $\nu^* = 0$, approached “from the correct side” by iterative solvers). The informative part of the dual approach is the *approximate* solution and its sign.

Proposition 2.2. *Under the hypothesis of Lemma 2.1: (a) the optimal set of the box relaxation $\min_{z \in C} \frac{1}{2}\|Az - y\|^2$ (whose Fenchel dual is (1)) is $Q(y)$, with value 0; (b) $\mu = 0$ minimises $\|A^\top \mu\|_1 + \frac{1}{2}\|\mu - y\|^2$ if and only if $y \in AC$, and the set of $z \in C$ witnessing the optimality condition $0 \in (\mu - y) + A\partial\|\cdot\|_1(A^\top \mu)$ at $\mu = 0$ is exactly $Q(y)$; (c) hence whatever the optimality system of the dual implies about a pixel p holds for all $z \in Q(y)$, and it can imply a value at p only if $p \in F(y)$.*

Proof. (a) is the definition. (b): $\partial\|\cdot\|_1(0) = C$, so the condition at $\mu = 0$ reads $y \in AC$, with witnesses $z \in C$, $Az = y$. (c) follows, since every $z \in Q(y)$, in particular every binary solution, witnesses the same optimum $\mu^* = 0$. $\square$

The numerical reading. The protocol of [KvL20], Section IV, and of the authors' public script is: (1) remove linearly dependent rows of A (pivoted QR) and scale A by its spectral norm, so that (1) becomes $\min_\mu \frac{1}{2}\|\mu - y\|^2 + \|A^\top \mu\|_1$ (problem (14) of [KvL20]); (2) solve it with a convex solver (CVX with SDPT3 or Gurobi), giving $\hat{\mu}$; (3) put $\hat{\nu} = A^\top \hat{\mu}$, set entries with $|\hat{\nu}_p| < \theta$ to 0 ($\theta = 10^{-9}$ in

[KvL20], 10^{-10} in the script), and output $\hat{x} = \text{sign } \hat{\nu}$. This is the *numerical reading*; in Section 5 we also report the output without threshold. The conjecture, in this reading, is that $\hat{x} = x$ whenever $S(y) = \{x\}$, and $\hat{x} = \iota(y)$ in general. Its solver-independent *fixed-pixel form* is: $Q(y) = \{x\}$ whenever $S(y) = \{x\}$ (part (i')), and $F(y) = I(y)$ (part (ii')). By Proposition 2.2(c) this is the most that an output based on the optimality system can achieve; the numerical reading adds the choice of the point of $Q(y)$ that the solver selects.

3 The central path and what interior-point output looks like

Interior-point codes for the dual approach follow, approximately, the central path of the quadratic program

$$\min_{\mu, t} \frac{1}{2} \|AA^\dagger(\mu - y)\|^2 + \mathbf{1}^\top t \quad \text{s.t.} \quad -t \leq A^\top \mu \leq t, \quad (2)$$

which is a quadratic-programming form of (1). For a matrix A of full row rank, $AA^\dagger = I$ and (2) is the form (14) of [KvL20], the form of the authors' script and of the solver runs below.

Lemma 3.1 (central path). *Let $y \in \text{range } A$. For $\tau > 0$ let $\psi_\tau(z) = \frac{1}{2} \|Az - y\|^2 - \tau \sum_p \log(1 - z_p^2)$ on $(-1, 1)^N$, and let $z(\tau)$ be its (unique) minimiser. The logarithmic-barrier problem $\min \frac{1}{2} \|AA^\dagger(\mu - y)\|^2 + \mathbf{1}^\top t - \tau \sum_p [\log(t_p - \nu_p) + \log(t_p + \nu_p)]$, $\nu = A^\top \mu$, of (2) has as minimisers exactly the μ with $AA^\dagger \mu = y - Az(\tau)$; if A has full row rank, the unique minimiser has*

$$\mu(\tau) = y - Az(\tau),$$

and in all cases

$$\nu_p(\tau) = \frac{2\tau z_p(\tau)}{1 - z_p(\tau)^2} \quad (p = 1, \dots, N).$$

In particular $\text{sign } \nu_p(\tau) = \text{sign } z_p(\tau)$. Moreover $\|Az(\tau) - y\| \rightarrow 0$ as $\tau \downarrow 0$ if $Q(y) \neq \emptyset$, and all limit points of $z(\tau)$ lie in $Q(y)$.

Proof. ψ_τ is strictly convex and tends to $+\infty$ at the boundary of the cube, so $z(\tau)$ exists and is unique. Put $w = y - Az(\tau) \in \text{range } A$, $\nu = A^\top w$, $\lambda_p^\pm = (1 \pm z_p)/2$ and $t_p = \frac{\tau}{2}(1/\lambda_p^+ + 1/\lambda_p^-)$. Stationarity of ψ_τ , $A^\top(Az - y) + 2\tau z/(1 - z^2) = 0$, gives $\nu_p = 2\tau z_p/(1 - z_p^2) = \frac{\tau}{2}(1/\lambda_p^- - 1/\lambda_p^+)$, so $t_p - \nu_p = \tau/\lambda_p^+ > 0$ and $t_p + \nu_p = \tau/\lambda_p^- > 0$. Since $y \in \text{range } A$, the objective of the barrier problem depends on μ only through $w' = AA^\dagger \mu \in \text{range } A$ (because $A^\top \mu = A^\top w'$) and on t ; it is convex, and strictly convex in (w', t) because $A^\top$ is injective on $\text{range } A$. Its stationarity conditions are $1 = \tau(1/(t_p - \nu_p) + 1/(t_p + \nu_p))$, which is $\lambda^+ + \lambda^- = 1$, and $w' - y + A(\lambda^+ - \lambda^-) = 0$, which is $w' = y - Az$ (an equation in $\text{range } A$). Both hold for (w, t) , so (w, t) is the unique minimiser in (w', t) . For the last claim let $\bar{z} \in Q(y)$ and $\delta \in (0, 1)$. From $\psi_\tau(z(\tau)) \leq \psi_\tau((1 - \delta)\bar{z})$ and $-\log(1 - z_p^2) \geq 0$,

