

On a Determinant of Teufl and Wagner Related to Set Partitions: Divisibility by the Catalan Power of $\det A$

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Abstract

Teufl and Wagner posed the following problem (Problem 1 of their contribution to Oberwolfach Report 23/2018, pp. 1457–1458). Let $A = (a_{ij})$ be an $n \times n$ matrix of indeterminates, and let $T = T(A)$ be the matrix, indexed by the set partitions of an n -set, whose entry (P, Q) is the sum of the weights of the transitions from P to Q ; a transition is a set of edges between two copies X, Y of the n -set which completes spanning trees of the blocks of P in X to a forest all of whose components meet Y and induce Q on Y . Is $\det T$ always divisible by $(\det A)^{C_n}$, where C_n is the n -th Catalan number? The report states $\det T = (\det A)^{B_n}$ for $n \leq 3$ (B_n the Bell number) and $\det T = (\det A)^{14}$ per A for $n = 4$, an experimental factorisation for $n = 5$ and experimental evidence for $n = 6$. We prove that the answer is yes for every n : in $\mathbb{Z}[a_{ij}]$ one has $\det T = (\det A)^{C_n} R_n$, where R_n is the determinant of $T(A)$ on an invariant subspace K of dimension $B_n - C_n$ which does not depend on A , and the map induced by $T(A)$ on the quotient, which has the non-crossing partitions as a basis, has determinant exactly $(\det A)^{C_n}$. The proof uses the Grassmann-algebra representation of forests of Caracciolo, Sokal and Sportiello, in which $T(A)$ becomes an integral operator $G(A)$ with $G(-A^\top)G(A) = (-1)^n(\det A)^2$. These authors announced, without proof, that the image of the representation has the non-crossing partitions as a basis; we did not find their proof in print (the source file of their preprint contains a draft) and give one for the case needed. The subspace K , the dimension and the planar basis of the quotient and the integrality of the projection are already contained in the work of Kenyon and Wilson on boundary partitions in trees, together with the meander determinant of Di Francesco, Golinelli and Guitter. The note adds the determinant on the quotient, hence the divisibility, the factorisation over $\mathbb{Z}[a_{ij}]$, a proof of the formula for $n = 4$, and the exactness of the exponent C_n for $4 \leq n \leq 8$ (computer-assisted for $n \geq 5$). A second proof, found by one of the independent verification runs, uses the tree pairing of Kenyon and Wilson and the matrix-tree theorem. Factors of R_5 and R_6 are only tested; exactness for $n \geq 9$ and a description of R_n remain open. The wording of the UnsolvedMath record OWR-16164-020 mixes in the title of the next problem of the report. A literature search found no answer, which is not a proof of novelty; the divisibility is close to the published structure theory and may be known to specialists. This is an unrefereed note.

1 Introduction

1.1 The problem

In the problem session of the Oberwolfach workshop *Enumerative Combinatorics* of May 2018, Teufl and Wagner described a family of matrices which arises, according to the source, from transfer matrices in the enumeration of spanning trees of ladder-like graphs [TW18]. We recall the definitions exactly as they are given there.

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Let $n \geq 1$, let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ be disjoint sets of vertices, and let $A = (a_{ij})$ be an $n \times n$ matrix of independent indeterminates; a_{ij} is the weight of the edge $x_i y_j$. Let $\mathcal{P}$ be a set partition of X and $\mathcal{Q}$ a set partition of Y . A set S of edges $x_i y_j$ (the empty set is allowed) is a *transition from $\mathcal{P}$ to $\mathcal{Q}$* if the following holds: take a spanning tree on every block of $\mathcal{P}$ and add the edges of S to the union of these trees; the result must be a forest on $X \cup Y$ such that every component contains at least one vertex of Y and such that the components induce the partition $\mathcal{Q}$ on Y . The *weight* of S is $w(S) = \prod_{x_i y_j \in S} a_{ij}$. The matrix $T = T(A)$ is the square matrix of size B_n , the n -th Bell number, with the entries

$$t_{\mathcal{P}, \mathcal{Q}} = \sum_{S \text{ transition from } \mathcal{P} \text{ to } \mathcal{Q}} w(S),$$

where the partitions of X index the rows, the partitions of Y index the columns, and rows and columns are ordered in the same way. All this is on p. 1457 of [TW18], together with the example $n = 2$,

$$T = \begin{pmatrix} a_{11}a_{22} + a_{11}a_{21} + a_{12}a_{22} + a_{12}a_{21} & a_{11}a_{12}a_{21} + a_{11}a_{12}a_{22} + a_{11}a_{21}a_{22} + a_{12}a_{21}a_{22} \\ a_{11} + a_{12} + a_{21} + a_{22} & a_{11}a_{22} + a_{11}a_{12} + a_{12}a_{21} + a_{21}a_{22} \end{pmatrix}, \quad (1)$$

whose first row and first column belong to the partitions into singletons. On p. 1458 the source names the matrix A , states that $\det T = (\det A)^{B_n}$ for $n = 1, 2, 3$ and that $\det T = (\det A)^{14} \cdot \text{per } A$ for $n = 4$, reports as an experimental observation that $\det T = (\det A)^{42} \cdot P_1 \cdot P_2$ for $n = 5$ with homogeneous polynomials P_1, P_2 of degrees 20 and 30, and asks:

Problem 1.1 ([TW18, Problem 1]). “Is it true that $(\det A)^{C_n}$ is always a factor of $\det T$ ”, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number?

The source adds that there is experimental evidence for $n = 6$ and nothing beyond, because of the size of the matrices. It cites no reference, and it does not ask whether the exponent C_n is best possible. We have $B_n = 1, 2, 5, 15, 52, 203, 877, 4140$ and $C_n = 1, 2, 5, 14, 42, 132, 429, 1430$ for $n = 1, \dots, 8$; so $n = 4$ is the first case with $C_n < B_n$.

Identification. Ordering rows and columns in the same way presupposes the identification $x_i \leftrightarrow y_i$. We therefore index rows and columns by the set Π_n of all set partitions of $[n] = \{1, \dots, n\}$: a partition $P \in \Pi_n$ stands for the partition $\{\{x_i : i \in B\} : B \in P\}$ of X and for the corresponding partition of Y , and we write $t_{P, Q}$ for $P, Q \in \Pi_n$. Another identification $y_i \leftrightarrow x_{\pi(i)}$ permutes the columns of T and changes $\det T$ at most by a sign.

