

A Finite Cover of the Genus-Two Surface in Which Every Four-Holed Sphere Has Connected Preimage: A Partial Answer to a Question of D. Gekhtman

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Abstract

In the problem session of the Oberwolfach workshop “New Trends in Teichmüller Theory and Mapping Class Groups” (2018), D. Gekhtman asked: given a finite cover $f: S_h \rightarrow S_g$ of closed orientable surfaces with $h > g \geq 2$, is there always a four-holed sphere $H \subset S_g$ with essential boundary components such that $f^{-1}(H)$ is disconnected, and is there always such a three-holed sphere? We give a partial answer. For the four-holed question the answer is no in genus 2: there is a cover $S_{10} \rightarrow S_2$ of degree 9, with monodromy group $3^2:Q_8$ of order 72, such that the preimage of every four-holed sphere with essential boundary components is connected. The mechanism is a lifting obstruction: the monodromy homomorphism does not lift to the central extension $3^{1+2}:Q_8$, whereas it would lift if some four-holed sphere had disconnected preimage. This settles the four-holed question in genus 2 only, where every such four-holed sphere is the complement of two disjoint curves, so that its boundary curves are isotopic in pairs; if the four boundary curves are required to be pairwise non-isotopic, there is no such subsurface in genus 2, the question begins in genus 3, and there it remains open. The example is not a regular cover, and its Galois closure is not an example. It belongs to a family of covers of S_2 of degree p^2 with group $p^2:Q_8$, for every prime $p \equiv 3 \pmod{4}$; the same groups give no example in genus at least 3. In the positive direction we find, in every genus, three-holed and four-holed spheres with disconnected preimage for covers which factor through a non-trivial abelian cover, for abelian-by-cyclic monodromy groups, for regular covers with characteristic kernel, and when the genus is at least three times the order of the monodromy group. Exact computations in genus 2 show that 9 is the smallest degree of such an example, that in degrees 9 and 10 the only examples are the covers with group $3^2:Q_8$ whose monodromy does not lift, and that there is none among the regular covers with deck group of order less than 12180. The three-holed question remains open (our covers do have pairs of pants with disconnected preimage), and so do the four-holed question in genus at least 3 and both questions for regular covers. This is an unrefereed note.

1 Introduction

1.1 The problem

Throughout, S_g is a closed connected orientable surface of genus g , and a *cover* $f: S_h \rightarrow S_g$ is a covering map of finite degree with connected total space; its degree is $n = (h-1)/(g-1)$. Problem 12 of the problem session of the Oberwolfach workshop “New Trends in Teichmüller Theory and Mapping Class Groups” (September 2018) was proposed by Dmitri Gekhtman [Gek18]. It fixes a finite cover $f: S_h \rightarrow S_g$ with $h > g \geq 2$ and asks: “Is there always a sphere minus four disks H with essential

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boundary components” in S_g such that the interior of $f^{-1}(H)$ is disconnected? The source then asks the same for a sphere minus three discs, and adds that one may consider only regular covers. As motivation it says that, for the map of Teichmüller spaces induced by f , the proposer can show that the smaller space is not a holomorphic retract of the bigger one if such an H exists, and that the existence of H may be an unnecessary requirement of the proof. The problem is record OWR-16415-018 of the UnsolvedMath collection [ula26], where it is listed as open. We write (Q4) and (Q3) for the question about four-holed and about three-holed spheres. This note gives a partial answer: (Q4) has a negative answer in genus 2, where every four-holed sphere with essential boundary components is the complement of two disjoint curves, so that its boundary curves are isotopic in pairs. If the four boundary curves are required to be pairwise non-isotopic, the question begins in genus 3. The example is not a regular cover, and its Galois closure is not an example. (Q3), (Q4) in genus at least 3, and both questions for regular covers remain open.

Dictionary. Fix a base point $* \in S_g$ and let $\pi = \pi_1(S_g, *)$. A cover of degree n is the same as a homomorphism $\varphi: \pi \rightarrow \text{Sym}(\Omega)$, $|\Omega| = n$, with transitive image, up to conjugation (Section 2.1); and for a connected subsurface H the preimage $f^{-1}(H)$ is connected if and only if the image under φ of the fundamental group of H is transitive on Ω (Lemma 2.1). So (Q4) asks whether for every transitive φ there is a four-holed sphere whose group has intransitive image.

Readings. A simple closed curve in S_g is *essential* if it does not bound a disc in S_g . A *four-holed sphere* (a *pair of pants*) in S_g is a compact subsurface homeomorphic to a sphere minus four (three) open discs. The condition on the boundary components in the question can be read in two ways.

(G) *Literal reading.* Each boundary curve of H is essential. Two boundary curves of H may be isotopic in S_g .

(S) *Strict reading.* Moreover the boundary curves of H are pairwise non-isotopic in S_g .

In genus 2 the difference is drastic: every four-holed sphere in S_2 with essential boundary components is the complement of two disjoint annuli, so its four boundary curves are isotopic in pairs (Lemma 3.1). Under (S) there is no admissible H in S_2 at all, and (Q4) begins in genus 3. The source is most naturally read with (G), for two reasons internal to its text: it allows $g = 2$, and it speaks of the *interior* of $f^{-1}(H)$. For a compact subsurface with disjoint boundary curves this precaution is superfluous; it has a point if H is thought of as the surface obtained by cutting S_g along a system of curves, in which two boundary circles may come from the same curve and whose interior is the open complementary region. This is exactly the situation in genus 2. It is an inference from the wording, not a fact of the source, which still makes sense under (S). The proposer’s theorem on holomorphic retracts and its exact hypothesis could not be located: they are neither in his thesis [Gek19] nor in the abstracts of his papers that were found. So the motivation does not decide between the readings, and we do not speculate. Both readings are treated wherever they differ.

1.2 Results

Convention for permutations. Permutations are composed as maps, $(pq)(i) = p(q(i))$, and $[p, q] = pqp^{-1}q^{-1}$. The group $\pi_1(S_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ is written with loops multiplied in the order of traversal (Sections 2.1 and 2.2).

Let $V = \mathbb{F}_3^2$ (column vectors $z = (z_1, z_2)$),

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q_8 = \langle I, J \rangle < \text{SL}(2, 3),$$

a quaternion group of order 8 ($I^2 = J^2 = -1$, $IJ = -JI$), and let $M_9 = \{z \mapsto Az + t : A \in Q_8, t \in V\} \cong 3^2:Q_8$, a group of order 72 acting on the nine points of V . Consider the affine maps

$$T(z) = z + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad U(z) = -Iz = \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}, \quad W(z) = IJz + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2z_1 + z_2 + 1 \\ z_1 + z_2 \end{pmatrix}. \quad (1)$$

Theorem 1.1 (a cover of S_2 with all four-holed spheres connected). (a) *The assignment $a_1 \mapsto T$, $b_1 \mapsto U$, $a_2 \mapsto T$, $b_2 \mapsto W$ defines a homomorphism φ_0 of $\pi_1(S_2)$ onto M_9 , hence a cover $f_0: S_{10} \rightarrow S_2$ of degree 9.*

(b) *For every four-holed sphere $H \subset S_2$ with essential boundary components, $f_0^{-1}(H)$ is connected.*

(c) *The same holds for the cover defined by any epimorphism $\pi_1(S_2) \rightarrow M_9$ which does not lift to the central extension $\tilde{G}_3 \cong 3^{1+2}:Q_8$ of M_9 defined in Section 4.2.*

So the answer to (Q4) is negative for $g = 2$. The scope of this statement is exactly the following. It concerns genus 2 only. There the four-holed spheres with essential boundary components are the complements of two disjoint curves; their boundary curves are isotopic in pairs, so the statement is one about reading (G), the only reading under which (Q4) is not void in genus 2. The cover f_0 is not regular, and its Galois closure is not a counterexample (Remark 5.3). (Q4) for $g \geq 3$, where four-holed spheres with pairwise non-isotopic boundary curves exist, is not touched.

With the points of V numbered $z \mapsto z_1 + 3z_2 \in \{0, \dots, 8\}$, the maps (1) are the permutations (image lists and cycles; see Figure 1)

$$\begin{aligned} T &= 120453786 = (012)(345)(678), \\ U &= 063174285 = (1623)(4785), \\ W &= 138570624 = (0135)(2847). \end{aligned} \tag{2}$$

The convention matters. The relation $[T, U][T, W] = 1$ holds for composition of maps and $[p, q] = pqp^{-1}q^{-1}$. It also holds in the convention of the system GAP (products read from left to right and $[p, q] = p^{-1}q^{-1}pq$); the subgroup tested in (b) is then the same set of permutations, written $\langle x, u^{-1}xu, v, w^{-1}vw \rangle$ instead of (3). In the two mixed conventions (maps with $p^{-1}q^{-1}pq$; left to right with $pqp^{-1}q^{-1}$) the relation fails for the permutations (2); a reader who multiplies from left to right and keeps $[p, q] = pqp^{-1}q^{-1}$ must replace the four permutations by their inverses. The affine maps (1) are the definition.

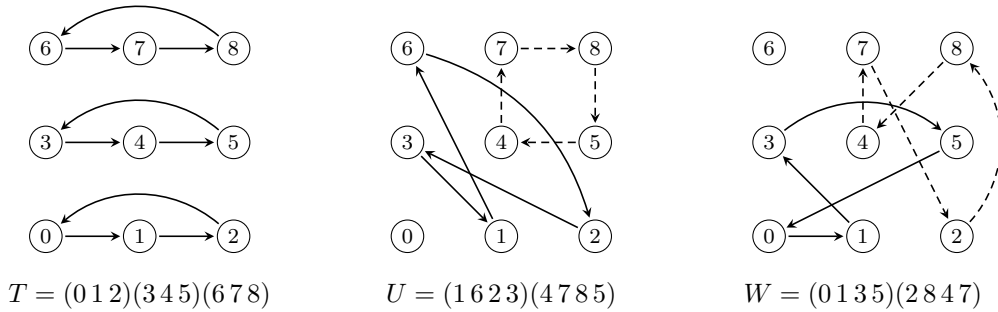


Figure 1: The nine sheets of f_0 , labelled by the points of $\mathbb{F}_3^2$ (the point (z_1, z_2) has the number $z_1 + 3z_2$ and is drawn at the position (z_1, z_2)), with the monodromy of the generators: $\varphi_0(a_1) = \varphi_0(a_2) = T$ is the translation by $(1, 0)$, $\varphi_0(b_1) = U$ fixes 0, and $\varphi_0(b_2) = W$ fixes 6. An arrow $i \rightarrow j$ means $i \mapsto j$; the two 4-cycles are drawn solid and dashed. Schematic.

For a homomorphism $\varphi: \pi_1(S_2) \rightarrow \Gamma$ we write $(x, u, v, w) = \varphi(a_1, b_1, a_2, b_2)$ and

$$J(x, u, v, w) = \langle x, uxu^{-1}, v, vvw^{-1} \rangle = \langle x, [x, u], v \rangle \tag{3}$$

(the two groups agree because $[x, u][v, w] = 1$). The proof of Theorem 1.1 has three steps. *Topology* (Lemma 3.1): every four-holed sphere with essential boundary components in S_2 has, for a suitable

standard generating system, fundamental group with image $J(x, u, v, w)$. *Obstruction* (Lemma 4.1): for a central extension $\tilde{G} \rightarrow \Gamma$, whether φ lifts to $\tilde{G}$ does not depend on the generating system; φ_0 does not lift to $\tilde{G}_3$. *Algebra* (Lemma 5.1): if (x, u, v, w) generates M_9 and $J(x, u, v, w)$ is intransitive, then φ lifts to $\tilde{G}_3$. The obstruction is the classical invariant of a surface group homomorphism in the second homology of the target; see Remark 4.2.

Theorem 1.2 (the family). *Let $p \equiv 3 \pmod{4}$ be a prime, $V = \mathbb{F}_p^2$, $Q_8 = \langle I, J \rangle < \mathrm{SL}(2, p)$ with I as above and $J = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, $\alpha^2 + \beta^2 = -1$, and $\Gamma_p = V \rtimes Q_8$, of order $8p^2$, acting on the p^2 points of V ; let $\tilde{G}_p \rightarrow \Gamma_p$ be the central extension by $\mathbb{Z}/p$ of Section 4.2.*

- (a) *For every epimorphism $\varphi: \pi_1(S_2) \rightarrow \Gamma_p$ which does not lift to $\tilde{G}_p$, every four-holed sphere with essential boundary components in S_2 has connected preimage in the corresponding cover $S_{p^2+1} \rightarrow S_2$ of degree p^2 .*
- (b) *Such epimorphisms exist: for $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b = \frac{1}{2} \begin{pmatrix} \beta - \alpha - 1 \\ 1 - \alpha - \beta \end{pmatrix}$ and $c_1 \in \mathbb{F}_p$ the assignment*

$$a_1 \mapsto (z \mapsto z + a), \quad b_1 \mapsto (z \mapsto Iz + \begin{pmatrix} c_1 \\ 0 \end{pmatrix}), \quad a_2 \mapsto (z \mapsto z + b), \quad b_2 \mapsto (z \mapsto Jz)$$

defines an epimorphism onto Γ_p whose obstruction is $\kappa = -c_1 - \frac{1+\alpha}{2} \in \mathbb{F}_p$; it does not lift unless $c_1 = -\frac{1+\alpha}{2}$.

- (c) (a) *holds more generally for $V \rtimes Q$, p an odd prime, whenever $Q < \mathrm{SL}(2, p)$ is a non-abelian subgroup of order prime to p in which every element with an eigenvalue in $\mathbb{F}_p$ is ± 1 (Lemma 5.1).*

For $p \equiv 1 \pmod{4}$ the conclusion of (a) fails for $p = 5$ and $p = 13$ (Proposition 6.1).

For $p = 3$, $\alpha = \beta = 1$, this is the group $M_9 = \Gamma_3$ of Theorem 1.1. The same groups give nothing in higher genus, under either reading:

Proposition 1.3 (the same groups in genus ≥ 3). *Let $g \geq 3$, let p be an odd prime, let $\Gamma_p = \mathbb{F}_p^2 \rtimes Q_8$ be defined as in Theorem 1.2 (for any odd p), and let $\varphi: \pi_1(S_g) \rightarrow \Gamma_p$ be a homomorphism with transitive image. Then*

- (a) *there is a four-holed sphere in S_g whose boundary curves are four pairwise non-isotopic non-separating curves and whose preimage is disconnected (it is admissible under both readings), and*
- (b) *there is a four-holed sphere in S_g with essential boundary components, two of which are isotopic, whose preimage is disconnected.*

For the positive results we use two kinds of pairs of pants (see Lemma 2.3): those of *nonseparating type*, whose three boundary curves are pairwise non-isotopic and nonseparating, and those of *handle type*, which lie in a one-holed torus and have boundary curves a , a , $[a, b]$.