$$\frac{1}{2} \|Az(\tau) - y\|^2 \leq \frac{1}{2} \delta^2 \|y\|^2 + \tau N \log \frac{1}{2\delta - \delta^2},$$

and the right side tends to 0 for $\delta = \sqrt{\tau}$. □

An interior-point method stays in a neighbourhood of this path and stops at a final duality measure proportional to τ . We use the following standard facts. The primal-dual pair is a convex quadratic program, the set of multipliers (the variable z) at the optimum is $Q(y)$ (Proposition 2.2), and strict complementarity (Goldman and Tucker [GT56]) fails at the fixed pixels, where one of the multipliers $\lambda_p^\pm = (1 \pm z_p)/2$ vanishes on all of $Q(y)$. Write $F = F(y)$ and let P be the other pixels.

Proposition 3.2 (cited, not re-proved; tested in Section 5). *Assume $Q(y) \neq \emptyset$. As $\tau \downarrow 0$: (a) $z_P(\tau) \rightarrow z_P^*$, where z^* is the unique maximiser of $\sum_{p \in P} \log(1 - z_p^2)$ over $Q(y)$ (the analytic centre of $Q(y)$), and $z_p^* \in (-1, 1)$ for $p \in P$; (b) $|z_p(\tau)| \rightarrow 1$ for $p \in F$, and $|\nu_p(\tau)|$ is of order $\sqrt{\tau}$; (c) $\nu_p(\tau)/\tau \rightarrow 2z_p^*/(1 - (z_p^*)^2)$ for $p \in P$.*

References: the convergence of the central path to the analytic centre of the optimal set is due to Sonnevend [Son86], McLinden [McL80], Güler and Ye [GY93] (linear programming) and Monteiro and Zhou [MZ98] (convex programming); the order $\sqrt{\tau}$ where strict complementarity fails is studied for monotone linear complementarity problems in [MT96]. For tomography linear programs the centre property is stated in [FSSV97, Sec. 7]. Only Lemma 3.1 is proved here; (a)–(c) are cited.

Corollary 3.3 (thresholded output, assuming Proposition 3.2). *Let $\theta > 0$ and $Q(y) \neq \emptyset$. There are $\tau_1 < \tau_2$, of order θ^2 and θ respectively, such that for the central point at $\tau \in (\tau_1, \tau_2)$ the thresholded sign output equals the fixed-pixel pattern $\phi(y)$; for $\tau \geq \tau_2$ the pixels of P also survive and carry the sign of $z_p(\tau)$, which tends to sign z_p^* ; for $\tau \leq \tau_1$ all entries are zero. In particular, if $P \neq \emptyset$ the output is never the image x unless sign $z_p^* = x_p$ on P and the solver stops at $\tau \geq \tau_2$.*

For $\theta = 10^{-9}$ the window is roughly $10^{-18} \lesssim \tau \lesssim 10^{-9}$, which contains the final accuracy of the interior-point runs with tolerances 10^{-8} to 10^{-12} in Section 5; for such runs one expects the output $\phi(y)$, and this is what we observe.

4 Exact results

We pass freely between the ± 1 coding of Section 2 and the 0/1 coding $z \mapsto (z + \mathbf{1})/2$, which maps $Q(y)$ to $\{z \in [0, 1]^N : Az = b\}$ with $b = (y + A\mathbf{1})/2$. Images are drawn in the 0/1 coding; fixed and common pixels are the same in both.

4.1 Two directions

Theorem 4.1 (two directions). *Let A be the line-sum matrix of any two of the four directions, on any $r \times c$ grid, and let $y = Ax$ for an image x . Then $Q(y)$ is the convex hull of $S(y)$. Consequently $S(y) = \{x\}$ implies $Q(y) = \{x\}$, and $F(y) = I(y)$; both halves of the conjecture hold in the fixed-pixel form for all grid sizes.*

Proof. Each pixel lies on exactly one line of each of the two families, and two lines of different families meet in at most one pixel. So A is the vertex–edge incidence matrix of a simple bipartite graph (lines as vertices, pixels as edges), hence totally unimodular [HT56, HK56]. In the 0/1 coding the data $b = (y + A\mathbf{1})/2$ are integral, so the polytope $\{z \in [0, 1]^N : Az = b\}$ has integral vertices. Therefore $Q(y)$ is the convex hull of $S(y)$. A common pixel has the same value at all vertices, hence at all points: $I(y) \subseteq F(y)$, and the reverse inclusion always holds. $\square$

For uniqueness this is also a case of Theorem 4 of [FSSV97]; for rows and columns see [FLRS91].

4.2 The 5×5 counterexample

Theorem 4.2. *Let $n = 5$ with rows, columns and diagonals ($j - i$ constant), and let x and c be the images*

$$\begin{array}{cccccc}
 0 & 0 & 0 & 1 & 0 & \\
 0 & 1 & 0 & 0 & 0 & \\
 x = 1 & 0 & 0 & 0 & 1 & \\
 1 & 0 & 0 & 0 & 0 & \\
 0 & 0 & 1 & 0 & 0 &
 \end{array}
 \quad
 \begin{array}{ccccc}
 1 & 0 & 0 & -1 & 0 \\
 0 & -2 & 0 & 1 & 1 \\
 c = 0 & 0 & 1 & 0 & -1, \\
 -1 & 1 & 0 & 0 & 0 \\
 0 & 1 & -1 & 0 & 0
 \end{array}$$

x in the 0/1 picture ($1 \leftrightarrow +1$, $0 \leftrightarrow -1$; six ones) and c an integer image with $Ac = 0$. The 0/1 row sums of x are 1, 1, 2, 1, 1, the column sums 2, 1, 1, 1, 1, and the diagonal sums for $j - i = -4, \dots, 4$ are 0, 1, 2, 0, 1, 0, 1, 1, 0. Then:

