

A Bounded Convex Hilbert Domain with Vanishing Squeezing

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Abstract

We construct a bounded convex balanced domain in the complex Hilbert space $\mathbb{C} \oplus \ell^2$ that is not holomorphic homogeneous regular. The domain is $\Omega = \{(z, w) : |z| + \|w\|_\infty < 1, \|w\|_2 < 1\}$. Its squeezing function at $(t, 0)$ is bounded above and below by positive constant multiples of $1 - t$ as t increases to 1. The upper bound follows from explicit holomorphic retractions onto polydiscs of increasing dimension and an elementary Hilbert-space Schwarz argument. This gives a negative answer to the convex Hilbert domain question in Kim's 2018 Oberwolfach report. No boundary smoothness is claimed, and the separate question about squeezing constants of arbitrary bounded homogeneous domains is not addressed.

1 The domain and the result

Let H be a complex Hilbert space and write $\mathbb{B}_H$ for its open unit ball. For a bounded domain $D \subset H$, define

$$\sigma_D(p) = \sup\{r > 0 : F(p) = 0, \quad r\mathbb{B}_K \subset F(D) \subset \mathbb{B}_K\},$$

where the supremum is over holomorphic embeddings F into Hilbert balls. As in [2], an embedding is biholomorphic onto its image, which is a domain. The choice of the Hilbert target does not affect the invariant. The domain is *holomorphic homogeneous regular* (HHR) if $\inf_{p \in D} \sigma_D(p) > 0$.

Every bounded convex domain in $\mathbb{C}^n$ is HHR [5]. In contrast, Question 5 in Kim's report [4, p. 2271] asks whether bounded convex domains in infinite dimensional Hilbert spaces are HHR. The positive theorem for uniformly elliptic Hilbert domains [2, Theorem 5.3] has an additional geometric hypothesis. The following example answers Question 5 negatively.

Equip $H = \mathbb{C} \oplus \ell^2(\mathbb{N})$ with $\|(z, w)\|_H^2 = |z|^2 + \|w\|_2^2$ and set

$$\Omega = \{(z, w) \in H : |z| + \|w\|_\infty < 1, \quad \|w\|_2 < 1\}. \quad (1)$$

Theorem 1. *The set Ω is a bounded convex balanced domain. For $0 \leq t < 1$, put $p_t = (t, 0)$ and $\delta = 1 - t$. Then*

$$\frac{\delta}{\sqrt{10}} \leq \sigma_\Omega(p_t) \leq \frac{1}{\sqrt{\lfloor \delta^{-2} \rfloor}} \leq \sqrt{2} \delta. \quad (2)$$

Moreover, $\text{dist}_H(p_t, \partial\Omega) = \delta/\sqrt{2}$. In particular, Ω is not HHR.

The supremum norm is continuous on ℓ^2 , so both strict inequalities in (1) define open sets in the Hilbert topology. Their defining functions are convex, and the set is balanced. Also

$$\frac{1}{\sqrt{2}}\mathbb{B}_H \subset \Omega \subset \sqrt{2}\mathbb{B}_H, \quad (3)$$

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because $|z| + \|w\|_\infty \leq |z| + \|w\|_2 \leq \sqrt{2}\|(z, w)\|_H$, whereas membership in Ω implies $|z| < 1$ and $\|w\|_2 < 1$. Thus Ω has nonempty interior and is connected. It is the unit ball of the equivalent complex norm

$$\|(z, w)\|_* = \max\{|z| + \|w\|_\infty, \|w\|_2\}.$$

The conclusion therefore holds even for a balanced convex domain uniformly sandwiched between two concentric Hilbert balls.

2 A polydisc retract bound

We use the classical Hilbert-space obstruction behind ball–polydisc embedding estimates; see Alexander [1] and the infinite dimensional discussion in [2, Lemma 4.1]. The formulation below avoids assumptions about boundary extensions.

Lemma 2. *Let X be a complex domain admitting holomorphic maps $i : \mathbb{D}^n \rightarrow X$ and $R : X \rightarrow \mathbb{D}^n$ with $R \circ i = \text{id}_{\mathbb{D}^n}$. If $p = i(0)$, then every Hilbert-ball squeezing radius of X at p is at most $1/\sqrt{n}$.*

Proof. Let $F : X \rightarrow \mathbb{B}_K$ be an admissible embedding with $F(p) = 0$, and suppose $r\mathbb{B}_K \subset F(X)$ for $r > 0$. Write $G = F^{-1}|_{r\mathbb{B}_K}$ and $A = (F \circ i)'(0) : \mathbb{C}^n \rightarrow K$. For every sign vector $\varepsilon \in \{-1, 1\}^n$, the map

$$\zeta \mapsto F(i(\zeta\varepsilon))$$

is holomorphic from $\mathbb{D}$ to $\mathbb{B}_K$ and sends zero to zero. Applying the scalar Schwarz lemma after any norm-one continuous linear functional on K gives $\|A\varepsilon\|_K \leq 1$. The Hilbert norm and sign cancellation yield

$$\sum_{j=1}^n \|Ae_j\|_K^2 = 2^{-n} \sum_{\varepsilon \in \{-1, 1\}^n} \|A\varepsilon\|_K^2 \leq 1. \quad (4)$$