Theorem 1.4 (partial positive results in every genus). *Let $f: S_h \rightarrow S_g$, $g \geq 2$, be a cover of degree $n \geq 2$ with monodromy group Γ and point stabiliser Γ_0 .*

- (C1) *Abelian factor. Suppose $\Gamma_0[\Gamma, \Gamma] \neq \Gamma$, that is, f factors through a regular cover of S_g with non-trivial abelian deck group.*
- (C2) *Abelian-by-cyclic. Or suppose that Γ has an abelian normal subgroup N with Γ/N cyclic.*

Then there are a pair of pants of nonseparating type, a pair of pants of handle type, and a four-holed sphere with essential boundary components (two of them isotopic) whose preimages are disconnected; if $g \geq 3$ there is also a four-holed sphere with four pairwise non-isotopic nonseparating boundary curves and disconnected preimage.

- (C3) Characteristic kernel. *If f is regular with deck group $G \neq 1$ and the kernel of $\pi_1(S_g) \rightarrow G$ is invariant under the automorphisms induced by Dehn twists (for instance, if it is characteristic), then every pair of pants and every four-holed sphere with essential boundary components has disconnected preimage.*
- (C4) Large genus. *If $g \geq |\Gamma|$ there is a pair of pants of handle type with disconnected preimage; if $g \geq 2|\Gamma|$ there are also a pair of pants of nonseparating type and a four-holed sphere with essential boundary components (two of them isotopic) with disconnected preimage; if $g \geq 3|\Gamma|$ there is a four-holed sphere with four pairwise non-isotopic nonseparating boundary curves and disconnected preimage.*

Table 2 in Section 8 lists, for each case, which reading of “four-holed sphere” is served. (C1) contains the regular covers with non-perfect deck group and the covers with nilpotent monodromy group. The next three statements are finite and were obtained by exhaustive computation; Sections 9 and 11 say exactly what was computed and on which cited theorems the computations rely.

Theorem 1.5 (computation; degree at most 8). *Every cover of S_2 of degree $2 \leq n \leq 8$ has a four-holed sphere with essential boundary components, and inside it pairs of pants of both types, with disconnected preimage. Hence 9 is the smallest degree of a cover of S_2 all of whose four-holed spheres with essential boundary components have connected preimage.*

Theorem 1.6 (computation; degrees 9, 10, 11). (a) *A cover of S_2 of degree 9 has no four-holed sphere with essential boundary components and disconnected preimage if and only if it is defined by an epimorphism onto a conjugate of M_9 in $\text{Sym}(9)$ which does not lift to $\tilde{G}_3$. There are 1440 such covers up to isomorphism; they form one orbit of the mapping class group of S_2 .*

- (b) *Every cover of S_2 of degree 10 has a four-holed sphere with essential boundary components and disconnected preimage.*
- (c) *So has every cover of S_2 of degree 11 whose monodromy group does not contain the alternating group A_{11} .*

Theorem 1.7 (computation; regular covers). *Every regular cover of S_2 with deck group G of order $1 < |G| < 12180$ has a four-holed sphere with essential boundary components, and inside it pairs of pants of both types, with disconnected preimage.*

Proposition 1.8 (the three-holed question). (a) *In genus 2, a cover all of whose pairs of pants with essential boundary components have connected preimage would be a counterexample to (Q4).*

- (b) *For every odd prime p and every homomorphism $\pi_1(S_2) \rightarrow \Gamma_p$ with transitive image there are pairs of pants of both types with disconnected preimage. For f_0 explicit ones are given in Section 10.*
- (c) (computation) *No cover of S_2 of degree at most 10 has the property of (a).*

Question of the source	Status after this note
(Q4), $g = 2$, reading (G)	no : Theorems 1.1 and 1.2; smallest degree 9 (Theorems 1.5 and 1.6)
(Q4), $g = 2$, reading (S)	void : no such subsurface exists (Lemma 3.1)
(Q4), $g \geq 3$, either reading	open ; yes in the cases of Theorem 1.4 and for the groups Γ_p (Proposition 1.3)
(Q3), all $g \geq 2$	open ; yes in the cases of Theorem 1.4; in genus 2 yes for degree ≤ 10 and for the classes of Theorems 1.6 and 1.7 and Proposition 1.8
regular covers, both questions	open ; yes for non-perfect deck groups, for characteristic kernels, and in genus 2 for $ G < 12180$

Table 1: The questions of Problem 12 and what is established here. The only negative answer is in the first row: genus 2, where the boundary curves of every admissible four-holed sphere are isotopic in pairs.

1.3 Related work, and what is not claimed

We found no answer to Problem 12 in print, no example like f_0 , and not this use of the lifting obstruction; the searches are described in the paragraph “Scope and priority” at the end. A search that finds nothing is not a proof of novelty, and no priority is claimed.

The obstruction itself is classical. For an epimorphism $\varphi: \pi_1(S_g) \rightarrow \Gamma$ onto a finite group, the image of the fundamental class in $H_2(\Gamma; \mathbb{Z})$ is the invariant of free actions of Γ on surfaces up to bordism; by a theorem of Livingston [Liv85], reproved algebraically by Zimmermann [Zim87], it is the complete invariant of stable equivalence, and Edmonds [Edm82, Edm83] had shown for abelian and for some metacyclic groups that no stabilisation is needed. Dunfield and Thurston [DT06, §6] use it to describe the orbits of the mapping class group on epimorphisms in large genus, and compute it exactly as we do, by lifting the images of the generators to a central extension [DT06, proof of Lemma 6.13]. That the non-vanishing of such an invariant forces a property of *all* members of a class of covers is the idea of the lifting invariant of Hurwitz theory, due to Fried and Serre [Ser90, Fri10] for covers of the sphere; the corresponding description of braid group orbits is the theorem of Conway and Parker, written down and improved by Fried and Völklein [FV91]. This is our model. What we add is the observation that in genus 2 this invariant vanishes as soon as the group of some four-holed sphere is intransitive, for the groups of Theorem 1.2.

For Theorem 1.4: the generating systems adapted to a cyclic quotient go back to Nielsen (see [Liv00, Lemma 4.1], with a proof in [Edm82]), and the first step of (C2) is the argument, due to Edmonds, of [Liv00, Theorem 4.2], where it is shown that for abelian-by-cyclic groups every epimorphism of $\pi_1(S_2)$ has an essential simple closed curve in its kernel. The pigeonhole argument of (C4) is the one in the proof of [DT06, Proposition 6.16], which has the bound $g > |\Gamma|$; Livingston’s theorem [Liv85] contains no bound. Finite simple characteristic quotients of surface groups, to which (C3) applies, were constructed by Funar and Lochak [FL18, Theorem 1.4].

Neighbouring questions about simple closed curves and finite covers have a large literature: whether the homology of a cover is generated by lifts of simple closed curves, or of curves lying in subsurfaces [PW13, KS16, MP19, BPS24, Klu25]. None of these papers, as far as they were read (see “Scope and priority”), states or answers the question of connectedness of preimages of subsurfaces.

What remains open: (Q3) in every genus; (Q4) for $g \geq 3$ under both readings; both questions for regular covers. The first open cases in genus 2 are the covers of degree 11 with monodromy group A_{11} or S_{11} and the covers of degree at least 12.

2 Preliminaries

2.1 Covers, monodromy and subsurfaces

Loops are multiplied in the order of traversal. Let $f: S' \rightarrow S_g$ be a covering of degree n and $\Omega = f^{-1}(*)$. For $j \in \Omega$ and a loop γ at $*$ let $j \cdot \gamma$ be the end point of the lift of γ starting at j , and put $\varphi(\gamma)(j) = j \cdot \gamma^{-1}$. Then $\varphi: \pi \rightarrow \text{Sym}(\Omega)$ is a homomorphism for composition of maps, the *monodromy*; S' is connected if and only if $\Gamma = \varphi(\pi)$ is transitive, and every transitive homomorphism $\pi \rightarrow \text{Sym}(n)$ arises from a cover, unique up to isomorphism; conjugate homomorphisms give isomorphic covers [Hat02, §1.3]. The cover is regular if and only if a point stabiliser Γ_0 is trivial, and then Γ is the deck group.

For a connected subsurface $H \subset S_g$ the image of $\pi_1(H) \rightarrow \pi$ is defined up to conjugacy (choose a base point in H and a path to $*$); we call it *the group of H* . The number of orbits of its image under φ is well defined.

Lemma 2.1. *Let f be a cover with monodromy φ and $H \subset S_g$ a compact connected subsurface.*

- (a) *The components of $f^{-1}(H)$, and those of its interior $f^{-1}(\text{int } H)$, correspond bijectively to the orbits on Ω of the image under φ of the group of H .*
- (b) *If f factors as $S_h \rightarrow S' \rightarrow S_g$ and the preimage of H in S' is disconnected, then $f^{-1}(H)$ is disconnected.*
- (c) *If $P \subset H$ is a connected subsurface and $f^{-1}(H)$ is disconnected, then $f^{-1}(P)$ is disconnected.*

Proof. (a) We may assume $*$ $\in \text{int } H$. The restriction $f^{-1}(H) \rightarrow H$ is a covering with fibre Ω , each of its components meets Ω , and two points of Ω lie in the same component if and only if they are joined by the lift of a loop in H . The inclusion $\text{int } H \subset H$ is a homotopy equivalence, so the same holds for the interior. (b) $f^{-1}(H)$ maps onto the preimage in S' . (c) Every component of $f^{-1}(H)$ maps onto H , hence meets $f^{-1}(P)$. $\square$

2.2 Standard systems

We fix a model: S_g is obtained from a $4g$ -gon P whose sides $s_1, \dots, s_{4g}$, in counterclockwise order, carry the labels $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$; the side s_k runs from the corner v_{k-1} to v_k ($v_{4g} = v_0$), and sides with the same letter are identified according to the labels. All corners become the base point $*$, the sides become simple loops $a_1, b_1, \dots, a_g, b_g$ meeting only at $*$, and

$$\pi = \langle a_1, b_1, \dots, a_g, b_g \mid R_g \rangle, \quad R_g = [a_1, b_1] \cdots [a_g, b_g].$$

Let δ_i be the loop given by the diagonal of P from v_{4i-4} to v_{4i} ; it is simple, meets the other loops only at $*$, and $\delta_i = [a_i, b_i]$ in π . For a loop c through $*$ let c^- be its initial and c^+ its final half-edge at $*$. The corner of P at v_k lies between the end of s_k and the beginning of s_{k+1} ; chaining the $4g$ corners one finds that the half-edges at $*$ follow each other in the cyclic order

$$a_1^-, b_1^+, a_1^+, b_1^-, a_2^-, b_2^+, a_2^+, b_2^-, \dots, a_g^-, b_g^+, a_g^+, b_g^-, \quad (4)$$

and that δ_i^- lies between b_{i-1}^- and a_i^- (indices modulo g), and δ_i^+ between b_i^- and a_{i+1}^- . (For sides numbered counterclockwise, (4) is the clockwise order. The sense plays no role below: reversing it replaces each boundary word of Lemma 2.2(b) by its inverse, and only unoriented curves are used.)

A *standard system* is a tuple $(a'_1, b'_1, \dots, a'_g, b'_g) = (\Phi(a_1), \dots, \Phi(b_g))$ with $\Phi \in \text{Aut}(\pi)$. It generates π with the single defining relation $[a'_1, b'_1] \cdots [a'_g, b'_g] = 1$. We use the Dehn–Nielsen–Baer theorem

[FM12, Chapter 8]: *every automorphism Φ of π is of the form $\gamma \mapsto cF_*(\gamma)c^{-1}$ for a homeomorphism F of S_g fixing $*$ and some $c \in \pi$.* Consequently, if $H \subset S_g$ is a subsurface whose group is conjugate to a subgroup $L \leq \pi$, then $F(H)$ is a subsurface whose group is conjugate to $\Phi(L)$; it is homeomorphic to H , and its boundary curves are essential, nonseparating, or pairwise non-isotopic if those of H are. Thus every statement below about a subsurface defined by the standard generators holds for every standard system. A substitution of the generators by words defines a standard system as soon as it is invertible and maps R_g to a conjugate of $R_g^{\pm 1}$ in the free group; this is the criterion we use, and it is the Dehn–Nielsen–Baer theorem which guarantees that such a substitution is induced by a homeomorphism.

Handle moves. The substitutions $b_i \mapsto b_i a_i^{\pm 1}$ and $a_i \mapsto a_i b_i^{\pm 1}$ (all other generators fixed) fix $[a_i, b_i]$, because $[a, ba] = [a, b] = [ab, b]$, and are invertible. On the exponent sums in a_i, b_i they act by the elementary matrices, which generate $\text{SL}(2, \mathbb{Z})$; hence for every primitive $(j, k) \in \mathbb{Z}^2$ a composition of handle moves turns a standard system into one whose new a_i is a word in the old a_i, b_i with exponent sums (j, k) , and whose other handles are unchanged.

Homology. $H_1(S_g; \mathbb{Z})$ carries the unimodular intersection form ω , and the classes of a standard system form a symplectic basis for ω or for $-\omega$. We use that the mapping class group maps onto $\text{Sp}(2g, \mathbb{Z})$ [FM12, Chapter 6]: *every symplectic basis of $H_1(S_g; \mathbb{Z})$ consists of the classes of a standard system.* The form induced on $H_1(S_g; \mathbb{F}_2)$ is denoted ω_2 .

2.3 Regular neighbourhoods of wedges of loops

All subsurfaces that we construct are regular neighbourhoods of wedges of simple loops, and their groups and boundary curves are given by the following lemma.