- (a) x is the unique binary solution: $S(Ax) = \{x\}$.
- (b) $Q(Ax) = \{x + sc : 0 \leq s \leq 1\}$, a segment. Its far endpoint $x + c$ has entries in $\{-1, 0, 1\}$ (half-integral in the 0/1 coding): pixel (1, 1) is flipped and the other ten pixels of $\text{supp } c$ are 0.
- (c) The fixed pixels are the 14 pixels outside $\text{supp } c$. The 11 pixels of $\text{supp } c$ are common to all (one) binary solutions but not fixed; pixel (1, 1), where $x_{(1,1)} = +1$, takes every value in $[-1, 1]$ on Q .
- (d) The analytic centre of Q is $x + s^*c$ with $s^* = (45 - \sqrt{89})/44 \approx 0.808$; its pixel (1, 1) is $(\sqrt{89} - 23)/22 \approx -0.617 < 0$. The minimum-norm point of Q is $x + \frac{6}{7}c$, with pixel (1, 1) equal to $-\frac{5}{7} < 0$.

Proof. (a) Write $b_{ij} \in \{0, 1\}$ for the value at (i, j) in a binary solution. The four diagonals with sum 0 ($j - i = -4, -1, 1, 4$) force $b_{40} = b_{10} = b_{21} = b_{32} = b_{43} = b_{01} = b_{12} = b_{23} = b_{34} = b_{04} = 0$. The remaining line sums read: rows $b_{00} + b_{02} + b_{03} = 1$, $b_{11} + b_{13} + b_{14} = 1$, $b_{20} + b_{22} + b_{24} = 2$, $b_{30} + b_{31} + b_{33} = 1$, $b_{41} + b_{42} + b_{44} = 1$; columns $b_{00} + b_{20} + b_{30} = 2$, $b_{11} + b_{31} + b_{41} = 1$, $b_{02} + b_{22} + b_{42} = 1$, $b_{03} + b_{13} + b_{33} = 1$, $b_{14} + b_{24} + b_{44} = 1$; diagonals $b_{30} + b_{41} = 1$ ($j - i = -3$), $b_{20} + b_{31} + b_{42} = 2$ (-2), $b_{00} + b_{11} + b_{22} + b_{33} + b_{44} = 1$ (0), $b_{02} + b_{13} + b_{24} = 1$ (2), $b_{03} + b_{14} = 1$ (3). If $b_{03} = 1$: $b_{00} = b_{02} = 0$ (row 0), $b_{14} = 0$ (diagonal 3), $b_{13} = b_{33} = 0$ (column 3), $b_{11} = 1$ (row 1), $b_{22} = b_{44} = 0$ (main diagonal), $b_{24} = 1$ (column 4), $b_{20} = 1$ (row 2), $b_{30} = 1$ (column 0), $b_{31} = 0$ (row 3), $b_{41} = 0$ (diagonal -3), $b_{42} = 1$ (diagonal -2); this is x . If $b_{03} = 0$: $b_{14} = 1$, so $b_{11} = b_{13} = 0$ (row 1), $b_{24} = b_{44} = 0$ (column 4), $b_{20} = b_{22} = 1$ (row 2), $b_{02} = b_{42} = 0$ (column 2), $b_{00} + b_{30} = 1$ (column 0), and the main diagonal gives $b_{00} = b_{33} = 0$, so $b_{30} = 1$; but then row 0 has sum $b_{00} + b_{02} + b_{03} = 0 \neq 1$, a contradiction.

(b)–(c) Let $z \in Q$ and $e = z - x$, so $Ae = 0$, $e_p \geq 0$ where $x_p = -1$ and $e_p \leq 0$ where $x_p = +1$. Define $f(i, j) = \rho_i + \gamma_j + \delta_{j-i}$ with $\rho = (1, 0, -1, 1, 0)$, $\gamma = (-1, 0, 1, 0, 1)$ and $\delta_{-4}, \dots, \delta_4 = (2, 0, -1, 2, 0, 2, 0, -1, 0)$:

$$\begin{array}{ccccc}
 & 0 & 3 & 2 & 0 & 2 \\
 & 1 & 0 & 3 & 0 & 0 \\
 f = -3 & 1 & 0 & 1 & 0. \\
 & 0 & 0 & 4 & 1 & 4 \\
 & 1 & 0 & 0 & 2 & 1
 \end{array}$$

This table satisfies $f_p \geq 0$ where $x_p = -1$, $f_p \leq 0$ where $x_p = +1$, and $f_p = 0$ exactly on $\text{supp } c$. Since $\sum_p f_p e_p = \sum_i \rho_i(\text{row sum}) + \sum_j \gamma_j(\text{column sum}) + \sum_d \delta_d(\text{diagonal sum}) = 0$ and every term $f_p e_p$ is ≥ 0 , each term vanishes, so $e_p = 0$ whenever $f_p \neq 0$, that is outside $\text{supp } c$. On the eleven pixels of $\text{supp } c$ the line sums of e vanish: writing $e_{(0,0)} = s$, the rows, columns and diagonals force $e_{(0,3)} = e_{(3,0)} = -s$, $e_{(1,3)} = e_{(3,1)} = s$, then, with $w = e_{(2,4)}$, the values $e_{(1,4)} = e_{(4,1)} = e_{(2,2)} = -w$, $e_{(4,2)} = w$, $e_{(1,1)} = w - s$, and the diagonal $j - i = -3$ gives $w = -s$. So $e = sc$. The constraints $x + sc \in C$ and the sign conditions give $0 \leq s \leq 1$, the binding one being pixel (1, 1): $1 - 2s \geq -1$. Conversely every such point lies in Q because $Ac = 0$. This proves (b), and (c) follows, since $c_p = \pm 1$ gives a range of length 1 and $c_{(1,1)} = -2$ gives $[-1, 1]$.