Trees. The condition does not depend on the choice of the spanning trees. Indeed, let F_P be a union of spanning trees of the blocks of P in X . The components of $F_P \cup S$ are the classes of the equivalence relation generated by the blocks of P and the edges of S , and $F_P \cup S$, which has $n - |P| + |S|$ edges, is a forest if and only if this number equals $2n$ minus the number of components. In particular, a transition from P to Q has exactly

$$|S| = n + |P| - |Q| \quad (2)$$

edges, where $|P|$ denotes the number of blocks. Hence $t_{P, Q}$ is zero or homogeneous of degree $n + |P| - |Q|$, and every term of the Leibniz expansion of $\det T$ is zero or homogeneous of degree $\sum_P (n + |P| - |\sigma(P)|) = nB_n$.

Factor. We read “factor” as divisibility in the polynomial ring $\mathbb{Z}[a_{ij}]$. Since $\det A$ is a primitive polynomial, this is the same as divisibility in $\mathbb{Q}[a_{ij}]$, by Gauss’s lemma. A divisibility in $\mathbb{Z}[a_{ij}]$ persists after every specialisation of A in a commutative ring.

The problem is record OWR-16164-020 of the UnsolvedMath collection [ula26], where it is listed as open. The statement of this record places the matrix T in a conjectured uniform presentation

of the pure braid group and prints the exponent on the line. That phrase is the title of the next contribution to the problem session, by McCammond and Williams, which begins on p. 1458 directly after Problem 1.1 and is unrelated to it; the matrix of the record is the matrix $T(A)$ defined above, and the question is the divisibility by $(\det A)^{C_n}$.

1.2 Results

Let $V := \mathbb{Q}^{\Pi_n}$ be the space of row vectors with the basis $(e_P)_{P \in \Pi_n}$, on which $T(A)$ acts by $v \mapsto vT(A)$, so that $e_P T(A) = \sum_Q t_{P,Q} e_Q$. A partition is *non-crossing* if there are no $a < b < c < d$ such that a, c lie in one block and b, d in another block; $\text{NC}_n \subseteq \Pi_n$ denotes the set of non-crossing partitions. It has C_n elements [Kre72].

Theorem 1.2. *Let $n \geq 1$. There is a subspace $K \subseteq V$ of dimension $B_n - C_n$, which does not depend on A , with the following properties.*

- (a) *K is invariant under $T(A)$: for $v \in K$ the vector $vT(A)$ is a combination of elements of K with coefficients in $\mathbb{Q}[a_{ij}]$.*
- (b) *The classes of the e_P , $P \in \text{NC}_n$, form a basis of V/K , and the endomorphism of V/K induced by $T(A)$ has the determinant $(\det A)^{C_n}$.*
- (c) *In $\mathbb{Z}[a_{ij}]$,*

$$\det T = (\det A)^{C_n} \cdot R_n, \quad R_n := \det(T(A)|_K),$$

where $R_n \in \mathbb{Z}[a_{ij}]$ is homogeneous of degree $n(B_n - C_n)$ and $R_n(I) = 1$ for the identity matrix I .

In particular $(\det A)^{C_n}$ is a factor of $\det T$ for every n : the answer to Problem 1.1 is yes.

Explicitly, K is the kernel of the linear map Φ of Section 3. It is also the span of the vectors of “Rule 1” of Kenyon and Wilson and the radical of their tree pairing (Proposition 6.1); see Section 1.3 for what is taken from the literature.

Corollary 1.3. (a) *For $n \leq 3$ we have $K = 0$ and $\det T = (\det A)^{B_n}$.*

(b) *For $n = 4$ the space K is spanned by*

$$\kappa := e_{13|24} + e_{12|34} + e_{14|23} - e_{1|234} - e_{2|134} - e_{3|124} - e_{4|123}, \quad (3)$$

we have $\kappa T(A) = \text{per}(A) \kappa$, and $\det T = (\det A)^{14} \cdot \text{per } A$.

So the two statements of the source for $n \leq 4$ are theorems. For $n = 5$ the degree of the cofactor is $5 \cdot (52 - 42) = 50 = 20 + 30$, in agreement with the observation of the source.

Proposition 1.4 (exactness of the exponent; computer-assisted for $n \geq 5$). *For $4 \leq n \leq 8$ the polynomial R_n is not divisible by $\det A$. Hence C_n is the exact power of $\det A$ in $\det T$ for these n .*

For $n = 4$ this follows from Corollary 1.3. For $5 \leq n \leq 8$ it is proved by exact computations at explicit integer matrices (Section 7.3, Table 1); these are computer-assisted proofs. For $n \geq 9$ exactness is open. The factors of R_5 and R_6 described in Section 7.4 are only tested.

1.3 What is known, what is new, and what stays open

The method. Spanning forests have a representation by anticommuting variables, introduced in [CJS⁺04] and developed, for hypergraphs, in [CSS07]: to a set B of vertices one attaches an even element f_B of a Grassmann algebra such that the product of the f_e over the edges of a forest is the product of the f_C over its components, the product over the edges of a graph with a cycle is 0 [CSS07, Corollary 4.4], and integrating out a vertex removes it from its component [CSS07, Lemma 5.1]. With $\Phi(P) := \prod_{B \in P} f_B$ this turns $T(A)$ into an integral operator $G(A)$ (Proposition 3.1), which $G(-A^T)$ inverts up to the factor $(-1)^n(\det A)^2$ (Proposition 3.2). So the determinant of $G(A)$ on the image W of Φ is $(\det A)^{\dim W}$ (Section 5), and $\dim W = C_n$ (Section 4).

What was known. (i) The representation and its elements are from [CJS⁺04, CSS07]; Lemmas 2.1 and 2.2 below are special cases of [CSS07, Lemma 4.1, Corollaries 4.3 and 4.4, Lemma 5.1]. The numbers refer to version 2 of the arXiv text of [CSS07].

(ii) At the end of Section 4 of [CSS07], after Corollary 4.5 and around equation (4.22), the authors announce the following for the subalgebra generated by their elements $f_A^{(\lambda)}$ with a fixed λ : the products f_C over the partitions $\mathcal{C}$ of the vertex set are linearly dependent as soon as there are four vertices; those with $\mathcal{C}$ non-crossing form a basis, so that the dimension is the Catalan number; and all relations are generated by a relation between four vertices, which for $\lambda = 0$ is (8) below. The proofs are deferred there to a paper in preparation [CSS07, reference 39]. We did not find that paper, and we could not determine whether the proof has appeared. The TeX source of both arXiv versions of [CSS07] contains, in lines which are commented out and are not part of the text, a draft of an appendix on this subalgebra: the statement, a verification of the relation between four vertices by expansion, and a sketch of a proof in two steps, a reduction of crossings by this relation and a distinguished monomial for every non-crossing partition. The draft was found in the last verification run, after Section 4 had been written. Section 4 must therefore be read as a proof of an announced statement, in the case $\lambda = 0$ which we need, and not as a new result; the proof of Proposition 4.2 is close in idea to the second step of the draft, and the bound of Proposition 4.3 is obtained in another way.