Lemma 2.2. *Let $c_1, \dots, c_k$ be simple loops in S_g based at $*$ which meet pairwise only at $*$. Let D be a small disc around $*$, let $p_i^-, p_i^+ \in \partial D$ be the points where c_i leaves and re-enters D , and suppose that the loops are unlinked: for $i \neq j$ the points $p_j^{\pm}$ lie in one component of $\partial D \setminus \{p_i^-, p_i^+\}$. Let N be a regular neighbourhood of $c_1 \cup \dots \cup c_k$. Then:*

- (a) *N is a sphere with $k + 1$ holes, $\pi_1(N, *)$ is free on $c_1, \dots, c_k$, and the group of N is $\langle c_1, \dots, c_k \rangle$.*
- (b) *The chords $[p_i^-, p_i^+]$ of D are pairwise disjoint and cut D into $k + 1$ regions, and the boundary curves of N correspond to these regions: the curve C_R of the region R is freely homotopic in N to $c_{i_1}^{\varepsilon_1} \dots c_{i_r}^{\varepsilon_r}$, where $[p_{i_1}^-, p_{i_1}^+], \dots, [p_{i_r}^-, p_{i_r}^+]$ are the chords on ∂R in their counterclockwise order around R , and $\varepsilon_j = 1$ if the j -th of them is traversed from p^- to p^+ and $\varepsilon_j = -1$ otherwise.*

Proof. N is the union of D and of k bands $B_i \cong [0, 1] \times [-1, 1]$, the band B_i being a neighbourhood of the arc of c_i outside D , attached to ∂D along two arcs $I_i^- \ni p_i^-$ and $I_i^+ \ni p_i^+$. It deformation retracts onto the wedge, which gives the statements on π_1 and $\chi(N) = 1 - k$. Write $I_i^- = [x_1, x_2]$ and $I_i^+ = [y_1, y_2]$ in the counterclockwise order of ∂D . Since N is orientable, the orientation of D extends over B_i ; this means that one of the two long sides of B_i joins x_2 to y_1 and the other joins y_2 to x_1 . The boundary of N consists of the long sides of the bands and of the arcs of ∂D between consecutive attaching arcs. Let R be a region. Following ∂R counterclockwise, each time we reach a chord $[p_i^-, p_i^+]$ at p_i^- we arrive on ∂D at x_1 , follow the long side from x_1 to y_2 , and continue on ∂D beyond p_i^+ ; each time we reach it at p_i^+ we arrive at y_1 and follow the other long side to x_2 . This describes a closed curve $C_R \subset \partial N$; every long side and every arc of ∂D between attaching arcs lies on exactly one C_R , so ∂N has exactly $k + 1$ components, and $1 - k = 2 - 2 \text{genus}(N) - (k + 1)$ gives genus 0. A long side is homotopic in its band to the arc of c_i outside D , traversed forwards in the first case and backwards in the second, and an arc of ∂D is homotopic in D to a path through $*$. This gives (b). $\square$

By (4) the loops $a_1, \dots, a_g$ are unlinked, with p^-, p^+ of each of them adjacent among their $2g$ points; and δ_i is unlinked with all a_j , the chord of a_i being nested inside that of δ_i and those of the a_j , $j \neq i$, lying outside. Lemma 2.2 then gives the subsurfaces that we use (see Figure 2, right, for the third one).

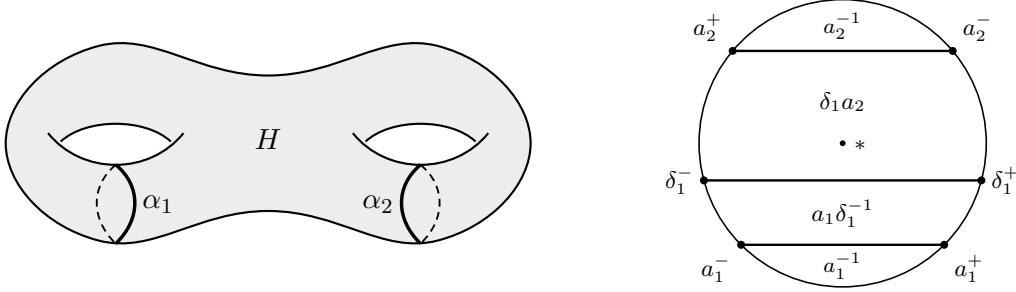


Figure 2: Left: in genus 2 every four-holed sphere H with essential boundary components is the complement of two disjoint annuli; their cores α_1, α_2 are nonseparating and not isotopic to each other, and the boundary curves of H are isotopic in pairs (Lemma 3.1). Right: the disc D around the base point with the chords of the loops $a_1, \delta_1 = [a_1, b_1], a_2$, in the cyclic order (4); each of the four regions gives one boundary curve of $H_0 = N(a_1 \vee \delta_1 \vee a_2)$, in the indicated free homotopy class (Lemma 2.2). In $\pi_1(S_2)$, $a_1 \delta_1^{-1}$ is conjugate to $b_1 a_1 b_1^{-1}$ and $\delta_1 a_2 = b_2 a_2 b_2^{-1}$.

Lemma 2.3 (standard subsurfaces). *Let $g \geq 2$ and $i \neq j$.*

- (a) $P(a_i, a_j) = N(a_i \vee a_j)$ is a pair of pants with group $\langle a_i, a_j \rangle$; its boundary curves are freely homotopic to $a_i, a_j, a_i a_j$, and are nonseparating and pairwise non-isotopic (nonseparating type).
- (b) $P(a_i, b_i) = N(a_i \vee \delta_i)$ is a pair of pants with group $\langle a_i, [a_i, b_i] \rangle = \langle a_i, b_i a_i b_i^{-1} \rangle$; its boundary curves are freely homotopic to $a_i, b_i a_i b_i^{-1}$ and $[a_i, b_i]$, and are essential (handle type).
- (c) $H(a_i, b_i; a_j) = N(a_i \vee \delta_i \vee a_j)$ is a four-holed sphere with group $\langle a_i, [a_i, b_i], a_j \rangle$; its boundary curves are freely homotopic to $a_i, b_i a_i b_i^{-1}, a_j$ and $[a_i, b_i] a_j$, and are essential. The first two are isotopic.
- (d) If $g \geq 3$ and i, j, l are distinct, $H(a_i, a_j, a_l) = N(a_i \vee a_j \vee a_l)$ is a four-holed sphere with group $\langle a_i, a_j, a_l \rangle$ whose boundary curves, freely homotopic to a_i, a_j, a_l and to the product of the three in their cyclic order, are nonseparating and pairwise non-isotopic.
- (e) More generally, let $B_1, \dots, B_r$ ($1 \leq r \leq 3$) be disjoint sets of consecutive indices in $\{1, \dots, g\}$, in increasing order, and $c_m = \prod_{i \in B_m} a_i$ (increasing order). Then $c_1, \dots, c_r$ are freely homotopic to pairwise disjoint nonseparating simple closed curves $\alpha_1, \dots, \alpha_r$ with connected complement $S_g \setminus \bigcup \alpha_m$; and if $r \geq 2$ there is a sphere with $r + 1$ holes in S_g with group $\langle c_1, \dots, c_r \rangle$ whose boundary curves, freely homotopic to $c_1, \dots, c_r$ and $c_1 \cdots c_r$, are nonseparating and pairwise non-isotopic.

Here “freely homotopic” refers to unoriented curves.

Proof. The groups and the homotopy classes of the boundary curves are read off from Lemma 2.2: in (a) and (d) the regions are the caps cut off by the chords and one central region; in (b) and (c) the regions are the cap of a_i , with curve a_i^{-1} , the region between the chords of a_i and δ_i , with curve $a_i \delta_i^{-1}$, which is conjugate to $b_i a_i b_i^{-1}$, the cap of a_j , and the remaining region, with curve δ_i in (b) and $\delta_i a_j$ in (c). A curve bounding a disc is null-homotopic, and a separating curve is null-homologous. The

classes of a_i , $a_i a_j$, $a_i a_j a_l$ and $\delta_i a_j$ in $H_1(S_g; \mathbb{F}_2)$ are non-zero, and in (a) and (d) pairwise distinct; so these curves are essential, resp. nonseparating and pairwise non-homotopic. Finally $[a_1, b_1] \neq 1$ in π , and likewise $[a_i, b_i] \neq 1$: the homomorphism of π to the free group on s, t with $a_1, b_2 \mapsto s$, $b_1, a_2 \mapsto t$ and all other generators trivial maps it to $[s, t]$. Two disjoint essential simple closed curves which are freely homotopic are isotopic [FM12, §1.2].

(e) Let $N = N(\bigvee_{i \in B_1 \cup \dots \cup B_r} a_i)$, the union of D and of bands as in the proof of Lemma 2.2, and let C be its boundary curve belonging to the central region. For each m let ℓ_m be the loop which runs from $*$ along a radius of D to the point of C just before the first band of the block B_m , then along C , in the direction of increasing indices, past all bands of this block, and back to $*$ along a radius. Inserting a radial excursion to $*$ between consecutive bands shows that $\ell_m = c_m$ in π . Pushed slightly into the interior of N , the loops $\ell_1, \dots, \ell_r$ are simple, meet only at $*$, and are unlinked, the two half-edges of each being adjacent. For $r = 1$ this shows that c_1 is freely homotopic to a simple closed curve α_1 . For $r \geq 2$, Lemma 2.2 gives the sphere with $r + 1$ holes $N(\ell_1 \vee \dots \vee \ell_r)$, with group $\langle c_1, \dots, c_r \rangle$ and with boundary curves $\alpha_1, \dots, \alpha_r$, freely homotopic to $c_1, \dots, c_r$, and a last one, freely homotopic to $c_1 \cdots c_r$. The classes of $c_1, \dots, c_r$ in $H_1(S_g; \mathbb{F}_2)$ are sums of basis vectors over disjoint sets, so they and their sum are non-zero and distinct, and $c_1, \dots, c_r$ are linearly independent. If $S_g \setminus \bigcup \alpha_m$ were disconnected, the boundary of one of its components would be a non-empty sum of the α_m which is zero in $H_1(S_g; \mathbb{F}_2)$. $\square$

2.4 Change of coordinates and adapted systems

Lemma 2.4 (change of coordinates). *Let $\alpha_1, \dots, \alpha_k$ and $\alpha'_1, \dots, \alpha'_k$ be two families of pairwise disjoint simple closed curves in S_g whose complements $S_g \setminus \bigcup \alpha_i$ and $S_g \setminus \bigcup \alpha'_i$ are connected, with disjoint closed annular neighbourhoods $A_i \supset \alpha_i$, $A'_i \supset \alpha'_i$. Then there is an orientation preserving homeomorphism F of S_g with $F(A_i) = A'_i$ and $F(\alpha_i) = \alpha'_i$ for all i .*

Proof. This is the change of coordinates principle [FM12, §1.3]. $\Sigma = S_g \setminus \bigcup \text{int } A_i$ and the corresponding Σ' are compact connected oriented surfaces with $2k$ boundary circles and the same Euler characteristic, hence homeomorphic by the classification of surfaces; and there is an orientation preserving homeomorphism $h: \Sigma \rightarrow \Sigma'$ which maps ∂A_i to $\partial A'_i$ for each i , since every bijection between the sets of boundary circles is induced by such a homeomorphism. On ∂A_i the orientation induced from Σ is opposite to the one induced from A_i , and likewise for A'_i ; so h respects the orientations of ∂A_i and $\partial A'_i$ as boundaries of the annuli, and therefore extends to an orientation preserving homeomorphism $A_i \rightarrow A'_i$ mapping the core to the core. $\square$

Lemma 2.5 (adapted systems). *Let $\chi: \pi \rightarrow \mathbb{Z}/m$ be an epimorphism, $m \geq 2$. Then there is a standard system with $\chi(b_1)$ a generator of $\mathbb{Z}/m$ and $\chi(a_1) = \chi(a_i) = \chi(b_i) = 0$ for all $i \geq 2$. Moreover:*

- (a) *every vector of $(\mathbb{Z}/m)^n$, $n \geq 2$, whose coordinates generate $\mathbb{Z}/m$ is the reduction of a primitive vector of $\mathbb{Z}^n$;*
- (b) *every primitive vector of $H_1(S_g; \mathbb{Z})$ is the first vector of a symplectic basis;*
- (c) *every homomorphism $\mathbb{Z}^2 \rightarrow C$ to a finite cyclic group has a primitive vector in its kernel.*

Proof. (a) Let $c \in \mathbb{Z}^n$ with $\gcd(c_1, \dots, c_n, m) = 1$. If $c_2 = \dots = c_n = 0$ replace c_2 by m . Then $d = \gcd(c_2, \dots, c_n) \neq 0$ and $\gcd(c_1, m, d) = 1$. A prime q dividing d and m does not divide c_1 ; for the primes q dividing d but not m choose t , by the Chinese remainder theorem, with $c_1 + tm \not\equiv 0 \pmod{q}$. Then $\gcd(c_1 + tm, c_2, \dots, c_n) = 1$. (b) If e_1 is primitive, unimodularity gives f_1 with $\omega(e_1, f_1) = 1$; the orthogonal complement of $\langle e_1, f_1 \rangle$ is unimodular, and one continues by induction.

(c) If the homomorphism is $(j, k) \mapsto j\lambda + k\mu$ modulo m' with $(\lambda, \mu) \neq (0, 0)$ and $d = \gcd(\lambda, \mu)$, take $(\mu/d, -\lambda/d)$; if $\lambda = \mu = 0$ take any primitive vector.

Now χ factors through $H_1(S_g; \mathbb{Z})$, lifts to a homomorphism to $\mathbb{Z}$, and is therefore of the form $\omega(\cdot, c)$ modulo m ; as χ is onto, the coordinates of c generate $\mathbb{Z}/m$, and by (a) we may take c primitive. By (b) and Section 2.2 there is a standard system whose classes (e_i, f_i) form a symplectic basis with $e_1 = c$. Then $\chi(a_1) = \omega(c, c) = 0$, $\chi(b_1) = \omega(f_1, c) = \mp 1$, and $\chi(a_i) = \chi(b_i) = 0$ for $i \geq 2$. $\square$

For $g = 2$ and cyclic targets the first statement is a lemma of Nielsen; see [Liv00, Lemma 4.1] (with a_1 and b_1 interchanged).