(d) On the segment, the pixel (1, 1) is $1 - 2s$ and each of the other ten pixels of $\text{supp } c$ is $\mp(1 - s)$. Maximising $\sum_{p \in \text{supp } c} \log(1 - z_p^2) = 10 \log(s(2 - s)) + \log(4s(1 - s))$ gives $22 - 45s + 22s^2 = 0$, so $s^* = (45 - \sqrt{89})/44$. The minimum of $\|x + sc\|^2 = 25 + 2s(-12) + 14s^2$ is at $s = 6/7$. $\square$

Corollary 4.3. *For every $n \geq 5$ there is an $n \times n$ image with a unique binary solution for rows, columns and diagonals, for which Q is a segment with a half-integral endpoint and exactly $n^2 - 11$ pixels are fixed.*

Proof. Put x in the upper left corner and fill the rest with 0 (in the 0/1 picture). A pixel outside the corner has a row or column index ≥ 5 whose 0/1 sum is 0, so it is 0 in every binary solution and, as $z \geq 0$ in the 0/1 coding, in every point of Q . Lines meeting the corner keep their sums, so the corner problem is that of Theorem 4.2. $\square$

Remark 4.4. Fishburn et al. [FSSV97, Sec. 4] give an 11-point set $S \subset \mathbb{Z}^2$ in a 49×70 box, unique among sets for the directions $(1,0), (0,1), (1,1)$ but not among fuzzy sets. We read the coordinates from the printed list; they are confirmed by the 13 triples printed there, each of which closes up to a grid point outside S on a row, a column and a diagonal of points of S (the count 13 alone is not a sufficient check: a misreading of one coordinate also gave 13 such points). An exhaustive search over the 2^{24} subsets of the resulting support finds S as the only binary solution, and 23 of the 24 support points are not fixed over Q . In an $n \times n$ grid with $n \geq 71$ this is another counterexample to part (i'); the example of Theorem 4.2 is much smaller.

4.3 Four directions, the tested range, and a criterion

Theorem 4.5. (a) *The 6×6 image with rows 000010, 000010, 100010, 001000, 000101, 011000 (nine ones) has a unique binary solution for rows, columns, diagonals and anti-diagonals. The set Q is a segment whose far endpoint has entries in $\{0, \frac{1}{2}, 1\}$ (0/1 coding) and differs from the image on 18 pixels; 18 of the 36 pixels are fixed. The example extends to every $n \geq 6$ by padding as in Corollary 4.3. (b) For 5×5 and all four directions, $\ker A$ has dimension 4 and its 15 circuits are ternary, so by Theorem 4.7 both halves hold in the fixed-pixel form: there is no counterexample, and the first failure for four directions on square grids is at $n = 6$.*

Part (a) is a computation (an exhaustive depth-first search finds one binary solution; linear programs give the rest); (b) is an exact enumeration of the circuits in rational arithmetic.

Theorem 4.6 (inside the range of [KvL20]). *For $n = 4$ and rows, columns and diagonals, the 65536 images fall into 59256 projection classes; 54272 images have a unique binary solution, and for each of them Q is a point. The other 11264 images lie in 4984 classes with several solutions. In exactly 112 of these classes, containing 448 images, some pixel common to all binary solutions is not fixed. Each of the 112 classes has 4 solutions; in 64 of them exactly one common pixel is not fixed and in the other 48 exactly two. The same holds for rows, columns and anti-diagonals. For $n \leq 3$ with every set of at least two directions, and for $n = 4$ with rows and columns or with all four directions, neither (i') nor (ii') fails; among all eleven sets of at least two directions with $n = 4$, (ii') fails only for the two sets above, and (i') never fails.*

Method. All 2^{16} images were grouped by line-sum vector; per class one linear program (maximise the total distance of z from the common values on the common pixels, over Q) decides whether a common pixel moves. The count was reproduced by independent code in both verification runs (Section 6).

Example. The image 1100/1001/0010/0100 has row sums 2, 2, 1, 1, column sums 2, 2, 1, 1 and diagonal sums 0, 1, 1, 2, 1, 1, 0, and exactly four binary solutions: 0110/1100/1000/0001, 1010/1100/0001/0100, 1100/0101/1000/0010, 1100/1001/0010/0100. They agree on the four pixels $(0,3), (1,2), (2,1), (3,0)$, all equal to 0. The point of Q (0/1 coding) with rows $(1, \frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0, \frac{1}{2})$, checked in exact arithmetic, has pixels $(1,2)$ and $(2,1)$ equal to $\frac{1}{2}$, so only $(0,3)$ and $(3,0)$ are fixed.

The integer vectors with all line sums zero are described in [HT01] for arbitrary direction sets; the circuits of $\ker A$ are the minimal ones among them.

Theorem 4.7 (unimodular kernel). *If every circuit of $\ker A$ (a kernel vector of minimal support, scaled to be primitive and integral) has entries in $\{-1, 0, 1\}$, then for every image x : $Q(Ax) = \{x\}$ whenever $S(Ax) = \{x\}$, and $F(Ax) = I(Ax)$.*

Proof. Let $z \in Q(Ax)$, $z \neq x$, and $e = z - x$, so $Ae = 0$, $e_p \geq 0$ where $x_p = -1$ and $e_p \leq 0$ where $x_p = +1$. By the conformal decomposition theorem of Rockafellar [Roc69], $e = \sum_k \lambda_k c^k$ with $\lambda_k > 0$ and circuits c^k that are conformal to e , that is, $c_p^k e_p \geq 0$ for all p . Hence $c_p^k \geq 0$ where $x_p = -1$ and $c_p^k \leq 0$ where $x_p = +1$. By assumption $c_p^k \in \{-1, 0, 1\}$, so $x^k = x + 2c^k \in \{-1, 1\}^N$ and $Ax^k = Ax$: a second binary solution, which differs from x exactly on $\text{supp } c^k \neq \emptyset$. So $Q(Ax) \neq \{x\}$ implies $S(Ax) \neq \{x\}$, which is the first claim. For the second, $F \subseteq I$ always; if $p \notin F$ there is $z \in Q$ with $z_p \neq x_p$, hence some $c_p^k \neq 0$, and x^k differs from x at p , so $p \notin I$. $\square$