(iii) Kenyon and Wilson [KW11] met the same quotient in the study of groves in electrical networks. Their “Rule 1” [KW11, §1.2] is the relation between four vertices, with sets of vertices in place of the vertices. By [KW11, Theorem 1.2] every partition is transformed by repeated application of Rule 1 into a combination of planar, that is non-crossing, partitions, which does not depend on the choices made. By [KW11, Theorem 2.6] the coefficients are integers, and the combination is characterised by a bilinear pairing on partitions [KW11, (3), (4), §2.3], whose Gram matrix on the planar partitions is non-singular by the meander determinant of Di Francesco, Golinelli and Guitter [DFGG97, (5.6), (5.14)–(5.18)], as quoted in [KW11, §2.1–2.2]. (The numbers refer to version 4 of the arXiv text of [KW11] and to the arXiv text of [DFGG97].) Hence the span of the Rule 1 relations is the radical of the pairing, the quotient has the dimension C_n and the classes of the planar partitions as a basis, and the projection is integral. Our K is this subspace (Proposition 6.1); so the structure of K and of V/K is not new.

(iv) According to their abstracts, the space of planar electrical networks with n boundary vertices embeds into a Grassmannian [Lam18], in fact into a Lagrangian Grassmannian [CGS21, BGKT23], which lies in the projectivisation of a space of dimension C_n carrying a fundamental representation of a symplectic group [BGKT23]. This suggests a reason for the exponent C_n ; none of this is used here.

What is new, to our knowledge. The determinant of the map induced by $T(A)$ on V/K , which rests on Proposition 3.2 or, in the second proof, on the identity $T(A) M T(-A)^T = (-1)^n(\det A)^2 M$ for the tree pairing M (Proposition 6.5); hence the divisibility asked for in Problem 1.1; the factorisation over $\mathbb{Z}[a_{ij}]$ with the cofactor $R_n = \det(T(A)|_K)$; the proof of the formula for $n = 4$; and Proposition 1.4. Given (ii) and (iii) the argument is short, and the divisibility may be known to specialists. We make no claim of priority; the searches and their limits are described in the paragraph “Scope and priority”.

What stays open. Exactness of the exponent for $n \geq 9$, and a description of the cofactor R_n for $n \geq 5$ (Section 7.4).

2 Grassmann variables and forests

Throughout, $\mathbb{k}$ denotes the field $\mathbb{Q}$ or the polynomial ring $\mathbb{Q}[a_{ij}]$. For a finite set S let Λ_S be the exterior algebra over $\mathbb{k}$ of the free module with the basis $\psi_v, \bar{\psi}_v$ ($v \in S$). It is free of rank $4^{|S|}$ with the monomials in the generators, in a fixed order, as a basis; it is graded by the degree; elements of even degree are central, and elements of odd degree anticommute. For $S' \subseteq S$ we regard $\Lambda_{S'}$ as a subalgebra of Λ_S . For a $\mathbb{Q}$ -vector space U we write $U_{\mathbb{k}} := U \otimes_{\mathbb{Q}} \mathbb{k}$.

We use two rules. For elements $a_1, \dots, a_k, \bar{a}_1, \dots, \bar{a}_k$ of degree 1,

$$\prod_{t=1}^k (\bar{a}_t a_t) = (-1)^{k(k-1)/2} (\bar{a}_1 \cdots \bar{a}_k)(a_1 \cdots a_k), \quad (4)$$

because moving $\bar{a}_t$ to the left across $a_1, \dots, a_{t-1}$ costs the sign $(-1)^{t-1}$. And if $b_i = \sum_{j=1}^m m_{ij} c_j$ ($1 \leq i \leq k$) with elements c_j of degree 1 and $m_{ij} \in \mathbb{k}$, then

$$b_1 \cdots b_k = \sum_{J \subseteq [m], |J|=k} \det(m_{ij})_{i \in [k], j \in J} c_J, \quad (5)$$

where c_J is the product of the c_j , $j \in J$, in increasing order. In particular $b_1 \cdots b_k = 0$ if one of the b_i is a combination of the others.

For $u \neq v$ in S put

$$\eta_{uv} := \psi_u - \psi_v, \quad \bar{\eta}_{uv} := \bar{\psi}_u - \bar{\psi}_v, \quad f_{uv} := \bar{\eta}_{uv} \eta_{uv} = (\bar{\psi}_u - \bar{\psi}_v)(\psi_u - \psi_v).$$

Then $f_{uv} = f_{vu}$ is even and $f_{uv}^2 = 0$. For a set H of edges on S put $f_H := \prod_{e \in H} f_e$, with $f_{\emptyset} = 1$; the order of the factors is irrelevant.

Lemma 2.1 (forests). (a) *If H contains a cycle, then $f_H = 0$.*

(b) *If τ and τ' are trees with the same vertex set $B \subseteq S$, then $f_{\tau} = f_{\tau'}$. We denote this element by f_B ; thus $f_B = 1$ if $|B| = 1$.*

(c) *If H is a forest on S whose components have the vertex sets $C_1, \dots, C_m$, isolated vertices included, then $f_H = \prod_{k=1}^m f_{C_k}$.*

Proof. (a) By (4), $f_H = \pm (\prod_{e \in H} \bar{\eta}_e) (\prod_{e \in H} \eta_e)$, where every edge is given an orientation. If $v_0, v_1, \dots, v_l = v_0$ is a cycle of H , then $\sum_t (\psi_{v_t} - \psi_{v_{t+1}}) = 0$; so one of the η_e is a combination of the others with coefficients ± 1 , and $\prod_{e \in H} \eta_e = 0$.