3 Four-holed spheres and pairs of pants in genus two

Lemma 3.1 (four-holed spheres in genus 2). (a) *Let $H \subset S_2$ be a four-holed sphere with essential boundary components. Then $S_2 \setminus \text{int } H$ is the union of two disjoint annuli, each joining two boundary curves of H . Their cores α_1, α_2 are disjoint nonseparating simple closed curves, not isotopic to each other, with $S_2 \setminus (\alpha_1 \cup \alpha_2)$ connected. In particular the boundary curves of H are isotopic in pairs, and S_2 contains no four-holed sphere with four pairwise non-isotopic essential boundary curves. Conversely, the complement of disjoint open annular neighbourhoods of two disjoint simple closed curves with connected complement is a four-holed sphere with essential boundary components.*

(b) *For any two such four-holed spheres H, H' there is a homeomorphism F of S_2 with $F(H) = H'$. So the mapping class group of S_2 acts transitively on their isotopy classes.*

(c) *For every such H there is a standard system (a_1, b_1, a_2, b_2) such that the group of H is conjugate to*

$$\langle a_1, [a_1, b_1], a_2 \rangle = \langle a_1, b_1 a_1 b_1^{-1}, a_2, b_2 a_2 b_2^{-1} \rangle.$$

Hence for a cover with monodromy φ , $f^{-1}(H)$ is connected if and only if the group $J(x, u, v, w)$ of (3) is transitive, where (x, u, v, w) are the images under φ of this system.

Proof. (a) $C = S_2 \setminus \text{int } H$ is a compact surface with $\partial C = \partial H$ and $\chi(C) = \chi(S_2) - \chi(H) = 0$. No component of C is closed, and none is a disc, because the boundary curves of H are essential. So every component C_j , of genus g_j with $b_j \geq 1$ boundary circles, has $\chi(C_j) = 2 - 2g_j - b_j \leq 0$; the sum being 0, all C_j are annuli, and there are two of them. $S_2 \setminus (\alpha_1 \cup \alpha_2)$ deformation retracts onto H and is connected; hence neither α_i separates, and there is a closed curve which crosses α_1 once and misses α_2 , so that α_1 and α_2 have different classes in $H_1(S_2; \mathbb{F}_2)$ and are not homotopic. For the converse: the complement is connected with four boundary circles and $\chi = -2$, hence of genus 0, and its boundary curves are nonseparating.

(b) Apply Lemma 2.4 with $k = 2$ to the two pairs of annuli.

(c) By Lemma 2.3(c), $H_0 = H(a_1, b_1; a_2) = N(a_1 \vee \delta_1 \vee a_2)$ is a four-holed sphere with essential boundary components and group $\langle a_1, [a_1, b_1], a_2 \rangle$ (Figure 2). Let F be as in (b) with $F(H_0) = H$ and Φ the automorphism F_* followed by a change of base point; then the group of H is conjugate to $\Phi \langle a_1, [a_1, b_1], a_2 \rangle$. The second description follows from $[a_1, b_1] = a_1(b_1 a_1 b_1^{-1})^{-1}$ and $b_2 a_2 b_2^{-1} = [a_1, b_1] a_2$, which is the defining relation. The last statement is Lemma 2.1. $\square$

Lemma 3.2 (pairs of pants in genus 2). *Let $P \subset S_2$ be a pair of pants with essential boundary components. Then $S_2 \setminus \text{int } P$ is either a pair of pants, and the boundary curves of P are nonseparating and pairwise non-isotopic, or the union of an annulus and a one-holed torus, and two boundary curves of P are isotopic while the third is separating. Every four-holed sphere $H \subset S_2$ with essential boundary components contains pairs of pants of both kinds with essential boundary components.*

Proof. The complement has Euler characteristic -1 , three boundary circles and, as in the proof of Lemma 3.1, components with $\chi \leq 0$; so it has one component with $\chi = -1$, a pair of pants or a one-holed torus, and the others are annuli. In the first case any two of the boundary curves have connected complement, hence independent classes modulo 2. For the last statement write $\partial H = \{c_1, c'_1, c_2, c'_2\}$ with c_i, c'_i parallel to α_i . A simple closed curve in H separating $\{c_1, c_2\}$ from $\{c'_1, c'_2\}$ is nonseparating in S_2 and cuts H into two pairs of pants of the first kind; one separating $\{c_1, c'_1\}$ from $\{c_2, c'_2\}$ bounds one-holed tori on both sides and cuts H into two pairs of pants of the second kind. $\square$

The two kinds are the nonseparating type and the handle type of Lemma 2.3(a), (b).

4 The lifting obstruction

4.1 Definition and invariance

Lemma 4.1. *Let $1 \rightarrow Z \rightarrow \tilde{G} \rightarrow \Gamma \rightarrow 1$ be a central extension of groups and $\varphi: \pi_1(S_g) \rightarrow \Gamma$ a homomorphism. For a standard system (a'_i, b'_i) choose lifts $\tilde{x}_i, \tilde{u}_i \in \tilde{G}$ of $\varphi(a'_i), \varphi(b'_i)$ and put*

$$\kappa = [\tilde{x}_1, \tilde{u}_1] \cdots [\tilde{x}_g, \tilde{u}_g].$$

- (a) κ lies in Z and does not depend on the choice of the lifts.
- (b) $\kappa = 1$ if and only if φ lifts to a homomorphism $\pi_1(S_g) \rightarrow \tilde{G}$.
- (c) Consequently the property $\kappa \neq 1$ does not depend on the standard system: it holds for all of them or for none.

Proof. (a) The image of κ in Γ is $\varphi([a'_1, b'_1] \cdots [a'_g, b'_g]) = 1$, and a commutator does not change if its entries are multiplied by central elements. (b) The system generates π with the single relation $[a'_1, b'_1] \cdots [a'_g, b'_g] = 1$. If $\kappa = 1$, the lifts satisfy this relation and define a lift of φ . Conversely the values of a lift $\tilde{\varphi}$ on the system are lifts, so $\kappa = \tilde{\varphi}([a'_1, b'_1] \cdots [a'_g, b'_g]) = 1$. (c) The criterion in (b) does not mention the system. $\square$

We write $\kappa(\varphi)$ for the value computed from the standard generators.

Remark 4.2 (this is the classical invariant). $\kappa(\varphi)$ is the image of the fundamental class under $H_2(\pi_1(S_g); \mathbb{Z}) \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow Z$, the second map being the one attached to the central extension. The class $\varphi_*[S_g] \in H_2(\Gamma; \mathbb{Z})$ is the bordism invariant of [Liv85, Zim87] and the invariant c_f of [DT06, §6.12], where it is computed by exactly this recipe for a Schur cover $\tilde{G}$. An automorphism of $\pi_1(S_g)$ acts on H_2 by ± 1 , so κ is preserved or inverted; we only need Lemma 4.1(c). For the groups Γ_p and $\tilde{G}_p$ defined below, $H_2(\Gamma_p; \mathbb{Z}) \cong \mathbb{Z}/p$ and $\tilde{G}_p$ is a Schur cover (the p -part of H_2 is the group of coinvariants of $H_2(V; \mathbb{Z}) = \Lambda^2 V \cong \mathbb{F}_p$, on which $Q_8 < \mathrm{SL}(2, p)$ acts trivially, and $H_2(Q_8; \mathbb{Z}) = 0$); this is not used. No novelty is claimed for the invariant.

4.2 The affine groups and their central extensions

Let p be an odd prime, $V = \mathbb{F}_p^2$ and $\omega(s, t) = s_1 t_2 - s_2 t_1$. For $A \in \mathrm{GL}(2, p)$ and $t \in V$ we write $(A | t)$ for the affine map $z \mapsto Az + t$; so $(A | t)(A' | t') = (AA' | t + At')$. Let $t_c = (1 | c)$ be the translation by c and $\rho_c = (-1 | 2c)$ the reflection in the point c . Then

$$(A | t) t_c (A | t)^{-1} = t_{Ac}, \quad \rho_c \rho_{c'} = t_{2(c-c')}, \quad t_d \rho_c = \rho_{c+d/2}. \quad (5)$$

For a subgroup $Q \leq \mathrm{SL}(2, p)$ let $\Gamma(Q) = \{(A|t) : A \in Q, t \in V\} \cong V \rtimes Q$, acting on the p^2 points of V ; we identify V with its group of translations. The linear part $(A|t) \mapsto A$ is a homomorphism $\Gamma(Q) \rightarrow Q$.

Let E_p be the Heisenberg group: the set $V \times \mathbb{F}_p$ with $(t, s)(t', s') = (t+t', s+s' + \frac{1}{2}\omega(t, t'))$. It has order p^3 and exponent p , and $\mathrm{SL}(2, p)$ acts on it by $A \cdot (t, s) = (At, s)$, because $\omega(At, At') = \omega(t, t')$. Let $\tilde{G}(Q) = E_p \rtimes Q$; we write its elements as $[t, s; A]$, so that

$$[t, s; A][t', s'; A'] = [t + At', s + s' + \frac{1}{2}\omega(t, At'); AA'], \quad [t, s; A]^{-1} = [-A^{-1}t, -s; A^{-1}]. \quad (6)$$

The map $[t, s; A] \mapsto (A|t)$ is a homomorphism of $\tilde{G}(Q)$ onto $\Gamma(Q)$ whose kernel $Z = \{[0, s; 1]\} \cong \mathbb{Z}/p$ is central. Two consequences of (6): for $\tilde{g} = [c, s; A]$ and $a \in V$,

$$\tilde{g}[a, 0; 1]\tilde{g}^{-1} = [Aa, \omega(c, Aa); 1], \quad (7)$$

$$[[a, 0; 1], \tilde{g}] = [(1 - A)a, -\omega(c, Aa) - \frac{1}{2}\omega(a, Aa); 1]. \quad (8)$$

(For (7) compare $\tilde{g}[a, 0; 1] = [c + Aa, s + \frac{1}{2}\omega(c, Aa); A]$ with $[Aa, \omega(c, Aa); 1]\tilde{g}$; then $[[a, 0; 1], \tilde{g}] = [a, 0; 1] \cdot \tilde{g}[-a, 0; 1]\tilde{g}^{-1}$.)

The groups Γ_p . For every odd prime p there are $\alpha, \beta \in \mathbb{F}_p$ with $\alpha^2 + \beta^2 = -1$ (the sets of values of α^2 and of $-1 - \beta^2$ have $\frac{p+1}{2}$ elements each). With I and J as in Theorem 1.2 one has $I^2 = J^2 = -1$ and $IJ = -JI$, so that $Q_8 = \langle I, J \rangle = \{\pm 1, \pm I, \pm J, \pm IJ\}$ is a quaternion group in $\mathrm{SL}(2, p)$ in which every element of order 4 has square -1 . We write $\Gamma_p = \Gamma(Q_8)$ and $\tilde{G}_p = \tilde{G}(Q_8)$. For $p = 3$ and $\alpha = \beta = 1$ this is $M_9 = \Gamma_3$, and $\tilde{G}_3$ has order 216 (in $\mathbb{F}_3$, $\frac{1}{2} = -1$).

Lemma 4.3. *Let $Q \leq \mathrm{SL}(2, p)$ have order prime to p , $\Gamma = \Gamma(Q)$, $\tilde{G} = \tilde{G}(Q)$.*

- (a) *A subgroup of Γ is transitive on V if and only if it contains V .*
- (b) *Every $g \in \Gamma$ of order m prime to p has exactly one lift $\hat{g} \in \tilde{G}$ of order m , and $\tilde{h}\hat{g}\tilde{h}^{-1} = \widehat{hgh^{-1}}$ for every $\tilde{h} \in \tilde{G}$ over $h \in \Gamma$.*
- (c) *If $L \leq \Gamma$ has order prime to p , then its preimage in $\tilde{G}$ is $Z \times S$ for a subgroup S which contains every element of the preimage of order prime to p .*

Proof. (a) V is a normal Sylow p -subgroup of Γ , of order p^2 . A transitive subgroup has order divisible by p^2 , hence contains a Sylow p -subgroup of Γ , which is V . (b) The lifts of g are $\tilde{g}z$, $z \in Z$, and $(\tilde{g}z)^m = \tilde{g}^m z^m$ with $\tilde{g}^m \in Z$; as $z \mapsto z^m$ is bijective on Z , exactly one lift has m -th power 1, and its order is m . The second statement follows by uniqueness. (c) Z is a normal subgroup of the preimage $\tilde{L}$ whose order p is prime to its index $|\tilde{L}|$; by the Schur–Zassenhaus theorem [Rob96, §9.1] it has a complement S , and $\tilde{L} = Z \times S$ because Z is central. An element (z, s) has order prime to p only if $z = 1$. $\square$

When L is a 2-group, as for $Q = Q_8$, (c) needs only Sylow's theorem: S is the Sylow 2-subgroup of $\tilde{L}$.

5 The main lemma and the proof of Theorem 1.1

Consider, for a subgroup $Q \leq \mathrm{SL}(2, p)$, the hypothesis

- (E) Q is non-abelian, its order is prime to p , and every element of Q which has an eigenvalue in $\mathbb{F}_p$ is 1 or -1 .

It holds for Q_8 if $p \equiv 3 \pmod{4}$: an element of order 4 has square -1 , so its eigenvalues are square roots of -1 , which do not lie in $\mathbb{F}_p$. Geometrically, (E) says that no element of Q other than ± 1 leaves a line of V invariant.

Lemma 5.1 (main lemma). *Let p be an odd prime, let $Q < \mathrm{SL}(2, p)$ satisfy (E), and let $\Gamma = \Gamma(Q)$, $\tilde{G} = \tilde{G}(Q)$. Let $x, u, v, w \in \Gamma$ generate Γ and satisfy $[x, u][v, w] = 1$. If the group $J = J(x, u, v, w)$ of (3) is not transitive on V , then $\kappa = [\tilde{x}, \tilde{u}][\tilde{v}, \tilde{w}] = 1$ for lifts $\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w} \in \tilde{G}$; that is, the homomorphism $\pi_1(S_2) \rightarrow \Gamma$ defined by (x, u, v, w) lifts to $\tilde{G}$.*

Proof. Put $x' = uxu^{-1}$ and $v' = wvw^{-1}$, so that $J = \langle x, x', v, v' \rangle$, and note that for arbitrary lifts

$$\kappa = \tilde{x}(\tilde{u}\tilde{x}\tilde{u}^{-1})^{-1}\tilde{v}(\tilde{w}\tilde{v}\tilde{w}^{-1})^{-1}, \quad (9)$$

a product of four elements of the preimage $\tilde{J}$ of J , in which $\tilde{u}\tilde{x}\tilde{u}^{-1}$ and $\tilde{w}\tilde{v}\tilde{w}^{-1}$ are lifts of x' and v' of the same orders as $\tilde{x}$ and $\tilde{v}$. We show that there is a subgroup $S \leq \tilde{J}$ with $S \cap Z = 1$ and lifts $\tilde{x}, \tilde{v}$ such that all four factors lie in S ; then $\kappa \in S \cap Z = 1$. Let $A, B, C, D \in Q$ be the linear parts of x, u, v, w ; they generate Q , and $[A, B][C, D] = 1$. By Lemma 4.3(a), $W = J \cap V$ is a proper subgroup of V .