Counts of counterexamples. An image is a counterexample to (i') if it has a unique binary solution and Q is not a point. For 5×5 with rows, columns and diagonals (or anti-diagonals) there are exactly 1536, the smallest with 6 ones; for rows, diagonals and anti-diagonals (or columns, diagonals, anti-diagonals) exactly 8064, the smallest with 5 ones. Both counts come from a circuit and satisfiability decision procedure and, independently, from two exhaustive scans of all 2^{25} images (one by linear programs, one by line-sum keys and the circuits of $\ker A$). For 6×6 with four directions the total 463872 is not the result of an exhaustive scan of the 2^{36} images: it comes from the circuit and satisfiability procedure, on which two separately written implementations agree (also on the distribution by the number of ones: none with at most eight ones and exactly 56 with nine); an exhaustive scan of all images with at most nine ones agrees for these small numbers of ones. For $n \leq 4$ and every direction set of at least two directions, (i') never fails, and (ii') fails only in the case of Theorem 4.6.

5 Computations with real solvers

All solvers are called through CVXPY [DB16]: Clarabel [GC24], ECOS [DCB13], SCS [OCPB16], OSQP [SBG⁺20], HiGHS [HH18], CVXOPT [ADV26], PIQP [SJKJ23] and DAQP [ABA22] (versions used: CVXPY 1.9.3, Clarabel 0.11.1, ECOS 2.0.14, SCS 3.3.1, OSQP 1.1.3, HiGHS 1.15.1, CVXOPT 1.3.3, PIQP 0.6.4, DAQP 0.10.3). CVX with SDPT3 or Gurobi, used in [KvL20], was not available. Clarabel, ECOS, PIQP and CVXOPT are interior-point codes; SCS and OSQP are operator-splitting methods (OSQP with optional polishing, a crossover-like step); HiGHS and DAQP are active-set methods.

The problem is that of the authors' script: dependent rows are removed by pivoted QR, A is scaled by $\|A\|_2$, the problem $\min_p \frac{1}{2}\|p - y\|^2 + \|A^\top p\|_1$ is solved, $\hat{v} = A^\top \hat{p}$ is formed, and the thresholds are 10^{-9} , 10^{-10} and none. Form (13) with the projector and the unscaled problem were also run, with the same classes of outcomes.

5.1 The 5×5 image

Table 1 lists the outcome for the settings of the authors' script (form (14) of [KvL20] with A scaled): F is the fixed-pixel pattern ϕ (rows 0--0-, -0-00, +-0-0, 00---, -00--: eleven zeros, no wrong sign), W is all 25 signs with one wrong sign (at (1, 1), as in Corollary 3.3), Z all zeros, and O any other pattern, given as z/w (zeros/wrong signs). Over all 68 runs (eight solvers, three formulations; 61 returned a solution) and all three readings the image x is never returned.

The interior-point codes behave as Corollary 3.3 predicts: at tight tolerance with $\theta = 10^{-9}$ they return exactly $\phi(y)$, the pattern of the 14 fixed pixels; the smallest $|\hat{v}|$ on the fixed pixels is between $5 \cdot 10^{-8}$ and $6 \cdot 10^{-6}$, and the largest on the eleven others between $4 \cdot 10^{-14}$ and $8 \cdot 10^{-10}$. At loose tolerance, or without threshold, they return all 25 signs, wrong at (1, 1) (at tolerance 10^{-12} without threshold also at two further pixels, whose values are noise). The operator-splitting and active-set

solver	setting (tolerance)	$\theta = 10^{-9}$	$\theta = 10^{-10}$	no threshold
Clarabel	default (10^{-8})	F	O 2/1	W
Clarabel	10^{-10}	F	F	W
Clarabel	10^{-12} ; 10^{-6}	F; W	F; W	O 0/3; W
ECOS	default (10^{-8})	F	W	W
ECOS	10^{-10}	F	F	W
ECOS	10^{-12} ; 10^{-6}	F; W	F; W	W; W
PIQP	default; 10^{-12}	F	F	W
CVXOPT	default	W	W	W
SCS	default; 10^{-6}	O 0/5	O 0/5	O 0/5
SCS	10^{-9}	Z	O 18/0	O 0/6
OSQP	10^{-8} , no polish	O 1/6	O 1/6	O 0/6
OSQP	10^{-8} , polish	Z	Z	O 0/7
HiGHS (active set)	default	O 12/2	O 12/2	O 0/6
DAQP (active set)	default	Z	O 21/0	O 0/4

Table 1: Output of the dual approach on the 5×5 image of Theorem 4.2, form (14) of [KvL20] with A scaled. The image is never returned.

τ	$z_{(1,1)}(\tau)$	$\max_{p \in P} \nu_p /\tau$	$\min_{p \in F} \nu_p /\sqrt{\tau}$	output, $\theta = 10^{-9}$ (wrong signs / zeros)
1	-0.5523	1.589	1.158	5 / 0
10^{-6}	-0.61679	1.991	0.8108	1 / 0
10^{-8}	-0.616652	1.990	0.8111	1 / 0
10^{-10}	-0.616639	1.990	0.8112	0 / 11
10^{-18}	-0.616637	1.990	0.8112	0 / 12
10^{-20}	-0.616637	1.990	0.8112	0 / 25

Table 2: The central point for the 5×5 image (grey levels ± 1 ; the true value of pixel $(1, 1)$ is $+1$). The limit of $z_{(1,1)}$ is the analytic centre value $(\sqrt{89} - 23)/22 = -0.616637\dots$. The maximum of $|\nu_p|/\tau$ over the eleven pixels of P is attained at $(1, 1)$ and tends to $2|z^*|/(1 - z^{*2}) = 1.990$; $|\nu_p|/\sqrt{\tau}$ on the fixed pixels tends to a constant, as in Proposition 3.2.

codes converge towards the exact dual optimum $\hat{\nu} = 0$, so their signs are residual noise or all entries vanish (HiGHS: rounding noise of size about 10^{-8}).