(b) Let $k = |B| - 1$ and let M_B be the group of all $\sum_{v \in B} c_v \psi_v$ with $c_v \in \mathbb{Z}$ and $\sum_v c_v = 0$; it is free abelian of rank k . For $u, v \in B$ the element $\psi_u - \psi_v$ is a sum of elements $\pm \eta_e$ along the path from u to v in τ . So the k elements η_e ($e \in \tau$) generate M_B , and therefore form a basis of it. The same holds for τ' . Hence $\eta_e = \sum_{e' \in \tau'} m_{ee'} \eta_{e'}$ with an integer matrix m of determinant ± 1 , and the same matrix expresses the $\bar{\eta}_e$ by the $\bar{\eta}_{e'}$. By (5), $\prod_{e \in \tau} \eta_e = \det(m) \prod_{e' \in \tau'} \eta_{e'}$, and likewise with bars; by (4), with the same sign on both sides, $f_{\tau} = \det(m)^2 f_{\tau'} = f_{\tau'}$.

(c) H is the union of trees on the sets C_k , and the factors commute. $\square$

Next we define the integral over the two variables of a vertex. Fix $v \in S$. Every $g \in \Lambda_S$ can be written uniquely as

$$g = g_0 + \bar{\psi}_v g_1 + \psi_v g_2 + \bar{\psi}_v \psi_v g_3 \quad (g_i \in \Lambda_{S \setminus \{v\}}),$$

and we put $\int_v g := g_3 \in \Lambda_{S \setminus \{v\}}$. The following rules hold.

- (I1) If $h \in \Lambda_{S \setminus \{v\}}$, then $\int_v(hg) = h \int_v g$ and $\int_v(gh) = (\int_v g)h$. Indeed, for homogeneous h we have $hg = hg_0 \pm \psi_v hg_1 \pm \bar{\psi}_v hg_2 + \bar{\psi}_v \psi_v hg_3$, because $\bar{\psi}_v \psi_v$ is central; and similarly for gh .
- (I2) $\int_u \int_v = \int_v \int_u$ for $u \neq v$: both sides extract the coefficient of the central monomial $\bar{\psi}_u \psi_u \bar{\psi}_v \psi_v$. For a set U of vertices we write $\int_U$ for the composition of the $\int_v$, $v \in U$.
- (I3) $\int_v 1 = 0$, and $\int_v f_{vu} = 1$ for $u \neq v$, because $f_{vu} = \bar{\psi}_u \psi_u + \bar{\psi}_v(-\psi_u) + \psi_v \bar{\psi}_u + \bar{\psi}_v \psi_v \cdot 1$.
- (I4) $\int_v$ maps elements of degree d to elements of degree $d - 2$.

Lemma 2.2 (integration). *Let $v \in C \subseteq S$. If $|C| \geq 2$, then $\int_v f_C = f_{C \setminus \{v\}}$. If $C = \{v\}$, then $\int_v f_C = 0$.*

Proof. If $|C| \geq 2$, take a tree on C in which v is a leaf with the neighbour u . Then $f_C = f_{C \setminus \{v\}} f_{vu}$ with $f_{C \setminus \{v\}} \in \Lambda_{S \setminus \{v\}}$, and (I1), (I3) give the claim. If $C = \{v\}$, then $f_C = 1$. $\square$

The elements f_B are the elements $f_B^{(\lambda)}$ of [CSS07, (4.2)] for $\lambda = 0$, by [CSS07, (4.3c) and Corollary 4.3]. Lemma 2.1 is the case $\lambda = 0$ of [CSS07, Lemma 4.1, Corollaries 4.3 and 4.4] for graphs, and Lemma 2.2 is the case $t_i = \lambda = 0$ of [CSS07, Lemma 5.1(b)].

3 The representation Φ and the operator $G(A)$

Let $\Lambda_n := \Lambda_{[n]}$. For a partition $P \in \Pi_n$ put

$$\Phi(P) := \prod_{B \in P} f_B \in \Lambda_n,$$

and extend this to a $\mathbb{Q}$ -linear map $\Phi : V \rightarrow \Lambda_n$ (with $\mathbb{k} = \mathbb{Q}$). Let

$$K := \ker \Phi \subseteq V, \quad W := \Phi(V) \subseteq \Lambda_n.$$

The element $\Phi(P)$ is homogeneous of degree $2(n - |P|)$. Since $\mathbb{k}$ is free as a $\mathbb{Q}$ -module, the kernel and the image of the $\mathbb{k}$ -linear extension of Φ are $K_{\mathbb{k}}$ and $W_{\mathbb{k}}$.

Now let X and Y be as in Section 1.1 and work in $\Lambda_{X \cup Y}$. For an $n \times n$ matrix A with entries in $\mathbb{k}$ we define

$$G(A) : \Lambda_X \rightarrow \Lambda_Y, \quad G(A)g := \int_X \left(g \cdot \prod_{i,j=1}^n (1 + a_{ij} f_{x_i y_j}) \right). \quad (6)$$

By means of the isomorphisms $\Lambda_X \cong \Lambda_n \cong \Lambda_Y$ which rename x_i and y_i to i , we regard $G(A)$ as an endomorphism of Λ_n ; its matrix in the monomial basis has entries which are polynomials in the a_{ij} .

Proposition 3.1 (intertwining). *For every $P \in \Pi_n$ and every matrix A with entries in $\mathbb{k}$,*

$$G(A) \Phi(P) = \sum_{Q \in \Pi_n} t_{P,Q}(A) \Phi(Q), \quad \text{that is} \quad \Phi(vT(A)) = G(A) \Phi(v) \quad (v \in V_{\mathbb{k}}).$$

Consequently $K_{\mathbb{k}} T(A) \subseteq K_{\mathbb{k}}$ and $G(A) W_{\mathbb{k}} \subseteq W_{\mathbb{k}}$, and under the isomorphism $V/K \cong W$ induced by Φ the map induced by $T(A)$ on V/K corresponds to the restriction of $G(A)$ to W .