For each j , the function $R_j \circ G : r\mathbb{B}_K \rightarrow \mathbb{D}$ vanishes at zero. Schwarz's lemma on complex lines gives

$$\|\lambda_j\|_{K^*} \leq r^{-1}, \quad \lambda_j = (R_j \circ G)'(0).$$

The identity $R \circ G \circ F \circ i = \text{id}$ holds near zero. Differentiating it gives $\lambda_j(Ae_k) = \delta_{jk}$. In particular,

$$1 = |\lambda_j(Ae_j)| \leq r^{-1}\|Ae_j\|_K,$$

so $\|Ae_j\|_K \geq r$. Equation (4) now implies $nr^2 \leq 1$. Taking suprema over admissible embeddings and contained radii proves the assertion. $\square$

The proof uses only finitely many signs and coordinate functionals, even when K is infinite dimensional. In particular it uses neither compactness of a Hilbert unit sphere nor boundary regularity of X .

3 Increasing retract dimension and the upper bound

Fix $0 \leq t < 1$ and an integer $n \geq 1$ satisfying $n\delta^2 \leq 1$. Define

$$i_{t,n}(u) = (t, \delta(u_1, \dots, u_n, 0, \dots)), \quad u \in \mathbb{D}^n, \quad (5)$$

$$R_n(z, w) = \left(\frac{w_1}{1-z}, \dots, \frac{w_n}{1-z} \right), \quad (z, w) \in \Omega. \quad (6)$$

The first map takes values in Ω : for $u \in \mathbb{D}^n$,

$$t + \delta\|u\|_\infty < t + \delta = 1, \quad \delta\|u\|_2 < \delta\sqrt{n} \leq 1.$$

These are strict inequalities even when $n\delta^2 = 1$. For $(z, w) \in \Omega$, the denominator in (6) does not vanish and

$$|w_j| < 1 - |z| \leq |1 - z|.$$

Consequently R_n is holomorphic into $\mathbb{D}^n$. Direct substitution gives $R_n \circ i_{t,n} = \text{id}$ and $i_{t,n}(0) = p_t$. Lemma 2 proves

$$\sigma_\Omega(p_t) \leq \frac{1}{\sqrt{n}}.$$

Take $n = \lfloor \delta^{-2} \rfloor$. Since $0 < \delta \leq 1$, this is a positive integer and $n \geq \delta^{-2}/2$. The two upper bounds in (2) follow. In particular, at $t_m = 1 - 1/m$ the domain has a holomorphic retract of dimension m^2 centered at p_{t_m} and $\sigma_\Omega(p_{t_m}) \leq 1/m$.

4 The lower bound and boundary distance

If $|a|^2 + \|b\|_2^2 < \delta^2/2$, then

$$|t + a| + \|b\|_\infty \leq t + |a| + \|b\|_2 < t + \delta = 1, \quad \|b\|_2 < \delta/\sqrt{2} < 1.$$

Thus

$$B_H(p_t, \delta/\sqrt{2}) \subset \Omega. \tag{7}$$

The point $q_t = (t + \delta/2, (\delta/2)e_1)$ lies on $\partial\Omega$ and has $\|q_t - p_t\|_H = \delta/\sqrt{2}$. It belongs to the boundary because the first constraint is an equality, the second is strict, and points of Ω approach it. Together with (7), this proves the exact boundary-distance formula.

Every $(z, w) \in \Omega$ satisfies

$$|z - t| < 1 + t, \quad \|w\|_2 < 1.$$

Hence

$$F_t(z, w) = \frac{(z, w) - p_t}{\sqrt{(1+t)^2 + 1}}$$

is an admissible affine embedding into $\mathbb{B}_H$. By (7) its image contains the ball of radius

$$\frac{\delta}{\sqrt{2((1+t)^2 + 1)}} \geq \frac{\delta}{\sqrt{10}}.$$

This completes the proof of Theorem 1.

5 Scope and reproducibility

The boundary of (1) is not asserted to be smooth. Question 5 of [4] imposes no smoothness or uniform ellipticity. The two-sided estimate shows that squeezing vanishes at a linear rate along one explicit path to one boundary point, despite the fixed concentric-ball inclusions (3). Question 6 of the same report, concerning general bounded homogeneous domains, is separate and is not resolved here.

The proof does not invoke the nearest-boundary-point arguments in [2]; a correction to that paper is recorded in [3]. All boundary points needed above are given explicitly. A supplementary standard-library Python checker uses exact Gaussian rational arithmetic to test the retraction identities, ball inclusions, floor estimates, convex combinations, and Hilbert sign-average identity. Those finite checks supplement the proof and do not certify an infinite dimensional theorem by enumeration.

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References

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