5.2 The central path in high precision

To check Proposition 3.2 on the example, the central point of Lemma 3.1 was computed in 80-digit arithmetic (mpmath), by Newton’s method with continuation, for $\tau = 10^{-k/2}$, $0 \leq k \leq 80$, with the 19-row line-sum matrix A (form (13)). The identity $\nu = A^T(y - Az)$ holds to better than 10^{-30} at every τ . Table 2 shows a selection. The path for the 16 independent rows of the solver runs differs at finite τ (for example $\min_{p \in F} |\nu_p|/\sqrt{\tau} \rightarrow 0.610$ instead of 0.811, and $z_{(1,1)}(1) = -0.560$) and has the same limit.

Over the whole grid $z_{(1,1)}(\tau) < 0$ (the largest value is -0.552), so the unthresholded sign pattern has a wrong sign at $(1, 1)$ for every τ and the image is never returned. With threshold 10^{-9} the pattern ϕ (no wrong sign, eleven zeros) appears in the computed grid exactly for $3 \cdot 10^{-18} \leq \tau \leq 3 \cdot 10^{-10}$ (17 grid points), in agreement with the window $1.5 \cdot 10^{-18} \lesssim \tau \lesssim 5 \cdot 10^{-10}$ that Corollary 3.3 predicts from the constants 0.8112 and 1.99 of the table. These are computations on one instance, not a proof of Proposition 3.2.

Algorithm 2 of [KvL20], a Chambolle–Pock-type primal-dual iteration (in the form derived there, with the step size γ in front of A ; the pseudo-code of the arXiv version omits it), run from zero, converges in our implementation to the minimum-norm point $x + \frac{6}{7}c$ with pixel $(1, 1) = -\frac{5}{7}$ (with

(n, m)	total	paper	Clarabel 10^{-8}	Clarabel 10^{-10}	ECOS 10^{-8}	ECOS 10^{-10}	PIQP
(3, 2)	282	282	222	282	282	279	192
(4, 2)	58634	58541*	57690	58634	58130	58598	45314
(4, 3)	11264	10813*	10816	10816	10816	10810	10816
(4, 4)	512	512	512	512	512	512	512

Table 3: Multiple-solution cases of Table I of [KvL20]: number of images whose output equals the intersection. The cells $n = 2$, (3, 3), (3, 4) agree in all columns.

unscaled independent rows the limit is $4 \cdot 10^{-4}$ away, pixel -0.714), and the smooth L -BFGS variant of [KvL20] (smoothing parameter 10^{-2} to 10^{-8}) also has the wrong sign at $(1, 1)$; the first run reproduces the minimum-norm values of Theorem 4.2(d). Other primal–dual interior-point codes need not reach the analytic centre exactly (a Mehrotra code written for this purpose ends $4 \cdot 10^{-3}$ away from it in grey levels, and HiGHS’ interior-point method without presolve ends near the far endpoint of Q), but they also select a point of Q that is not the image.

5.3 Replication of Table I

For every $n \leq 4$ and $m \in \{2, 3, 4\}$ one problem per projection class was solved (about 140,000 solves per setting). Unique-solution cases are retrieved by Clarabel in every cell and at every tolerance tried; ECOS at tolerance 10^{-10} misses at most five images per cell. Table 3 gives the counts for the multiple-solution cases (images whose output equals the intersection, $\theta = 10^{-9}$).

For (4, 2), where Q is integral (Theorem 4.1), all failures are numerical and depend on solver and tolerance (0 at Clarabel 10^{-10} , 944 at its default); the 93 failures reported in [KvL20] are of this kind. For (4, 3) the number of failures is never below the 448 forced by Theorem 4.6; the reported 451 differs by 3, and other settings give 450 (ECOS 10^{-10} , $\theta = 10^{-10}$) or more. Up to a few images the star at (4, 3) is therefore the forced failure, not a defect of CVX.

5.4 The 112 failing classes

For each of the 112 failing classes of Theorem 4.6 the dual problem was solved in 15 solver settings. No interior-point or active-set setting returned the intersection pattern ι in any class. With Clarabel (default, 10^{-10}), ECOS (default, 10^{-10}) and PIQP at $\theta = 10^{-9}$ the output equals the fixed-pixel pattern ϕ in all 112; at loose tolerance (Clarabel 10^{-6} , CVXOPT) spurious nonzero pixels appear in all 112. One non-converged OSQP run (no polishing, $\theta = 10^{-10}$) matched ι in one class and had wrong signs in 22 others; we regard this as chance.

6 Remarks and limits

What is and is not shown. The binary solution of the 5×5 image is unique, and nothing here questions the identifiability of binary images from line sums. What fails is the dual approach: the optimality system of its relaxation does not distinguish the image from the other points of Q . Kadu and van Leeuwen also remark that h, v, d -convex images can be reconstructed in polynomial time and conjecture a polynomially solvable subclass of binary tomography; this is not affected (the 5×5 image is not h -convex: its row 2 is 10001). The numerical statements are computations for the listed solvers; another code or tolerance might return the image on a particular instance, but, assuming Proposition 3.2 (cited, not re-proved), by Corollary 3.3 an interior-point code would have to stop far from the window or select a point of Q with the correct sign at $(1, 1)$ and keep all eleven free pixels above the threshold.