Proof. Let F_P be a union of spanning trees of the blocks of P in X ; by Lemma 2.1(c) the element $\Phi(P)$, written in the variables of X , is f_{F_P} . The factors in (6) are even, so the product expands as $\sum_S w(S) f_S$ over all sets S of edges $x_i y_j$, and

$$G(A) \Phi(P) = \sum_S w(S) \int_X f_{F_P \cup S}.$$

By Lemma 2.1, $f_{F_P \cup S} = 0$ unless $F_P \cup S$ is a forest, and then it equals $\prod_k f_{C_k}$ over the components $C_k \subseteq X \cup Y$. We integrate the vertices of X one at a time. When $v \in C_k$ is integrated, the other factors are even and free of v ; so by (I1) and Lemma 2.2 only the factor of C_k changes: its set loses the vertex v if another vertex is left, and the factor becomes 0 if v was the last one. A component which contains a vertex of Y never loses its last vertex and ends as $f_{C_k \cap Y}$; a component inside X ends as 0. Hence $\int_X f_{F_P \cup S}$ equals $\prod_k f_{C_k \cap Y}$, which is $\Phi(Q)$ in the variables of Y , if S is a transition from P to Q , and it is 0 if S is not a transition from P to any partition. Summing over S gives the first formula. The consequences are immediate. $\square$

Proposition 3.2 (the inverse). *For every $n \times n$ matrix A with entries in $\mathbb{k}$,*

$$G(-A^\top) \circ G(A) = (-1)^n (\det A)^2 \cdot \text{id}_{\Lambda_n}.$$

Proof. We use three copies X, Y, Z of the vertex set and work in $\Lambda_{X \cup Y \cup Z}$. Let

$$E_1 := \prod_{i,j} (1 + a_{ij} f_{x_i y_j}), \quad E_2 := \prod_{i,j} (1 - a_{ij} f_{y_j z_i});$$

E_2 is the product which defines $G(-A^\top) : \Lambda_Y \rightarrow \Lambda_Z$, because the entry (j, i) of $-A^\top$ is $-a_{ij}$. For $g \in \Lambda_X$ we get from (I1) and (I2)

$$G(-A^\top)G(A)g = \int_Y \left(\left(\int_X g E_1 \right) E_2 \right) = \int_X \int_Y (g E_1 E_2) = \int_X (g \cdot J), \quad J := \int_Y (E_1 E_2).$$

Computation of J . For an even element ν with $\nu^2 = 0$ we have $1 + \nu = \exp(\nu)$, and the exponential of a sum of commuting nilpotent elements is the product of the exponentials. Hence $E_1 E_2 = \exp(\sum_{i,j} a_{ij} (f_{x_i y_j} - f_{y_j z_i}))$. Put

$$\begin{aligned} u_i &:= \psi_{x_i} - \psi_{z_i}, & \bar{u}_i &:= \bar{\psi}_{x_i} - \bar{\psi}_{z_i}, & \tau_i &:= \bar{\psi}_{x_i} \psi_{x_i} - \bar{\psi}_{z_i} \psi_{z_i}, \\ r_i &:= \sum_j a_{ij}, & \beta_j &:= \sum_i a_{ij} u_i, & \bar{\beta}_j &:= \sum_i a_{ij} \bar{u}_i. \end{aligned}$$

Expanding the two products one finds

$$f_{x_i y_j} - f_{y_j z_i} = \tau_i - \bar{u}_i \psi_{y_j} - \bar{\psi}_{y_j} u_i;$$

the terms $\bar{\psi}_{y_j} \psi_{y_j}$ cancel. So the exponent is $\sum_i r_i \tau_i - \sum_j \nu_j$ with $\nu_j := \bar{\beta}_j \psi_{y_j} + \bar{\psi}_{y_j} \beta_j$. All the τ_i and both summands of each ν_j are even, hence $E_1 E_2 = \exp(\sum_i r_i \tau_i) \prod_j \exp(-\nu_j)$. The two summands $a := \bar{\beta}_j \psi_{y_j}$ and $b := \bar{\psi}_{y_j} \beta_j$ of ν_j satisfy $a^2 = b^2 = 0$ and $ab = \bar{\beta}_j \psi_{y_j} \bar{\psi}_{y_j} \beta_j = \bar{\psi}_{y_j} \psi_{y_j} \beta_j \bar{\beta}_j$, so that

$$\exp(-\nu_j) = 1 - a - b + ab = 1 + \psi_{y_j} \bar{\beta}_j - \bar{\psi}_{y_j} \beta_j + \bar{\psi}_{y_j} \psi_{y_j} \beta_j \bar{\beta}_j, \quad \int_{y_j} \exp(-\nu_j) = \beta_j \bar{\beta}_j.$$

The other factors are even and free of y_j . By (I1) therefore $J = \exp(\sum_i r_i \tau_i) \prod_{j=1}^n (\beta_j \bar{\beta}_j)$. By (5) we have $\beta_1 \cdots \beta_n = \det(A) u_1 \cdots u_n$ and $\bar{\beta}_1 \cdots \bar{\beta}_n = \det(A) \bar{u}_1 \cdots \bar{u}_n$; using (4) twice we obtain

$$\prod_{j=1}^n (\beta_j \bar{\beta}_j) = (\det A)^2 \prod_{i=1}^n (u_i \bar{u}_i) = (-1)^n (\det A)^2 \delta, \quad \delta := \prod_{i=1}^n \bar{u}_i u_i = \prod_{i=1}^n f_{x_i z_i}.$$

Properties of δ . We have $u_i \delta = \bar{u}_i \delta = 0$. Let $\sigma : \Lambda_X \rightarrow \Lambda_Z$ be the isomorphism of algebras which renames x_i to z_i . For a generator γ of Λ_X the difference $\gamma - \sigma(\gamma)$ is one of the $u_i, \bar{u}_i$, so $\gamma \delta = \sigma(\gamma) \delta$.

For a monomial $\gamma_1 \cdots \gamma_k$ it follows by induction on k that $\gamma_1 \cdots \gamma_k \delta = \sigma(\gamma_1) \cdots \sigma(\gamma_k) \delta$: replace γ_k by $\sigma(\gamma_k)$, move $\sigma(\gamma_k)$ to the front, apply the induction hypothesis, and move it back; the two signs cancel. Hence $g\delta = \sigma(g)\delta$ for all $g \in \Lambda_X$. In particular $\tau_i \delta = 0$, so $\exp(\sum_i r_i \tau_i) \delta = \delta$ and $J = (-1)^n (\det A)^2 \delta$.

Conclusion. By (I1), (I3), applied to the factors of δ one after the other, $\int_X \delta = 1$. Therefore

$$G(-A^\top)G(A)g = (-1)^n (\det A)^2 \int_X \sigma(g) \delta = (-1)^n (\det A)^2 \sigma(g) \int_X \delta = (-1)^n (\det A)^2 \sigma(g),$$

and under the identification of Λ_Z with Λ_X the element $\sigma(g)$ is g . $\square$

4 The image of Φ

In this section $\mathbb{k} = \mathbb{Q}$. We prove that $W = \Phi(V)$ has the dimension C_n and the basis $\Phi(P)$, $P \in \text{NC}_n$. As explained in Section 1.3(ii), this is the case $\lambda = 0$ of a statement announced without proof in [CSS07]; in the equivalent form “the classes of the planar partitions are a basis of V/K ” it follows from [KW11, DFGG97], see Proposition 6.1.