Case $W = 0$. Then J is isomorphic to its image in Q and has order prime to p . Take S as in Lemma 4.3(c) and $\tilde{x} = \hat{x}$, $\tilde{v} = \hat{v}$; by Lemma 4.3(b) the other two factors are $\hat{x}'^{-1}$ and $\hat{v}'^{-1}$.

Case $|W| = p$. W is a line of V , and it is normalised by J ; by (5) it is invariant under the linear part of every element of J , so by (E) all these linear parts are ± 1 . In particular $A, C \in \{\pm 1\}$.

If $A = 1$, then $x = 1$. Indeed $x = t_a$ and $x' = t_{Ba}$ lie in W . If $a \neq 0$, then $W = \mathbb{F}_p a$ contains Ba , so a is an eigenvector of B and $B = \pm 1$ by (E). Then $[A, B] = 1$, hence $[C, D] = 1$, and $Q = \langle A, B, C, D \rangle$ is generated by the central elements A, B, C and the single element D ; so Q is abelian, a contradiction. In the same way $C = 1$ implies $v = 1$. *If $A = -1$, then x and x' are reflections in points.* So each of x, x', v, v' is trivial or an involution.

If all four are trivial, take trivial lifts. Otherwise J contains a reflection r ; as all linear parts in J are ± 1 and $J \cap V = W$, we have $J = W \rtimes \langle r \rangle$, of order $2p$. The preimage $\tilde{W}$ of W in $\tilde{G}$ consists of the elements $[c, s; 1]$, $c \in W$; it is elementary abelian of order p^2 , because ω vanishes on a line. Let $\tilde{r} = \hat{r}$. Conjugation by $\tilde{r}$ is an involution of the $\mathbb{F}_p$ -vector space $\tilde{W}$ which is trivial on Z and acts by -1 on $\tilde{W}/Z \cong W$; as p is odd, $\tilde{W} = Z \oplus W_-$ with W_- its (-1) -eigenspace. Then $S = W_- \rtimes \langle \tilde{r} \rangle$ is a subgroup of order $2p$ with $S \cap Z = 1$, and $\tilde{J} = Z \times S$. An element (z, s) of $Z \times S$ of order at most 2 has $z = 1$. Choosing $\tilde{x} = \hat{x}$ and $\tilde{v} = \hat{v}$, all four factors of (9) have order at most 2 and lie in S . $\square$

Proof of Theorem 1.1. (a) By (5), $[t_a, (B|c)] = t_a t_{-Ba} = t_{(1-B)a}$. With $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Ie = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $IJe = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ this gives $[T, U] = t_{(1+I)e} = t_{(1,1)}$ and $[T, W] = t_{(1-IJ)e} = t_{(2,2)}$, so $[T, U][T, W] = 1$ and φ_0 is a homomorphism. Its image maps onto Q_8 , since the linear parts $-I$ and IJ generate Q_8 , and contains $T \neq 1$; the translations in the image form a non-zero subgroup of V invariant under Q_8 , which is V because no line is invariant. So φ_0 is onto M_9 , the cover is connected of degree 9, and $\chi(S_h) = 9 \cdot (-2)$ gives $h = 10$.

φ_0 does not lift to $\tilde{G}_3$. Take the lifts $\tilde{x} = \tilde{v} = [e, 0; 1]$, $\tilde{u} = [0, 0; -I]$, $\tilde{w} = [e, 0; IJ]$. By (8), with $\omega(e, Ie) = \omega(e, IJe) = 1$ and $\frac{1}{2} = -1$ in $\mathbb{F}_3$,

$$[\tilde{x}, \tilde{u}] = [(1, 1), \frac{1}{2}\omega(e, Ie); 1] = [(1, 1), 2; 1], \quad [\tilde{v}, \tilde{w}] = [(2, 2), -\frac{3}{2}\omega(e, IJe); 1] = [(2, 2), 0; 1],$$

and by (6) $\kappa(\varphi_0) = [(1, 1), 2; 1][(2, 2), 0; 1] = [0, 2 + \frac{1}{2}\omega((1, 1), (2, 2)); 1] = [0, 2; 1] \neq 1$. By Lemma 4.1, φ_0 does not lift.

(b), (c) Let $\varphi: \pi_1(S_2) \rightarrow M_9$ be an epimorphism which does not lift to $\tilde{G}_3$, and H a four-holed sphere with essential boundary components. By Lemma 3.1(c) there is a standard system (a_1, b_1, a_2, b_2)

such that $f^{-1}(H)$ is connected if and only if $J(x, u, v, w)$ is transitive, where (x, u, v, w) are the images of the system. These generate M_9 and satisfy $[x, u][v, w] = 1$, and $\kappa \neq 1$ for them by Lemma 4.1(c). $Q_8 < \text{SL}(2, 3)$ satisfies (E). By Lemma 5.1, $J(x, u, v, w)$ is transitive. $\square$

Remark 5.2. (a) The hypothesis that x, u, v, w generate Γ is needed in Lemma 5.1: for $x = 1, v = t_{(1,0)}, w = t_{(0,1)}$ and any u the relation holds, $J = \langle v \rangle$ is intransitive, and $\kappa = [\tilde{v}, \tilde{w}] \neq 1$. So Theorem 1.1(c) is a statement about epimorphisms.

(b) For M_9 the criterion is sharp, by computation (Section 9): of the 1,036,800 epimorphisms $\pi_1(S_2) \rightarrow M_9$, the 622,080 which do not lift all have J transitive, while the 414,720 which lift form a single orbit under the mapping class group and conjugation, in which 25,920 have J intransitive. So an epimorphism onto M_9 defines a cover with all four-holed spheres connected if and only if it does not lift.

(c) A cover as in Theorem 1.1(c) has no essential simple closed curve in the kernel of its monodromy. A nonseparating one would give an intransitive J by Lemma 9.1 below. A separating one cuts S_2 into two one-holed tori and is therefore, by the classification of surfaces as in the proof of Lemma 2.4, freely homotopic to $[a_1, b_1]$ for some standard system; then $[x, u] = [v, w] = 1$, the linear parts of the two handles are cyclic and generate Q_8 , so both have order 4; the abelian groups $\langle x, u \rangle$ and $\langle v, w \rangle$ contain no translation other than 1 (it would commute with an element whose linear part has no eigenvalue 1), so they are cyclic of order 4, and $\kappa = [\hat{x}, \hat{u}][\hat{v}, \hat{w}] = 1$ by Lemma 4.3(c). Livingston [Liv00] shows that the smallest group admitting a homomorphism from $\pi_1(S_2)$ without essential simple closed curve in the kernel has order 32.

Remark 5.3 (the Galois closure). The cover f_0 is not regular. The regular cover with deck group M_9 defined by φ_0 is not a counterexample: M_9 is not perfect, so Theorem 1.4(C1) applies; concretely $J(\varphi_0) = \langle T, t_{(1,1)}, T \rangle = V$, so the preimage of H_0 in this regular cover has $[M_9 : V] = 8$ components.

6 The family

Proof of Theorem 1.2. (a) and (c). This is the proof of Theorem 1.1(b), (c) with M_9 replaced by $\Gamma(Q)$: Lemma 3.1(c), Lemma 4.1(c) and Lemma 5.1. The degree is p^2 and $\chi = -2p^2$ gives genus $p^2 + 1$. For Q_8 and $p \equiv 3 \pmod{4}$ hypothesis (E) was checked at the beginning of Section 5.

(b) Let $x = t_a, u = (I | c)$ with $c = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$, $v = t_b, w = (J | 0)$. By (5), $[x, u] = t_{(1-I)a}$ and $[v, w] = t_{(1-J)b}$, so the relation is $(1-I)a + (1-J)b = 0$. Now $(1-I)a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\det(1-J) = 1 - \alpha^2 - \beta^2 = 2$, and $(1-J)^{-1} = \frac{1}{2} \begin{pmatrix} 1+\alpha & \beta \\ \beta & 1-\alpha \end{pmatrix}$, whence $b = -(1-J)^{-1}(1-I)a = \frac{1}{2} \begin{pmatrix} \beta-\alpha-1 \\ 1-\alpha-\beta \end{pmatrix}$, as stated. The image contains $t_a \neq 1$ and maps onto Q_8 ; its translations form a non-zero Q_8 -invariant subgroup of V , which is V since by (E) there is no invariant line. So the homomorphism is onto. For the lifts $[a, 0; 1], [c, 0; I], [b, 0; 1], [0, 0; J]$, (8) gives

$$[\tilde{x}, \tilde{u}] = [(1-I)a, -\omega(c, Ia) - \frac{1}{2}\omega(a, Ia); 1], \quad [\tilde{v}, \tilde{w}] = [(1-J)b, -\frac{1}{2}\omega(b, Jb); 1].$$

The first components add up to 0, so by (6) the second components add up to κ . Here $Ia = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\omega(c, Ia) = c_1$, $\omega(a, Ia) = 1$, and a direct computation gives $Jb = \frac{1}{2} \begin{pmatrix} 1-\alpha+\beta \\ -1-\alpha-\beta \end{pmatrix}$ and $\omega(b, Jb) = \frac{1}{4}((\alpha + 1)^2 - \beta^2 - (1 - \alpha)^2 + \beta^2) = \alpha$. Hence $\kappa = -c_1 - \frac{1}{2} - \frac{\alpha}{2}$. $\square$

For $p \equiv 1 \pmod{4}$ every element of order 4 of Q_8 has two invariant lines. The proof of Lemma 5.1 then breaks down at the step “ $A = 1$ implies $x = 1$ ”: x can be a non-trivial translation along an invariant line of the linear part of u ; then $\tilde{x}$ and $\tilde{u}\tilde{x}\tilde{u}^{-1}$ have order p and are not distinguished among the lifts by their order. The conclusion fails as well:

Proposition 6.1. For $p = 5$ and $p = 13$ let $Q_8 = \langle I, J \rangle < \text{SL}(2, p)$ with $J = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$, $\beta = 2$, resp. $\beta = 5$, and consider in $\Gamma(Q_8)$

$$\begin{array}{lllll} p = 5 : & x = t_{(1,0)}, & u = (IJ | 0), & v = \rho_{(0,3)}, & w = (-J | (0, 3)); \\ p = 13 : & x = t_{(1,0)}, & u = (IJ | 0), & v = \rho_{(0,7)}, & w = (-J | (12, 7)). \end{array}$$

(Here $IJ = \text{diag}(-\beta, \beta)$, that is $\text{diag}(3, 2)$, resp. $\text{diag}(8, 5)$.) In both cases $[x, u][v, w] = 1$, the four elements generate $\Gamma(Q_8)$, the group $J(x, u, v, w)$ has order $2p$ and is intransitive, and $\kappa = 4 \neq 0$, resp. $\kappa = 3 \neq 0$. So the corresponding cover of S_2 of degree p^2 does not lift to $\tilde{G}(Q_8)$ and has a four-holed sphere with disconnected preimage.

Proof. A finite verification, carried out by the program of this note (Section 11, part H) and by the second verification run. For $p = 5$ it can be followed directly. $[x, u] = t_{(1-IJ)(1,0)} = t_{(3,0)}$; $wvw^{-1} = \rho_{w(0,3)} = \rho_{(4,3)}$ and $[v, w] = v(wvw^{-1})^{-1} = \rho_{(0,3)}\rho_{(4,3)} = t_{2(-4,0)} = t_{(2,0)}$, so the relation holds. $uxu^{-1} = t_{(3,0)}$, and $J = \langle t_{(1,0)}, \rho_{(0,3)}, \rho_{(4,3)} \rangle$ preserves the line $z_2 = 3$. The image contains $t_{(1,0)}$ and its linear parts IJ , -1 , $-J$ generate Q_8 , which leaves no line invariant; so the image contains V and is $\Gamma(Q_8)$. Finally, for the lifts $\tilde{x} = [(1, 0), 0, 1]$, $\tilde{u} = [0, 0, IJ]$, $\tilde{v} = [(0, 1), 0, -1]$, $\tilde{w} = [(0, 3), 0, -J]$, (8) gives $[\tilde{x}, \tilde{u}] = [(3, 0), 0, 1]$, and (6) gives $\tilde{v}\tilde{w} = [(0, 3), 0, J]$, $\tilde{w}\tilde{v} = [(3, 3), 3, J]$, hence $[\tilde{v}, \tilde{w}] = \tilde{v}\tilde{w}(\tilde{w}\tilde{v})^{-1} = [(2, 0), 4, 1]$ and $\kappa = [0, 4, 1]$. $\square$

Remark 6.2 (computed orbit counts). For $p \in \{3, 7, 11\}$ the second verification run enumerated all homomorphisms $\pi_1(S_2) \rightarrow \Gamma_p$ (1,592,136, 261,651,376 and 4,067,504,056 of them), and the first programs treated the same groups by a normalised search. In both computations the classes, modulo conjugation by Γ_p , of epimorphisms which do not lift form exactly $p-1$ orbits under the substitutions of Section 9, of 4320, 60,480 and 237,600 classes, one for each non-zero value of κ ; these are exactly the orbits without a four-holed sphere with disconnected preimage. For $p = 5$ and $p = 13$ (34,375,000 and 11,314,040,296 homomorphisms) there are $p-1$ orbits with $\kappa \neq 0$ and each contains tuples with J intransitive (240 of 21,600 and 1872 of 393,120 classes): no cover with group Γ_5 or Γ_{13} has all four-holed spheres connected. That this happens for every $p \equiv 1 \pmod{4}$ is not proved.

Remark 6.3 (other groups Q). An element of $\text{SL}(2, p)$ of order m prime to p has an eigenvalue in $\mathbb{F}_p$ if and only if m divides $p-1$. So (E) also holds for the generalised quaternion group $Q_{16} < \text{SL}(2, 7)$ and for the dicyclic group of order 12 and $\text{SL}(2, 3)$ in $\text{SL}(2, 11)$ (the program of this note checks (E) for one subgroup of each kind). The second verification run computed all orbits for these three groups and found, in agreement with Theorem 1.2(c), 6, 10 and 10 orbits of non-liftable epimorphisms, none with a four-holed sphere with disconnected preimage (degrees 49, 121, 121); pairs of pants of both types with disconnected preimage occur in each of them.