Crossover. A solver returning a basic solution (for example a barrier method with crossover) returns $\hat{p} = 0$ up to rounding for noiseless data, by Lemma 2.1, so the output is zero or rounding noise; the HiGHS and DAQP runs are of this kind. Gurobi was not tested. *Strong duality* (record OWR-16766-005): for noiseless data the binary problem and the dual of the relaxation both have value 0; in general there is a gap, e.g. $A = (1)$, $y = 0$: the binary problem has value $\frac{1}{2}$, the relaxation and its dual 0. By Theorem 2 of [FSSV97] the images with Q a point are exactly the *additive* ones; how common the non-additive uniquely determined images are is left open.

Verification. The claims were checked as follows.

1. Exact computations (Python 3.13; scripts and outputs in the package): brute force over all 2^{25} images and a depth-first search for Theorem 4.2(a); exact integer and rational checks of $Ac = 0$, the potential f , the endpoint and the 4×4 witness; linear programs (HiGHS) for fixed pixels; the closed forms of Theorem 4.2(d); padding to $n = 6, 7, 8$; the 6×6 example; exact circuits for 5×5 with four directions; exhaustive classification of all images for $n = 2, 3, 4$; exhaustive linear-programming scans of all 5×5 images (counts 1536, 8064) and of the 6×6 images with at most nine ones.
2. Real-solver runs through CVXPY (Section 5): the 5×5 image, the replication of Table I, the 112 classes, and the 80-digit central path.
3. Two independent verification runs, both AI-assisted, wrote separate code from the problem statement. The first recomputed the counts of Table I of [KvL20], the 2^{25} search, the fixed pixels, the segment and its endpoint, the 6×6 example with a SAT solver, the unimodular-kernel statement, the 112 classes and their 64/48 split, and an interior-point emulation; its findings produced the corrections in the text (the 64/48 split, the half-integral point as an endpoint of Q , the status of the numbers 451, 448, 93, and the scoping). The second ran the solvers of Section 5 with new scripts (reproducing Tables 1 and 3 and the 112 classes), computed the central path of Table 2 with its own code, recomputed the classification for all eleven direction sets and $n \leq 4$, the counts 1536 and 8064 by a new exhaustive method, the 6×6 total 463872 with a separately written implementation of the circuit and satisfiability procedure, the total unimodularity of two directions by minors, an interior-point code of its own, and the example of [FSSV97] against the list printed there; it also re-ran the package from the source archive. Its findings produced the corrections to Lemma 3.1, the statements on Algorithm 2 and on the central path, the provenance of 463872, the credit to earlier work and the software versions.
4. Every reference with a DOI, an arXiv number or a Zbl number was checked against Crossref, the arXiv API or zbMATH (once when the note was written and again in the second verification run); for the software entry [ADV26] and the corpus record [ula26] the web page was checked.

Scope and priority. Theorems 4.2 and 4.6 answer Conjecture 1 of [Kad19] as stated there and tested in [KvL20], for the direction sets of Table I and the numerical reading of Section 2; they do not show that binary tomography is not identifiable. We read the report [Kad19], the paper [KvL20] with its LaTeX source and the authors' public script, the paper [FSSV97] (a copy on a co-author's page) and the preprint of [KSP17] (arXiv:1703.03769). Of [HT01] we saw only the zbMATH review, and of [WSH03, GGP99, FS99, BDP13, BDP14] only the bibliographic records (the publisher pages of [BDP13, BDP14] were not accessible); the NP-hardness statement is quoted from the literature. We searched arXiv, Crossref, OpenAlex, Semantic Scholar and zbMATH and made three web searches in all; we found no published refutation of Conjecture 1, and the works citing [KvL20] (13 in OpenAlex, 15 in Semantic Scholar, October 2026) concern applications and variants of the method. As [FSSV97]

shows, the underlying phenomenon is old, and the small instances here may be known in the literature on non-additive sets of uniqueness [FS99, BDP13, BDP14], which we could not read; this note makes no priority claim for the phenomenon. Russian- and Hungarian-language literature was not checked, and a negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [ABA22] D. Arnström, A. Bemporad, and D. Axehill. A dual active-set solver for embedded quadratic programming using recursive LDL^T updates. *IEEE Transactions on Automatic Control*, 67(8):4362–4369, 2022. doi:10.1109/TAC.2022.3176430.
- [ADV26] M. S. Andersen, J. Dahl, and L. Vandenberghe. CVXOPT: a Python package for convex optimization. <https://cvxopt.org>, 2026. Software, version 1.3.3.
- [BDP13] S. Brunetti, P. Dulio, and C. Peri. On the non-additive sets of uniqueness in a finite grid. In *Discrete Geometry for Computer Imagery (DGCI 2013)*, volume 7749 of *Lecture Notes in Computer Science*, pages 288–299. Springer, 2013. doi:10.1007/978-3-642-37067-0_25.
- [BDP14] S. Brunetti, P. Dulio, and C. Peri. Non-additive bounded sets of uniqueness in $\mathbb{Z}^n$. In *Discrete Geometry for Computer Imagery (DGCI 2014)*, volume 8668 of *Lecture Notes in Computer Science*, pages 226–237. Springer, 2014. doi:10.1007/978-3-319-09955-2_19.
- [DB16] S. Diamond and S. Boyd. CVXPY: a Python-embedded modeling language for convex optimization. arXiv:1603.00943, 2016.
- [DCB13] A. Domahidi, E. Chu, and S. Boyd. ECOS: an SOCP solver for embedded systems. In *2013 European Control Conference (ECC)*, pages 3071–3076, 2013. doi:10.23919/ECC.2013.6669541.
- [FLRS91] P. C. Fishburn, J. C. Lagarias, J. A. Reeds, and L. A. Shepp. Sets uniquely determined by projections on axes II. Discrete case. *Discrete Mathematics*, 91(2):149–159, 1991. doi:10.1016/0012-365X(91)90106-C.
- [FS99] P. C. Fishburn and L. A. Shepp. Sets of uniqueness and additivity in integer lattices. In G. T. Herman and A. Kuba, editors, *Discrete Tomography: Foundations, Algorithms, and Applications*, pages 35–58. Birkhäuser, Boston, 1999. doi:10.1007/978-1-4612-1568-4_2.
- [FSSV97] P. Fishburn, P. Schwander, L. Shepp, and R. J. Vanderbei. The discrete Radon transform and its approximate inversion via linear programming. *Discrete Applied Mathematics*, 75(1):39–61, 1997. doi:10.1016/S0166-218X(96)00083-2.
- [GC24] P. J. Goulart and Y. Chen. Clarabel: an interior-point solver for conic programs with quadratic objectives. arXiv:2405.12762, 2024.
- [GGP99] R. J. Gardner, P. Gritzmann, and D. Prangenberg. On the computational complexity of reconstructing lattice sets from their X-rays. *Discrete Mathematics*, 202(1–3):45–71, 1999. doi:10.1016/S0012-365X(98)00347-1.