For $1 \leq t \leq n-1$ put $\chi_t := \psi_t - \psi_{t+1}$ and $\bar{\chi}_t := \bar{\psi}_t - \bar{\psi}_{t+1}$. These $2(n-1)$ elements of degree 1 are linearly independent, so the subalgebra $\Lambda' \subseteq \Lambda_n$ generated by them is an exterior algebra with the basis

$$\bar{\chi}_S \chi_{S'} := \prod_{s \in S} \bar{\chi}_s \cdot \prod_{s \in S'} \chi_s \quad (S, S' \subseteq [n-1]; \text{ products in increasing order}).$$

For $i < j$ we have $\eta_{ij} = \chi_i + \cdots + \chi_{j-1}$. Hence all f_{ij} and all $\Phi(P)$ lie in Λ' , that is $W \subseteq \Lambda'$.

4.1 Arcs

An *arc* of a partition $P \in \Pi_n$ is a pair (b, b') of elements $b < b'$ of one block such that no element of this block lies between them. P has $r = n - |P|$ arcs; different arcs have different left endpoints b and different right endpoints b' . The *interval* of the arc (b, b') is $I = \{b, b+1, \dots, b'-1\} \subseteq [n-1]$, and we put $\xi_I := \sum_{t \in I} \chi_t = \eta_{bb'}$ and $\bar{\xi}_I := \sum_{t \in I} \bar{\chi}_t$. Let $I_1, \dots, I_r$ be the intervals of the arcs of P , ordered by their left endpoints, and for an r -subset S of $[n-1]$ let

$$D_S(P) := \det(\mathbf{1}[s \in I_k])_{1 \leq k \leq r, s \in S} \in \mathbb{Z},$$

with the columns in the increasing order of s . Taking in every block the path through its elements in increasing order as the spanning tree, we obtain from (4) and (5)

$$\Phi(P) = \prod_{k=1}^r \bar{\xi}_{I_k} \xi_{I_k} = (-1)^{r(r-1)/2} \sum_{S, S'} D_S(P) D_{S'}(P) \bar{\chi}_S \chi_{S'}, \quad (7)$$

where the sum is over all r -subsets S, S' of $[n-1]$.

Let $\alpha(P) \subseteq [n-1]$ be the set of the left endpoints of the arcs of P , and let $\omega(P)$ be the set of the last points $b'-1$ of their intervals; both sets have r elements. For r -subsets S, S' of $[n-1]$ with the elements $s_1 < \cdots < s_r$ and $s'_1 < \cdots < s'_r$ we write $S \leq S'$ if $s_i \leq s'_i$ for all i . This holds if and only if $|S \cap [m, n]| \leq |S' \cap [m, n]|$ for all m , and if and only if $|S \cap [1, m]| \geq |S' \cap [1, m]|$ for all m . (If $s_i > s'_i$, then for $m = s_i$ the set S has at least $r - i + 1$ elements $\geq m$ and S' has at most $r - i$.)

Lemma 4.1. (a) *If P is non-crossing, then $D_{\alpha(P)}(P) = \pm 1$ and $D_{\omega(P)}(P) = \pm 1$.*

(b) *Let $Q \in \Pi_n$ have r arcs, and let S be an r -subset of $[n-1]$ with $D_S(Q) \neq 0$. Then $\alpha(Q) \leq S \leq \omega(Q)$.*

(c) A non-crossing partition P is determined by the pair $(\alpha(P), \omega(P))$.

Proof. (a) Let (a, a') and (b, b') be arcs of P with $a < b < a'$. Then $b' < a'$: we have $b' \neq a'$, and $a < b < a' < b'$ is impossible, both if the two arcs lie in one block, because a and a' are consecutive in it, and if they lie in different blocks, because P is non-crossing. Hence the intervals of two different arcs are disjoint, or one of them is properly contained in the other. It follows for two different arcs j and k that the left endpoint of I_j lies in I_k if and only if $I_j \subsetneq I_k$, and that the last point of I_j lies in I_k if and only if $I_j \subsetneq I_k$; for if $I_k \subsetneq I_j$, the two arcs would have the same left endpoint, respectively the same right endpoint. Order the arcs by the lengths of their intervals. Then the matrices $(\mathbf{1}[\min I_j \in I_k])_{k,j}$ and $(\mathbf{1}[\max I_j \in I_k])_{k,j}$ are triangular with the diagonal entries 1. Up to the order of rows and columns, their determinants are $D_{\alpha(P)}(P)$ and $D_{\omega(P)}(P)$.

(b) Some term of the Leibniz expansion of $D_S(Q)$ is not zero: there is a bijection $k \mapsto s_k$ from the arcs of Q onto S with $s_k \in I_k$. For every m , the arcs with a left endpoint $\geq m$ have $s_k \geq m$; so $|S \cap [m, n]| \geq |\alpha(Q) \cap [m, n]|$, that is $\alpha(Q) \leq S$. The arcs whose interval has its last point $\leq m$ have $s_k \leq m$; so $|S \cap [1, m]| \geq |\omega(Q) \cap [1, m]|$, that is $S \leq \omega(Q)$.

(c) The right endpoints of the arcs of P are the numbers c with $c - 1 \in \omega(P)$. Go through $c = 1, \dots, n$ and keep a list of the left endpoints whose arcs have not yet ended. If c is a right endpoint, then the arc which ends at c starts at the largest element b of the list: if it started at an element $a < b$ of the list, the arc (b, b') would satisfy $a < b < c < b'$, which contradicts the first step of the proof of (a). Remove b from the list; then add c to the list if $c \in \alpha(P)$. In this way $(\alpha(P), \omega(P))$ determines the arcs of P , hence P . $\square$

4.2 The basis of non-crossing partitions

Proposition 4.2. *The elements $\Phi(P)$, $P \in \text{NC}_n$, are linearly independent.*