7 The same groups in genus at least three

In this section p is an odd prime, $Q_8 = \langle I, J \rangle < \text{SL}(2, p)$ and $\Gamma_p = \Gamma(Q_8)$; we do not assume $p \equiv 3 \pmod{4}$. The quotient $Q_8/\{\pm 1\}$ is identified with $\mathbb{F}_2^2$, the classes of I and J being the basis, and for $A, B \in Q_8$ with classes $\bar{A}, \bar{B}$

$$[A, B] = (-1)^{\det(\bar{A}, \bar{B})}, \quad (10)$$

because the commutator subgroup of Q_8 is $\{\pm 1\}$ and two elements of Q_8 commute if and only if they lie in a common cyclic subgroup, that is, if their classes are linearly dependent. Let $D = \{(\pm 1 | t)\}$ be the group of translations and point reflections. By (5), for an affine line $\ell = c_0 + \mathbb{F}_p d$ the translations $t_{\lambda d}$ and the reflections ρ_c , $c \in \ell$, form a group which preserves ℓ ; in particular

$$\text{a subgroup of } D \text{ generated by such elements, for one line } \ell, \text{ is intransitive on } V. \quad (11)$$

Lemma 7.1. *Let $g \geq 2$ and let $\psi: \pi_1(S_g) \rightarrow Q_8$ be an epimorphism. Then there is a standard system with $\psi(a_1) = \psi(a_2) = -1$, $\psi(b_1)$ and $\psi(b_2)$ of order 4, and $\psi(a_i), \psi(b_i) \in \{\pm 1\}$ for $i \geq 3$.*

Proof. Let $\bar{\psi} = (\xi, \eta): H_1(S_g; \mathbb{F}_2) \rightarrow \mathbb{F}_2^2$ be induced by ψ , and write $\xi = \omega_2(\cdot, s)$, $\eta = \omega_2(\cdot, t)$; s, t are linearly independent because $\bar{\psi}$ is onto. For a standard system with classes (e_i, f_i) modulo 2, the relation and (10) give

$$0 = \sum_i \det(\bar{\psi}e_i, \bar{\psi}f_i) = \sum_i (\omega_2(e_i, s)\omega_2(f_i, t) + \omega_2(e_i, t)\omega_2(f_i, s)) = \omega_2(s, t),$$

the last equality being the expansion of s and t in the symplectic basis. So s, t span an isotropic plane. Choose a primitive lift $\tilde{e}_1 \in H_1(S_g; \mathbb{Z})$ of s and a symplectic basis $(\tilde{e}_1, \tilde{f}_1, \dots)$ (Lemma 2.5(a), (b)). Modulo $\mathbb{F}_2 s$, the class t is congruent to a non-zero vector t' of the reduction of the complement $\langle \tilde{e}_1, \tilde{f}_1 \rangle^\perp$, because $\omega_2(s, t) = 0$; lift t' to a primitive vector $\tilde{e}_2$ of this complement and complete to a symplectic basis $(\tilde{e}_i, \tilde{f}_i)$ of $H_1(S_g; \mathbb{Z})$. By Section 2.2 it is realised by a standard system. As $\langle s, t \rangle = \langle e_1, e_2 \rangle$ modulo 2, the kernel of $\bar{\psi}$ is the orthogonal complement of $\langle e_1, e_2 \rangle$, which is spanned by e_1, e_2 and e_i, f_i , $i \geq 3$; and $\bar{\psi}(f_1), \bar{\psi}(f_2) \neq 0$. So $\psi(a_1), \psi(a_2), \psi(a_i), \psi(b_i) \in \{\pm 1\}$ for $i \geq 3$, while $\psi(b_1), \psi(b_2)$ have order 4 and square -1 . If $\psi(a_i) = 1$ for $i = 1$ or 2 , replace a_i by $a_i b_i^2$ (two handle moves). $\square$

Proof of Proposition 1.3. Let $\psi: \pi \rightarrow Q_8$ be the linear part of φ . If ψ is not onto, its image is cyclic, $\varphi(\pi) \cap V$ is an abelian normal subgroup of the monodromy group $\varphi(\pi)$ with cyclic quotient, and Theorem 1.4(C2), whose proof in Section 8 does not use the present proposition, gives both (a) and (b). Let ψ be onto, and take a standard system as in Lemma 7.1. Then $\varphi(a_1) = \rho_{c_1}$ and $\varphi(a_2) = \rho_{c_2}$ are reflections, and $X = \varphi(a_3)$, $Y = \varphi(b_3)$ lie in D .

(a) We modify the third handle with handle moves until $\langle \rho_{c_1}, \rho_{c_2}, \varphi(a_3) \rangle$ satisfies (11); then $H(a_1, a_2, a_3)$ of Lemma 2.3(d) has disconnected preimage. If $c_1 = c_2$ nothing has to be done: a translation t_ε and ρ_{c_1} preserve the line $c_1 + \mathbb{F}_p \varepsilon$, and two reflections preserve the line through their centres. Let $c_1 \neq c_2$, $d = c_1 - c_2$, and let ℓ be the line through c_1 and c_2 . We need $\varphi(a_3)$ to be a translation along d or a reflection in a point of ℓ .

If $X = t_\lambda$ and $Y = t_\mu$ are translations, the third handle maps to an abelian group and a word with exponent sums (j, k) in a_3, b_3 maps to $t_{j\lambda + k\mu}$. By Lemma 2.5(c) there is a primitive (j, k) with $j\lambda + k\mu \in \mathbb{F}_p d$; make it the exponent vector of the new a_3 .

Otherwise the pair of linear parts of (X, Y) is not $(1, 1)$; handle moves act transitively on the three other pairs, so we may assume $X = t_\lambda$ and $Y = \rho_e$. If $\lambda \in \mathbb{F}_p d$ we are done. If not, the lines ℓ and $e + \mathbb{F}_p \lambda$ meet in exactly one point $e + \mu\lambda$. The handle moves $b_3 \mapsto b_3 a_3^k$ and then $a_3 \mapsto a_3 b_3$ replace a_3 by $a_3 b_3 a_3^k$, whose image is $t_\lambda \rho_e t_{k\lambda} = \rho_{e + (1-k)\lambda/2}$ by (5); choose k with $(1-k)/2 = \mu$.

(b) Using handle moves in the third handle we may assume that $\varphi(a_3) = t_\lambda$ is a translation. The group of the four-holed sphere $H(a_3, b_3; a_1)$ of Lemma 2.3(c) has image $\langle t_\lambda, Y t_\lambda Y^{-1}, \rho_{c_1} \rangle = \langle t_\lambda, t_{\pm\lambda}, \rho_{c_1} \rangle$, which preserves the line $c_1 + \mathbb{F}_p \lambda$ (any line through c_1 if $\lambda = 0$). $\square$

Remark 7.2 (why genus 2 is special). In genus 2 the complement of H consists of annuli, so the defining relation of $\pi_1(S_2)$ is the relation between the four boundary loops of H , and by (9) the obstruction is a product of lifts of elements of J ; moreover the kernel of $\bar{\psi}$ is a Lagrangian plane, so that no handle maps into D . For $g \geq 3$ a whole handle maps into D , and this is what the proof above exploits.

8 Partial positive results in every genus

Proof of Theorem 1.4. Let φ be the monodromy and $K = \ker \varphi$. A subgroup $L \leq \Gamma$ containing Γ_0 is transitive only if $L = \Gamma$.

(C1) Let N be a maximal subgroup of Γ containing $\Gamma_0[\Gamma, \Gamma]$. It is normal, Γ/N is cyclic of prime order, and N is intransitive. (The cover belonging to $\varphi^{-1}(\Gamma_0[\Gamma, \Gamma])$ is the abelian cover of the statement.) Take a standard system adapted to $\chi: \pi \rightarrow \Gamma/N$ (Lemma 2.5). The normal subgroup $\ker \chi$ contains a_1, a_2, a_3 and all commutators, hence the groups $\langle a_1, a_2 \rangle$, $\langle a_1, [a_1, b_1] \rangle$, $\langle a_1, [a_1, b_1], a_2 \rangle$ and $\langle a_1, a_2, a_3 \rangle$ of the subsurfaces $P(a_1, a_2)$, $P(a_1, b_1)$, $H(a_1, b_1; a_2)$ and, for $g \geq 3$, $H(a_1, a_2, a_3)$ of Lemma 2.3. Their images lie in N .

(C2) If N is intransitive, then $\Gamma_0[\Gamma, \Gamma] \subset \Gamma_0 N \neq \Gamma$ and (C1) applies; likewise if Γ is abelian. Let N be transitive, hence regular, and $\Gamma \neq N$. Take a standard system adapted to $\pi \rightarrow \Gamma/N$ and write $x_i = \varphi(a_i)$, $u_i = \varphi(b_i)$. Then x_1 and all x_i, u_i with $i \geq 2$ lie in the abelian group N , so $[x_i, u_i] = 1$ for $i \geq 2$ and the relation gives $[x_1, u_1] = 1$. If $\langle x_1 \rangle = N$, then u_1 centralises N and $\Gamma = \langle N, u_1 \rangle$ is abelian. Otherwise $N/\langle x_1 \rangle \neq 1$. In the second handle a word with exponent sums (j, k) maps to $x_2^j u_2^k$; we choose a primitive (j, k) with $\langle x_1, x_2^j u_2^k \rangle \neq N$: if $N/\langle x_1 \rangle$ is cyclic, by Lemma 2.5(c) with $x_2^j u_2^k \in \langle x_1 \rangle$; if it is not cyclic, any (j, k) will do. After handle moves, $\langle x_1, x_2 \rangle \neq N$. In the same way, if $g \geq 3$, we achieve $\langle x_1, x_2, x_3 \rangle \neq N$ with handle moves in the third handle. The four groups of (C1) now have the images $\langle x_1, x_2 \rangle$, $\langle x_1 \rangle$, $\langle x_1, x_2 \rangle$ and $\langle x_1, x_2, x_3 \rangle$, proper subgroups of the regular group N , hence intransitive.

(C3) Here $\Gamma = G$ acts regularly, so a subgroup is transitive only if it is G . Let H be a sphere with $k \geq 2$ holes with essential boundary components, $* \in H$, and suppose that the group of H maps onto G . Let e be a simple closed curve disjoint from H . The Dehn twist T_e is supported outside H , so $(T_e)_*$ fixes $\pi_1(H, *)$ elementwise; it preserves K by hypothesis and induces an automorphism of $G = \pi/K$ which fixes $\varphi(\pi_1(H, *)) = G$ pointwise. Hence $\varphi \circ (T_e)_* = \varphi$. If e is nonseparating, there is a loop $\gamma = \gamma_1 \gamma_2$ at $*$ crossing e once, and $(T_e)_*(\gamma) = \gamma_1 e^{\pm 1} \gamma_2$; so $\gamma_1 e^{\pm 1} \gamma_1^{-1} \in K$. Thus K contains (the conjugacy class of) every nonseparating simple closed curve disjoint from H . A boundary curve d of H is freely homotopic to such a curve if it is nonseparating. If it separates, it bounds on the side away from H a surface which is not a disc, hence of genus $r \geq 1$ with one boundary circle; d is a product of commutators of $2r$ simple loops in it which are nonseparating in S_g and can be pushed off H . In both cases $d \in K$. But $\pi_1(H, *)$ is generated by loops freely homotopic to boundary curves, so its image is trivial, not G .

(C4) Among the $|\Gamma| + 1$ elements $1, \varphi(a_1), \varphi(a_1 a_2), \dots, \varphi(a_1 \cdots a_{|\Gamma|})$ two coincide, so $c = a_{i+1} \cdots a_j \in K$ for some $0 \leq i < j \leq |\Gamma|$. If $g \geq r|\Gamma|$, $r \in \{1, 2, 3\}$, we find in this way $c_1, \dots, c_r \in K$, products over disjoint sets of consecutive indices. For $r \geq 2$ the subsurface of Lemma 2.3(e) has group $\langle c_1, \dots, c_r \rangle \subset K$; this is the pair of pants of nonseparating type ($r = 2$) and the four-holed sphere with pairwise non-isotopic boundary curves ($r = 3$). By Lemma 2.3(e) and Lemma 2.4 there is a homeomorphism which maps a simple closed curve freely homotopic to a_1 , resp. two disjoint ones freely homotopic to a_1 and a_2 , to α_1 , resp. α_1, α_2 , which are freely homotopic to c_1, c_2 ; this gives a standard system with $a_1 \in K$, resp. $a_1, a_2 \in K$, and the groups $\langle a_1, [a_1, b_1] \rangle$, resp. $\langle a_1, [a_1, b_1], a_2 \rangle$, of $P(a_1, b_1)$ and $H(a_1, b_1; a_2)$ lie in K . A trivial group is intransitive. $\square$

Remark 8.1. (a) By [FL18, Theorem 1.4], for every $g \geq 2$ the group $\pi_1(S_g)$ has infinitely many finite simple characteristic quotients. By (C3) the corresponding regular covers are as far from counterexamples as possible. (b) In case (C1) the kernel of the monodromy need not contain an essential simple closed curve: the homomorphism of $\pi_1(S_2)$ onto a group of order 32 in [Liv00] has none, and a 2-group is nilpotent. In case (C2) with $g = 2$ it always contains one [Liv00, Theorem 4.2].

	pants of nonseparating type	pants of handle type	four-holed sphere, two isotopic boundary curves	four-holed sphere, pairwise non-isotopic boundary curves
admissible under			(G) only	(G) and (S)
(C1), (C2)	yes	yes	yes	yes if $g \geq 3$
(C3)	all	all	all	all
(C4)	$g \geq 2 \Gamma $	$g \geq \Gamma $	$g \geq 2 \Gamma $	$g \geq 3 \Gamma $
groups Γ_p , $g \geq 3$ (Proposition 1.3)			yes	yes
groups Γ_p , $g = 2$ (Proposition 10.1)	yes	yes	see Theorem 1.2	void

Table 2: The subsurfaces with disconnected preimage provided by Theorem 1.4, Proposition 1.3 and Proposition 10.1. A four-holed sphere with two isotopic boundary curves is admissible only under the literal reading (G); one with pairwise non-isotopic boundary curves under both readings. For $g = 2$ the last column is void (Lemma 3.1).