- [GT56] A. J. Goldman and A. W. Tucker. Theory of linear programming. In *Linear Inequalities and Related Systems*, volume 38 of *Annals of Mathematics Studies*, pages 53–97. Princeton University Press, 1956. Zbl 0072.37601.
- [GY93] O. Güler and Y. Ye. Convergence behavior of interior-point algorithms. *Mathematical Programming*, 60(1–3):215–228, 1993. doi:10.1007/BF01580610.
- [HH18] Q. Huangfu and J. A. J. Hall. Parallelizing the dual revised simplex method. *Mathematical Programming Computation*, 10(1):119–142, 2018. doi:10.1007/s12532-017-0130-5.
- [HK56] A. J. Hoffman and J. B. Kruskal. Integral boundary points of convex polyhedra. In *Linear Inequalities and Related Systems*, volume 38 of *Annals of Mathematics Studies*, pages 223–246. Princeton University Press, 1956. Zbl 0072.37803.
- [HT56] I. Heller and C. B. Tompkins. An extension of a theorem of Dantzig. In *Linear Inequalities and Related Systems*, volume 38 of *Annals of Mathematics Studies*, pages 247–254. Princeton University Press, 1956. Zbl 0072.37804.
- [HT01] L. Hajdu and R. Tijdeman. Algebraic aspects of discrete tomography. *Journal für die reine und angewandte Mathematik*, 534:119–128, 2001. doi:10.1515/crll.2001.037.
- [Kad19] A. Kadu. A convex formulation for binary tomography. *Oberwolfach Reports*, 16(1):256–259, 2019. Abstract in Report 4/2019, “Tomographic Inverse Problems: Theory and Applications” (whole report pp. 209–303, published 2020); joint with T. van Leeuwen. doi:10.4171/OWR/2019/4.
- [KSP17] J. Kuske, P. Swoboda, and S. Petra. A novel convex relaxation for non-binary discrete tomography. In *Scale Space and Variational Methods in Computer Vision (SSVM 2017)*, volume 10302 of *Lecture Notes in Computer Science*, pages 235–246. Springer, 2017. doi:10.1007/978-3-319-58771-4_19.
- [KvL20] A. Kadu and T. van Leeuwen. A convex formulation for binary tomography. *IEEE Transactions on Computational Imaging*, 6:1–11, 2020. Preprint: arXiv:1807.09196 (version 3). doi:10.1109/TCI.2019.2898333.
- [McL80] L. McLinden. An analogue of Moreau’s proximation theorem, with application to the nonlinear complementarity problem. *Pacific Journal of Mathematics*, 88(1):101–161, 1980. doi:10.2140/pjm.1980.88.101.
- [MT96] R. D. C. Monteiro and T. Tsuchiya. Limiting behavior of the derivatives of certain trajectories associated with a monotone horizontal linear complementarity problem. *Mathematics of Operations Research*, 21(4):793–814, 1996. doi:10.1287/moor.21.4.793.
- [MZ98] R. D. C. Monteiro and F. Zhou. On the existence and convergence of the central path for convex programming and some duality results. *Computational Optimization and Applications*, 10(1):51–77, 1998. doi:10.1023/A:1018339901042.
- [OCPB16] B. O’Donoghue, E. Chu, N. Parikh, and S. Boyd. Conic optimization via operator splitting and homogeneous self-dual embedding. *Journal of Optimization Theory and Applications*, 169(3):1042–1068, 2016. doi:10.1007/s10957-016-0892-3.
- [Roc69] R. T. Rockafellar. The elementary vectors of a subspace of $\mathbb{R}^N$. In *Combinatorial Mathematics and its Applications (Proc. Conf., Univ. North Carolina, Chapel Hill, 1967)*, pages 104–127. Univ. North Carolina Press, 1969. Zbl 0229.90019.

- [SBG⁺20] B. Stellato, G. Banjac, P. Goulart, A. Bemporad, and S. Boyd. OSQP: an operator splitting solver for quadratic programs. *Mathematical Programming Computation*, 12(4):637–672, 2020. doi:10.1007/s12532-020-00179-2.
- [SJKJ23] R. Schwan, Y. Jiang, D. Kuhn, and C. N. Jones. PIQP: a proximal interior-point quadratic programming solver. In *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 1088–1093, 2023. doi:10.1109/CDC49753.2023.10383915.
- [Son86] G. Sonnevend. An “analytical centre” for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming. In *System Modelling and Optimization*, volume 84 of *Lecture Notes in Control and Information Sciences*, pages 866–875. Springer, 1986. doi:10.1007/BFb0043914.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Records OWR-16766-002 to -005.
- [WSH03] S. Weber, C. Schnörr, and J. Hornegger. A linear programming relaxation for binary tomography with smoothness priors. *Electronic Notes in Discrete Mathematics*, 12:243–254, 2003. doi:10.1016/S1571-0653(04)00490-1.