Proof. $\Phi(P)$ is homogeneous of degree $2(n - |P|)$. So we fix r and consider the set NC_n^r of the non-crossing partitions with r arcs. For $P \in \text{NC}_n^r$ and any $Q \in \Pi_n$ with r arcs let

$$c_{P,Q} := D_{\alpha(P)}(Q) \cdot D_{\omega(P)}(Q) \in \mathbb{Z};$$

by (7) this is, up to the sign $(-1)^{r(r-1)/2}$, the coefficient of $\bar{\chi}_{\alpha(P)}\chi_{\omega(P)}$ in $\Phi(Q)$. On NC_n^r we define $P \preceq Q$ if $\alpha(Q) \leq \alpha(P)$ and $\omega(P) \leq \omega(Q)$. This is a partial order; it is antisymmetric by Lemma 4.1(c). By Lemma 4.1(a), $c_{P,P} = \pm 1$. By Lemma 4.1(b), $c_{P,Q} \neq 0$ implies $\alpha(Q) \leq \alpha(P)$ and $\omega(P) \leq \omega(Q)$, that is $P \preceq Q$. Hence, if NC_n^r is enumerated along a linear extension of $\preceq$, the integer matrix $c = (c_{P,Q})_{P,Q \in \text{NC}_n^r}$ is triangular with the diagonal entries ± 1 . If $\sum_{Q \in \text{NC}_n^r} \lambda_Q \Phi(Q) = 0$, the coefficients of the elements $\bar{\chi}_{\alpha(P)}\chi_{\omega(P)}$ give $c\lambda = 0$, and so $\lambda = 0$. $\square$

Proposition 4.3. *$\dim W = C_n$. Hence $(\Phi(P))_{P \in \text{NC}_n}$ is a basis of W , $\dim K = B_n - C_n$, and the classes of the e_P , $P \in \text{NC}_n$, form a basis of V/K . Moreover $W = \{g \in \Lambda' : Eg = Fg = 0\}$ for the derivations E, F defined in the proof.*

Proof. A linear endomorphism of the space spanned by the generators of an exterior algebra extends uniquely to a derivation D of the algebra, that is $D(gh) = D(g)h + gD(h)$. Let E and F be the derivations of Λ_n with

$$E\bar{\psi}_v = \psi_v, \quad E\psi_v = 0, \quad F\psi_v = \bar{\psi}_v, \quad F\bar{\psi}_v = 0,$$

and let $H := EF - FE$. Then H is the derivation with $H\psi_v = \psi_v$ and $H\bar{\psi}_v = -\bar{\psi}_v$, and $[H, E] = 2E$, $[H, F] = -2F$, because both sides are derivations which agree on the generators. E, F and H map Λ' into itself, since $E\bar{\chi}_t = \chi_t$, $E\chi_t = 0$, $F\chi_t = \bar{\chi}_t$ and $F\bar{\chi}_t = 0$. On Λ' the operator H is diagonal:

$H \bar{\chi}_S \chi_{S'} = (|S'| - |S|) \bar{\chi}_S \chi_{S'}$. Let $m := n - 1$ and let $M_k \subseteq \Lambda'$ be the eigenspace of H for the eigenvalue k . Then $\dim M_k = \sum_s \binom{m}{s} \binom{m}{s+k} = \binom{2m}{m+k}$, and E maps M_k into M_{k+2} , F maps M_k into M_{k-2} .

W is annihilated by E and F . For $\eta = \sum_t c_t \chi_t$ and $\bar{\eta} = \sum_t c_t \bar{\chi}_t$ we have $E(\bar{\eta}\eta) = \eta\eta = 0$ and $F(\bar{\eta}\eta) = \bar{\eta}\bar{\eta} = 0$. So E and F annihilate every f_{ij} and, being derivations, every product of such elements.

The space $\{g \in \Lambda' : Eg = Fg = 0\}$ has dimension at most $\dim M_0 - \dim M_2$. If $Eg = Fg = 0$, then $Hg = 0$, so $g \in M_0$. Thus it suffices to show that $E : M_0 \rightarrow M_2$ is surjective. On M_k we have $EF = FE + k$. The non-zero eigenvalues of FE on M_k are eigenvalues of EF on M_{k+2} , since $FE|_{M_k}$ and $EF|_{M_{k+2}}$ are the two compositions of the maps $E : M_k \rightarrow M_{k+2}$ and $F : M_{k+2} \rightarrow M_k$. By descending induction on $k \geq 1$, which starts with $M_k = 0$ for $k > m$, all complex eigenvalues of EF on M_k are real and positive: they are of the form $\mu + k$, where μ is 0 or an eigenvalue of EF on M_{k+2} . Hence EF is invertible on M_2 , and $E : M_0 \rightarrow M_2$ is surjective.

Conclusion. We obtain $\dim W \leq \binom{2m}{m} - \binom{2m}{m+2} = \binom{2m+2}{m+1} - \binom{2m+2}{m+2} = C_{m+1} = C_n$, where the first equality follows from Pascal's rule applied twice. With Proposition 4.2 we get $\dim W = C_n$, and all the inclusions are equalities. $\square$

The operators E , F and $-H$ are the operators X_- , X_+ and X_0 of [CSS07, (7.5)]; for $\lambda \neq 0$ the invariance of the elements $f_A^{(\lambda)}$ under them is part of [CSS07, Section 7].

Corollary 4.4 (integrality). *For every $Q \in \Pi_n$ there are unique integers $\pi_{Q,P}$ ($P \in \text{NC}_n$) with $\Phi(Q) = \sum_{P \in \text{NC}_n} \pi_{Q,P} \Phi(P)$. The vectors*

$$n_Q := e_Q - \sum_{P \in \text{NC}_n} \pi_{Q,P} e_P \quad (Q \in \Pi_n \setminus \text{NC}_n)$$

form a basis of K , and together with the e_P , $P \in \text{NC}_n$, they form a basis of the lattice $\mathbb{Z}^{\Pi_n}$.

Proof. By Proposition 4.3 there are unique rational numbers $\pi_{Q,P}$ with this property. Let Q have r arcs; then only partitions P with r arcs occur. Comparing the coefficients of $\bar{\chi}_{\alpha(P')} \chi_{\omega(P')}$ for $P' \in \text{NC}_n^r$ gives $c_{P',Q} = \sum_{P \in \text{NC}_n^r} c_{P',P} \pi_{Q,P}$ in the notation of the proof of Proposition 4.2. The numbers $c_{P',Q}$ are integers, and the matrix $(c_{P',P})$ is integral and triangular with the diagonal entries ± 1 , so its inverse is integral. Hence $\pi_{Q,P} \in \mathbb{Z}$. The vectors n_Q lie in K and are linearly independent, and their number is $\dim K$; the matrix which expresses the n_Q and the e_P by the basis $(e_Q)_{Q \in \Pi_n}$ is triangular with the diagonal entries 1. $\square$

4.3 The relation between four vertices

Lemma 4.5. *For distinct vertices a, b, c, d ,*

$$f_{ab}f_{cd} + f_{ac}f_{bd} + f_{ad}f_{bc} = f_{\{b,c,d\}} + f_{\{a,c,d\}} + f_{\{a,b,d\}} + f_{\{a,b,c\}}. \quad (8)$$

Consequently, for pairwise disjoint non-empty subsets A, B, C, D of $[n]$ and a partition ϱ of the complement of their union, the vector

$$\kappa(A, B, C, D; \varrho) := e_{AC|BD|\varrho} + e_{AB|CD|\varrho} + e_{AD|BC|\varrho} - e_{A|BCD|\varrho} - e_{B|ACD|\varrho} - e_{C|ABD|\varrho} - e_{D|ABC|\varrho} \quad (9)$$

lies in K . Here $AC|BD|\varrho$ denotes the partition with the blocks $A \cup C$, $B \cup D$ and the blocks of ϱ , and so on.