9 Exact computations in genus two

9.1 Moves and what they prove

Consider the substitutions (generators not mentioned are fixed)

$$\begin{aligned}
A_1: b_1 &\mapsto b_1 a_1, & B_1: a_1 &\mapsto a_1 b_1, & C: b_1 &\mapsto a_2 a_1 b_1, & b_2 &\mapsto a_1 a_2 b_2, \\
B_2: a_2 &\mapsto a_2 b_2, & A_2: b_2 &\mapsto b_2 a_2; \\
G_1: a_1 &\mapsto a_1 b_1^{-1}, & G_2: b_1 &\mapsto b_1 a_1^{-1}, & G_3: a_1 &\mapsto b_2 b_1 a_1, & a_2 &\mapsto b_1 b_2 a_2, \\
G_4: b_2 &\mapsto b_2 a_2, & G_5: a_2 &\mapsto a_2 b_2.
\end{aligned} \tag{12}$$

Each is invertible and maps R_2 to a conjugate of itself in the free group (C and G_3 to $a_1 a_2 R_2 (a_1 a_2)^{-1}$, resp. $b_2 b_1 R_2 (b_2 b_1)^{-1}$, the others to R_2); so each defines an automorphism of $\pi_1(S_2)$ and, by the Dehn–Nielsen–Baer theorem (Section 2.2), maps standard systems to standard systems. On tuples (x, u, v, w) the first five act by $u \mapsto ux$; $x \mapsto xu$; $u \mapsto vxu$, $w \mapsto xvw$; $v \mapsto vw$; $w \mapsto wv$. We call a tuple *failing* if $J(x, u, v, w)$ is intransitive. By Lemma 3.1(c) and Lemma 2.3:

if a tuple obtained from φ by a sequence of the moves (12) and their inverses is failing, then the cover defined by φ has a four-holed sphere with essential boundary components and disconnected preimage, and pairs of pants of both types with disconnected preimage inside it. (13)

(The pairs of pants are $P(a_1, a_2)$ and $P(a_1, b_1)$ for that system, with groups $\langle x, v \rangle$ and $\langle x, uxu^{-1} \rangle$ contained in J .) This direction uses no theorem on generators of the mapping class group.

The moves A_1, B_1, C, B_2, A_2 are induced, up to inner automorphisms and inversion, by the Dehn twists about the simple closed curves a_1, b_1, c, b_2, a_2 , where c is the boundary curve of $P(a_1, a_2)$ homotopic to $a_1 a_2$: a twist fixes the loops disjoint from its curve and inserts the curve, read from the crossing point, into a loop crossing it once; by (4) a push-off of a_1 crosses only b_1 , one of b_1 only a_1 , and c crosses b_1 and b_2 at their initial half-edges, where it reads $a_2 a_1$, resp. $a_1 a_2$. These five curves form a chain (consecutive ones meet once, the others are disjoint). Likewise $G_1, \dots, G_5$ belong to the chain $b_1, a_1, b_1 b_2, a_2, b_2$. By theorems of Lickorish and Humphries [Lic64, Hum79], see [FM12, §4.4] and, for genus 2, Birman and Hilden [BH71], *the Dehn twists about a chain of five curves generate the mapping class group of S_2* . We use this theorem only in Lemma 9.2 below, that is, in the normalised enumerations of the first programs (orbits containing a tuple with $x = 1$ or $v = 1$ contain a failing tuple by Lemma 9.1); the identification of the substitutions with twists was in addition tested by the

programs (symplectic action generating $\mathrm{Sp}(4, \mathbb{Z}/2)$ and $\mathrm{Sp}(4, \mathbb{Z}/3)$, braid and commutation relations decided by Dehn's algorithm, invariance of the chain curves).

Lemma 9.1 (curves in the kernel). *Let $\varphi: \pi_1(S_2) \rightarrow \mathrm{Sym}(n)$, $n \geq 2$, be transitive. If $\ker \varphi$ contains a nonseparating simple closed curve, then there is a standard system whose tuple is failing.*

Proof. By Lemma 2.4 there is a standard system with $\varphi(a_2) = 1$. Then $[x, u] = 1$ and $J = \langle x \rangle$. Handle moves in the first handle replace x by $x^i u^j$ for any coprime (i, j) . If all the groups $\langle x^i u^j \rangle$ were transitive, x would be an n -cycle, $u = x^k$ a power of it, and $(i, j) = (-k, 1)$ would give the identity. $\square$

Lemma 9.2 (normalisation). *Let rk be a function on $\Gamma \leq \mathrm{Sym}(n)$ which is constant on conjugacy classes and invariant under inversion. Every orbit of the mapping class group of S_2 on conjugacy classes of homomorphisms $\pi_1(S_2) \rightarrow \Gamma$ which contains no tuple with $x = 1$ or $v = 1$ contains a tuple (x, u, v, w) such that x is a prescribed representative of its conjugacy class, u has minimal index among its conjugates under the centraliser of x , and $y \neq 1$, $\mathrm{rk}(y) \leq \mathrm{rk}(x)$ for all $y \in \{u, v, w, xu, xu^{-1}, vw, vw^{-1}, xv\}$.*

Proof. Choose a nonseparating simple closed curve whose image has maximal rank and, by Lemma 2.4, a mapping class moving a_1 to it; then conjugate. The listed elements are images of the nonseparating simple closed curves $b_1, a_2, b_2, a_1 b_1^{\pm 1}, a_2 b_2^{\pm 1}$ and c . None of them is in the kernel: otherwise the orbit would contain a tuple with $x = 1$. $\square$

9.2 The programs

Four independent families of programs were used (Section 11 gives the files, commands and running times).

The first programs. `g2search.c` enumerates tuples in $\mathrm{Sym}(n)$: in mode 0 all tuples with x a class representative, in mode 1 only the tuples normalised as in Lemma 9.2 with rk the number of cycles. From each transitive tuple it runs a walk of 3000 random moves and stops at the first failing tuple; tuples which survive are treated by `g2orbit.py`, which searches the orbit breadth-first. `regfull.c` treats a given permutation group G : from each normalised tuple it searches the orbit breadth-first until a failing tuple or a class already treated is found, or the orbit is exhausted; in its “regular” mode a tuple fails if x, uxu^{-1}, v, vvw^{-1} lie in a common member of a conjugation-closed family of proper subgroups, so that $J \neq G$. The enumerations of mode 0 are complete: the number of tuples equals $|\mathrm{Hom}(\pi_1(S_2), \mathrm{Sym}(n))| = n! \sum_{\chi} (n!/\chi(1))^2$ for $n \leq 7$.

The programs of the second verification run use the generators $G_1, \dots, G_5$, no normalisation and no list of groups. `hb.c` treats all homomorphisms to $\mathrm{Sym}(n)$, $n \leq 10$, in blocks. The handle moves in the first handle preserve $[x, u]$; for a representative c of each conjugacy class of even permutations the set $P_c = \{(x, u) : [x, u] = c\}$ is enumerated and decomposed into orbits of these moves. A *block* is a product $O_1 \times gO_2g^{-1}$ of two such orbits, where $gcg^{-1} = c^{-1}$; any two of its tuples differ by a sequence of such moves. Since J is generated by $\langle x, uxu^{-1} \rangle$ and $\langle v, vvw^{-1} \rangle$, a block contains a failing tuple if and only if some proper non-empty subset of the n points is invariant under $\langle x, uxu^{-1} \rangle$ for some $(x, u) \in O_1$ and under $\langle v, vvw^{-1} \rangle$ for some $(v, w) \in gO_2g^{-1}$; this is decided with bit sets. For the blocks of transitive tuples without such a subset, a tuple is followed along a random walk until it fails, and if the walk does not succeed the orbit is enumerated completely. Completeness is checked by the two counts in Table 3, and the block criterion was compared with a brute-force search over all tuples of all transitive blocks for $n \leq 7$. `orb.c` enumerates, for a given group, all homomorphisms modulo conjugation and computes all orbits; `kappa.py` adds the obstruction κ .

The programs of the first verification run treat M_9 only: all 1,592,136 homomorphisms with an independent model of $\tilde{G}_3$ as a group of 3×3 matrices over $\mathbb{F}_4$, the orbit of φ_0 , and a cut-and-glue model of the covers which does not use the formula for the group of H_0 .

The programs of the final verification run were written from the text of this note. `r2_blk.c` treats all homomorphisms to $\text{Sym}(n)$, $n \leq 10$, by blocks as above, with its own enumeration of the sets P_c and with walks in the substitutions $G_2, B_1, C^{-1}, B_2, A_2^{-1}$ of (12) and their inverses, which that run found by a search and tested in the same way; it reproduces every entry of Table 3. `r2_grp.c` applies the method to a permutation group given by generators, through its multiplication table; the criterion with invariant sets is used when $[x, u]$ has at most 14 cycles, and walks otherwise.

n	homomorphisms to $\text{Sym}(n)$	transitive	transitive blocks	by a walk	exceptional
2	16	15	3	0	0
3	486	440	10	1	0
4	34 176	31 650	115	6	0
5	3 858 240	3 626 064	516	16	0
6	824 354 640	792 600 480	8 471	120	0
7	268 020 990 720	260 690 336 640	57 335	196	0
8	135 486 004 792 320	132 905 092 496 400	1 108 649	2 713	0
9	97 519 295 217 254 400	96 124 376 168 956 800	12 119 792	8 405	14 blocks
10	97 018 295 368 129 632 000	95 938 557 393 889 324 800	244 527 705	45 844	0

Table 3: Output of `hb.c` (second verification run); every entry was reproduced by `r2_blk.c` (final verification run). The second column agrees with $n! \sum_{\lambda} (n!/f^{\lambda})^2$, the third with the value obtained from it by the usual recursion. A transitive block not counted in the last two columns (blocks settled by a walk, exceptional blocks) contains a failing tuple. The 14 exceptional blocks for $n = 9$ form one orbit of 1440 classes, with group of order 72; they contain $1440 \cdot 9! = 522,547,200$ homomorphisms.

9.3 Proofs of Theorems 1.5 to 1.7

All three are proofs by computation. By Lemma 2.1(b) a failing subsurface for a quotient cover is failing for the cover.

Proof of Theorem 1.5 (computation). By Table 3, for $2 \leq n \leq 8$ every block of transitive tuples contains a failing tuple or leads to one by a walk; apply (13). Independently, mode 0 of `g2search.c` finds a failing tuple on the walk from every transitive tuple for $n \leq 7$ (1,188,326,736 tuples for $n = 7$), and for $n = 8$ mode 1 treats 813,196,586 normalised tuples, of which 12 survive the walk; for these `g2orbit.py` finds explicit words in the moves leading to failing tuples. The second route uses Lemmas 9.1 and 9.2 for $n = 8$. The last sentence of the theorem follows with Theorem 1.1. $\square$

Proof of Theorem 1.6 (computation). (a), (b) By Table 3 every transitive homomorphism to $\text{Sym}(10)$, and every transitive homomorphism to $\text{Sym}(9)$ outside one orbit $\mathcal{O}$ of 1440 classes modulo $\text{Sym}(9)$, is related by moves to a failing tuple; apply (13). The tuples of $\mathcal{O}$ generate groups of order 72. On the other hand the 622,080 epimorphisms $\pi_1(S_2) \rightarrow M_9$ which do not lift (Remark 5.2) define covers without failing tuple in their orbits, by Theorem 1.1(c) and Lemma 4.1(c). The normaliser of M_9 in $\text{Sym}(9)$ is $\text{AGL}(2, 3)$, of order 432 (it normalises the characteristic subgroup V , whose normaliser is $\text{AGL}(2, 3)$, and Q_8 is normal in $\text{GL}(2, 3)$), and it acts freely on these epimorphisms, the centraliser of M_9 being trivial; so they give $622,080/432 = 1440$ classes modulo $\text{Sym}(9)$. Hence $\mathcal{O}$ consists exactly of these classes. It is one orbit under $G_1, \dots, G_5$, hence under the mapping class group, which preserves this set of classes. `r2_blk.c` finds the same numbers of blocks for $n = 9$ and $n = 10$, and again 14 exceptional blocks, with group of order 72 and $1440 \cdot 9!$ homomorphisms. The first programs find the

same orbit of 1440 classes, starting from φ_0 , and no other exhausted orbit among all homomorphisms to $\text{AGL}(2, 3)$, $\text{PTL}(2, 8)$, $\text{PTL}(2, 9)$ and S_5 acting on 10 points, which by the classical tables [DM96, Appendix B] contain all primitive groups of degree 9 and 10 other than the alternating and symmetric groups (this was not compared with a table, and it is not needed after Table 3).

(c) A transitive group of degree 11 not containing A_{11} is conjugate to a subgroup of $\text{AGL}(1, 11)$, to $\text{PSL}(2, 11)$ or to M_{11} : the eight transitive groups of degree 11 are four groups of orders 11, 22, 55, 110 inside $\text{AGL}(1, 11)$, and $\text{PSL}(2, 11)$, M_{11} , A_{11} , S_{11} [BM83, Table 11A]; the transitive $\text{PSL}(2, 11)$ lies in M_{11} . Subgroups of $\text{AGL}(1, 11)$ are abelian-by-cyclic (Theorem 1.4). For M_{11} on 11 points `orb.c` finds 65,714,472 classes of homomorphisms; the transitive ones form 8 orbits, each containing a failing tuple. `regfull.c` confirms this for M_{11} and $\text{AGL}(1, 11)$, and `r2_grp.c` for M_{11} : each of the 22,145 blocks of transitive tuples contains a failing tuple or leads to one by a walk. $\square$

Proof of Theorem 1.7 (computation). G has a simple quotient. If it is cyclic, G is not perfect and Theorem 1.4(C1) applies. Otherwise it is one of the 16 non-abelian simple groups of order less than $12180 = |\text{PSL}(2, 29)|$: A_5 , $\text{PSL}(2, 7)$, A_6 , $\text{PSL}(2, 8)$, $\text{PSL}(2, 11)$, $\text{PSL}(2, 13)$, $\text{PSL}(2, 17)$, A_7 , $\text{PSL}(2, 19)$, $\text{PSL}(2, 16)$, $\text{PSL}(3, 3)$, $\text{PSU}(3, 3)$, $\text{PSL}(2, 23)$, $\text{PSL}(2, 25)$, M_{11} , $\text{PSL}(2, 27)$ (a classical list, see [CCN⁺85]; their orders are the first sixteen terms of the entry A001034 of [OEI26], and by the entry A109379 no order below 20160 belongs to two simple groups). For each of them `regfull.c`, in its regular mode, finds no orbit of epimorphisms without a tuple with $J \neq G$. `orb.c` confirms this: for the first nine groups in the regular action, and for the other seven in a faithful permutation action, where an intransitive J is a proper subgroup. `r2_grp.c` confirms it for all sixteen groups in faithful transitive actions of degree at most 28. Apply (13) to the regular action. $\square$

The two lists of groups quoted in these proofs were compared by the final verification run with [BM83, Table 11A] and with [OEI26]; the orders of the groups which the programs generate are verified by the programs. Theorem 1.5 and parts (a) and (b) of Theorem 1.6 do not depend on any list.

10 The three-holed question

Proposition 10.1. *Let p be an odd prime and $\varphi: \pi_1(S_2) \rightarrow \Gamma_p = \mathbb{F}_p^2 \rtimes Q_8$ a homomorphism with transitive image. Then there are a pair of pants of nonseparating type and a pair of pants of handle type with disconnected preimage.*

Proof. If the linear part ψ is not onto Q_8 , the monodromy group is abelian-by-cyclic and Theorem 1.4(C2) applies. Otherwise take a standard system as in Lemma 7.1. Then $x = \varphi(a_1)$, $v = \varphi(a_2)$ and uxu^{-1} are reflections, and any two reflections preserve a line; by (11) the groups $\langle x, v \rangle$ and $\langle x, uxu^{-1} \rangle$ of $P(a_1, a_2)$ and $P(a_1, b_1)$ are intransitive. $\square$

Proof of Proposition 1.8. (a) follows from Lemma 3.2 and Lemma 2.1(c); (b) is Proposition 10.1. (c) By Theorems 1.5 and 1.6 a cover of degree at most 10 has a four-holed sphere with disconnected preimage, and then pairs of pants of both types with disconnected preimage, unless it is one of the covers of Theorem 1.1(c), for which (b) applies. $\square$

For f_0 itself: the group $\langle a_1, a_2 \rangle$ of $P(a_1, a_2)$ has image $\langle T \rangle$, with three orbits; and for the standard system $(a_1 b_1^2, b_1, a_2, b_2)$ the group of the pair of pants of handle type has image $\langle TU^2, U(TU^2)U^{-1} \rangle$, of order 6, with two orbits. On the orbit of φ_0 (1440 classes) the group $\langle x, v \rangle$ is intransitive for 190 classes and $\langle x, uxu^{-1} \rangle$ for 240 (computation).

By the same computations no counterexample to (Q3) exists in genus 2 among the covers of degree 11 whose monodromy group does not contain A_{11} , among the regular covers with deck group of order

less than 12180, among the covers whose monodromy group is one of the last seven groups in the proof of Theorem 1.7 in its action on 11 to 28 points, nor among the covers of Remark 6.3. We know of no mechanism that would force all pairs of pants to have connected preimage: in Lemma 5.1 the essential point is that the complement of H consists of annuli, and the complement of a pair of pants does not. (Q3) remains open.

11 Computations and reproducibility

Theorems 1.1 and 1.2, Propositions 1.3 and 1.8(a), (b), Propositions 6.1 and 10.1 and Theorem 1.4 are proved without computation; the finite verifications inside their proofs are printed above (the relation and the surjectivity of φ_0 , the formulas (6)–(8), the value of $\kappa(\varphi_0)$, and Proposition 6.1 for $p = 5$), except that Proposition 6.1 for $p = 13$ is a finite verification left to the programs. Theorems 1.5 to 1.7, Proposition 1.8(c) and Remark 5.2(b), Remarks 6.2 and 6.3 are computations. The reproducibility package has five parts.

The program of this note (`writing_stage/check_note.py`, standard library, a few seconds; 78 checks). It checks the statements as printed: (A) the maps (1), the permutations (2), the relation in the four conventions, and that T, U, W generate the group M_9 of order 72; (B) the product rule (6) for $p = 3$ (associativity on all 216^3 triples, the centre, the projection to M_9) and (7), (8); (C) $\kappa(\varphi_0) = [0, 2; 1]$ for all 81 choices of lifts; (D) all 1,592,136 homomorphisms $\pi_1(S_2) \rightarrow M_9$, the counts of Remark 5.2(b), the 68 subgroups of M_9 (62 intransitive), and the case distinction in the proof of Lemma 5.1 (19,008 generating tuples with $J \cap V = 0$ and 6912 with $|J \cap V| = 3$); (E) the ten substitutions (12) and the orbit of φ_0 (1440 classes for both sets of moves, J transitive on all); (F) the pairs of pants of Section 10 and Remark 5.3; (G) Theorem 1.2(b) for the 13 primes $p \equiv 3 \pmod{4}$ below 100; (H) Proposition 6.1; (I) the identities (5) and (10); (J) hypothesis (E) for the groups of Remark 6.3; (K) the cyclic order (4), the position of the diagonals δ_i , and the boundary curves of the subsurfaces of Lemma 2.3(a)–(d), by tracing the faces of the corresponding ribbon graphs.

The programs with which the results were first obtained (`original/`), with all recorded outputs: the checks `verify_m9.py`, `verify_family.py` and `verify_m9_polygon.py`; `validate_twists.py` (the moves, and a model of the cover built from n octagons, against which (3) was tested on 63,678 homomorphisms); the searches `g2search.c`, `g2orbit.py` and `regfull.c` with two predecessors of the latter. The long runs were `g2search` for $n = 7$ in mode 0 and for $n = 8$ in mode 1 (about 12 and 22 minutes on six processor cores).

The programs of the two verification runs (`verification_run_A/`, `verification_run_B/`), with their outputs; among them `hb.c` (for $n = 10$ about four minutes and 5.5 GB of memory, one process) and `orb.c` (the longest run, for Γ_{13} , about twelve minutes). C programs are compiled with `cc -O2 -o NAME NAME.c` (Apple clang 21.0.0 was used); for groups with more than 4095 elements `orb.c` is compiled with the additional option `-DWIDE`.

The programs of the final verification run are in the folder `independent_run_2/`, with their outputs: `r2_moves.py`, `r2_m9.py` (55 checks), `r2_ribbon.py`, `r2_family.py`, `r2_fam.c`, `r2_blk.c` (for $n = 10$ about two minutes and 1.8 GB), `r2_grp.c` (at most 18 seconds for each group) and `r2_lists.py`.

Re-runs. When this note was written, the quick programs of the first four parts were run again from a copy of the package (one process, about seven minutes); the script `run_quick.sh` repeats this, and the log is in the package. All outputs agree with the recorded ones. Four slow programs were re-run separately, with identical results: `hb.c` for $n = 10$, `orb.c` for M_{11} , and the genus-3 tests of the two verification runs. The final verification run repeated `run_quick.sh` from the archive, and `hb.c` for $n = 10$ and `orb.c` for M_{11} , again without any difference.

Verification. The results were first obtained with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs, the first run before it opened those of the first stage.

The first run examined the statement and the main example. It read the problem session in the source and the proposer’s thesis, and fixed the conventions and the two readings. It re-derived the classification of the four-holed spheres of S_2 , their transitivity under homeomorphisms, and the group of H_0 in two ways (by cutting the octagon along two segments, and by van Kampen’s theorem for the two handles); it proved the invariance of the obstruction, built the central extension independently as a group of unitary 3×3 matrices over $\mathbb{F}_4$ of order 216, found the same obstruction for φ_0 , and re-proved the main lemma for M_9 . Its programs enumerate all 1,592,136 homomorphisms to M_9 , compute the orbit of φ_0 (1440 classes) and all orbits on epimorphisms (three, of 4320, 4320 and 5760 classes modulo M_9), and count components in a cut-and-glue model of the cover, with one component for each of the 1440 classes and no disagreement with (3) on 10,500 random homomorphisms. It found no gap in the main claim and required five corrections, all made: the statement for $g \geq 3$ in both readings (Proposition 1.3, whose part (a) it proved); the convention next to the permutations; the qualification “genus 2, boundary curves isotopic in pairs” in every summary; a weaker wording for the choice of reading; and the citations for the generators of the mapping class group.

The second run examined the family, the positive results, the computations and the literature. It proved Lemma 5.1 in the present generality, Theorem 1.2, part (a) of Proposition 1.3 independently, and (C1)–(C4); it found the examples of Proposition 6.1 and Remark 6.3. With its programs `hb.c` and `orb.c`, which use another chain of curves and another method, it re-did all computations and extended them: all homomorphisms to $\text{Sym}(n)$ for $n \leq 10$ (Table 3), which removed a restriction on the monodromy group from Theorem 1.6; M_{11} ; the 16 simple groups; Γ_p for $p \leq 13$. All numbers which it compared agree with those of the first programs. It read the sources listed below and found no gap. It required the credits of Section 1.3 and six corrections, all made: the two readings in Proposition 1.3; the citations of [Liv00], [FL18] and [DT06] in their present form; the presentation of the obstruction as the classical invariant; and the statement of the Dehn–Nielsen–Baer theorem where substitutions are used. Its two suggestions, the unrestricted form of Theorem 1.6 and the general form of Lemma 5.1, were adopted.

When this note was written, the arguments were reorganised in four places: Lemma 2.2, from which all subsurfaces and their groups are now derived (the first run had checked the group of H_0 by the two derivations mentioned above); the definition of standard systems through automorphisms, with the Dehn–Nielsen–Baer theorem, and the lifting step in the proof of Lemma 7.1; the proof of (C4) through Lemma 2.3(e); and the closed form $\omega(b, Jb) = \alpha$ in Theorem 1.2(b) together with the printed computation in the proof of Proposition 6.1. The program `check_note.py` tests the printed statements.

A third independent verification run, also AI-assisted, was therefore made on the text of this note (the final verification run); its programs, in `independent_run_2/`, were written without reading those of the package. It examined the four places first: it recomputed the cyclic order (4) from the relator, re-derived Lemma 2.2 and, by tracing the boundaries of the ribbon graphs, the boundary curves of Lemma 2.3(a)–(e) for $g = 2, 3, 4$; confirmed the group of H_0 with a cut-and-glue model of its own (one component for f_0 ; the number of components agrees with (3) on 46,638 homomorphisms to M_9); and re-derived Lemmas 3.1, 4.1 and 7.1, (C4), Theorem 1.2(b) and every printed value in the proof of Proposition 6.1. It then read all other proofs line by line, read Problem 12 again in the source, compared the statements quoted from [Liv00, DT06, FL18] with the arXiv versions and the two lists of groups with [BM83, OEI26], and repeated the literature search. Its programs recompute all finite claims: the relation in the four conventions, M_9 , the central extension and $\kappa(\varphi_0)$; all homomorphisms to M_9 (Remark 5.2(b)); the orbits on epimorphisms under five substitutions found by a search (each

maps the relator to a conjugate of itself; together they act on $H_1(S_2; \mathbb{Z}/3)$ as $\mathrm{Sp}(4, \mathbb{Z}/3)$, with J transitive on the whole orbit of φ_0 ; all homomorphisms to Γ_p , $p \leq 13$ (the numbers of classes of Remark 6.2 for each value of κ); every entry of Table 3; M_{11} in degree 11; and the sixteen groups of Theorem 1.7. All numbers agree with those of the earlier programs. The run found no wrong theorem, proposition or lemma and no gap. It required twelve corrections of the text and of the package, which were made; none changes a statement (the scope was completed in the abstract and in the first paragraph, and the two lists of groups were compared with sources).

Theorem 1.6(a), (b) for monodromy groups containing A_9 or A_{10} had rested on one program, `hb.c`; it is now confirmed by `r2_blk.c`, which uses the same decomposition into blocks and was written independently. The orbit counts of Remark 6.3 rest on the programs of the second run alone.

Scope and priority. This note gives a partial answer to Problem 12 of [Gek18]. It answers the four-holed question in the negative in genus 2 only, where every four-holed sphere with essential boundary components is the complement of two disjoint curves and has boundary curves isotopic in pairs. Under the strict reading (four pairwise non-isotopic boundary curves) genus 2 contains no such subsurface, the question starts at genus 3, and there it is open under both readings. The example is not a regular cover and its Galois closure is not a counterexample. The three-holed question and the questions for regular covers remain open. The problem is due to D. Gekhtman. The obstruction is the classical invariant in the second homology of the group [Liv85, Zim87, Edm82, Edm83, DT06], and its use follows the model of the lifting invariant of Hurwitz theory [Ser90, Fri10, FV91]; adapted generating systems are due to Nielsen and the first step of (C2) to Edmonds [Liv00]; the pigeonhole argument of (C4) is in [DT06].

What was read. The problem session of [Gek18] (pp. 2525–2529) in the open file of the Oberwolfach repository, by the first run; its p. 2528 by the second run, again when this note was written, and by the final run. Of [Gek19] the abstract, and by the first two runs a text extraction searched for the retract statement and for the hypothesis on H , without result. [Liv00] in full in the arXiv version, to which the numbers of its Lemma 4.1 and Theorem 4.2 refer (its Section 4 again by the final run); the journal version was not compared. Of [DT06], in the arXiv version 3, whose numbering we use, Section 6 by the second run and §§6.12–6.18 again at the writing and by the final run. Of [Liv85] the first page. Of [FL18], arXiv version 2, the introduction with Theorem 1.4. Of [BPS24] the introduction, and of [MP19, KS16, Klu25, PW13] the abstracts, each with a keyword search of the full text of the arXiv version, by the second run. Of [Fri10] the abstract. Of [BM83], in the scanned copy linked from [OEI26], Tables 11A to 11C, and of [OEI26] the three entries named in the bibliography, by the final run. Not read: [Edm82, Edm83, Zim87, Ser90, FV91], which are cited as they are described in [Liv00, DT06] and in the abstract of [Fri10]; the books [FM12, DM96, CCN⁺85, Rob96, Hat02], which are cited by chapter or section for standard theorems (the theorem numbers of [FM12] for the generation by twists were not available); and the papers [Lic64, Hum79, BH71]. The proofs of the cited theorems were not checked. All bibliographic data were checked with Crossref, zbMATH or the arXiv.

The searches were made on 9 October 2026: queries to the arXiv (about forty in all, for the proposer, for holomorphic retracts of Teichmüller spaces, for preimages and lifts of subsurfaces, pairs of pants and simple closed curves in finite covers, for lifting invariants and for orbits of mapping class groups on homomorphisms to finite groups), to Crossref, OpenAlex, Semantic Scholar and zbMATH, including the lists of works citing the report (one citing work was listed, unrelated), and six web searches. The final run repeated the search on the same day in a smaller form (twenty queries and one web search). No work stating, answering or citing Problem 12 was found. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments,

to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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