Proof. We may assume $(a, b, c, d) = (1, 2, 3, 4)$. Put $q_t := \bar{\chi}_t \chi_t$ and $p_{st} := \bar{\chi}_s \chi_t + \bar{\chi}_t \chi_s$. These elements are even, and

$$q_t^2 = 0, \quad q_s p_{st} = q_t p_{st} = 0, \quad p_{12} p_{23} = -q_2 p_{13};$$

in the last product only the terms $\bar{\chi}_1 \chi_2 \bar{\chi}_2 \chi_3 = -q_2 \bar{\chi}_1 \chi_3$ and $\bar{\chi}_2 \chi_1 \bar{\chi}_3 \chi_2 = -q_2 \bar{\chi}_3 \chi_1$ survive. Since $\eta_{13} = \chi_1 + \chi_2$, $\eta_{24} = \chi_2 + \chi_3$ and $\eta_{14} = \chi_1 + \chi_2 + \chi_3$, we have

$$f_{12} = q_1, \quad f_{23} = q_2, \quad f_{34} = q_3, \quad f_{13} = q_1 + q_2 + p_{12}, \quad f_{24} = q_2 + q_3 + p_{23}, \quad f_{14} = q_1 + q_2 + q_3 + p_{12} + p_{23} + p_{13}.$$

By Lemma 2.1, $f_{\{2,3,4\}} = f_{23} f_{34}$, $f_{\{1,3,4\}} = f_{13} f_{34}$, $f_{\{1,2,4\}} = f_{12} f_{24}$ and $f_{\{1,2,3\}} = f_{12} f_{23}$. With the rules above, both sides of (8) expand to $2(q_1 q_2 + q_1 q_3 + q_2 q_3) + q_1 p_{23} + p_{12} q_3$.

For the second statement choose $a \in A$, $b \in B$, $c \in C$ and $d \in D$. By Lemma 2.1 we have $f_{A \cup C} = f_A f_C f_{ac}$, $f_{B \cup C \cup D} = f_B f_C f_D f_{\{b,c,d\}}$, and so on. Hence Φ maps the vector (9) to the product of $f_A f_B f_C f_D$, $\prod_{B' \in \mathcal{Q}} f_{B'}$ and the difference of the two sides of (8), which is 0. $\square$

The relation (8) is the relation $R_{abcd} = 0$ of [CSS07, (4.22)] for $\lambda = 0$, which is announced there, and it corresponds to [KW11, Lemma 2.3]. The vectors (9) are the differences of the two sides of Rule 1 of [KW11, §1.2]; the vector (9) is symmetric in A, B, C, D , and Rule 1 applies to it after the four sets have been named in the order of four elements chosen from them.

5 Proof of Theorem 1.2

We need two elementary facts.

Lemma 5.1. *The polynomial $\det A$ is irreducible in $\mathbb{Q}[a_{ij}]$.*

Proof. This is classical; here is a proof. Let $\det A = fg$. Since $\det A$ has degree 1 in each variable, each variable occurs in exactly one of f, g . Two variables of the same row cannot occur in different factors: if a_{ij} occurs in f and $a_{ij'}$ in g , write $f = a_{ij} f_1 + f_0$ and $g = a_{ij'} g_1 + g_0$ with $f_1, g_1 \neq 0$ and f_0, f_1, g_0, g_1 free of a_{ij} and $a_{ij'}$; then the part of fg which is divisible by $a_{ij} a_{ij'}$ is $a_{ij} a_{ij'} f_1 g_1 \neq 0$, whereas no monomial of $\det A$ contains two variables of one row. The same holds for columns. Hence all the variables occur in the same factor, and the other factor is constant. $\square$

Lemma 5.2. *For the identity matrix I we have $t_{P,P}(I) = 1$, and $t_{P,Q}(I) \neq 0$ only if Q is a refinement of P . Hence $T(I)$ is unipotent, and $\det T(I) = 1$.*

Proof. For $A = I$ only the edges $x_i y_i$ have a non-zero weight. A set S of such edges, which we identify with a subset of $[n]$, always gives a forest $F_P \cup S$; its components are the sets $\{x_i : i \in B\} \cup \{y_i : i \in S \cap B\}$ for the blocks B of P , and the isolated vertices y_i , $i \notin S$. So S is a transition if and only if it meets every block of P , and then Q consists of the sets $S \cap B$ and of singletons. Thus Q refines P , and $Q = P$ if and only if $S = [n]$. With respect to a linear extension of the refinement order, $T(I)$ is triangular with the diagonal entries 1. $\square$

Proof of Theorem 1.2. Let $K = \ker \Phi$. Part (a) is contained in Proposition 3.1, and $\dim K = B_n - C_n$ and the first statement of (b) are contained in Proposition 4.3.

Let $d := C_n$. We use the basis of V which consists of the vectors n_Q for the crossing partitions Q and of the vectors e_P , $P \in \text{NC}_n$ (Corollary 4.4). It arises from the basis (e_Q) by an integral matrix of determinant 1, and by (a) the matrix of $T(A)$ in this basis is block triangular. Hence

$$\det T(A) = R_n(A) \cdot p_n(A), \tag{10}$$

where $R_n(A) = \det(T(A)|_K)$ and $p_n(A)$ is the determinant of the map induced on V/K ; both are polynomials in the a_{ij} with integer coefficients. By Proposition 3.1, $p_n(A)$ is the determinant of the

restriction of $G(A)$ to $W_{\mathbb{k}}$, and this holds for every matrix with entries in $\mathbb{k} = \mathbb{Q}[a_{ij}]$; for the matrix $-A^T$ the determinant is $p_n(-A^T)$, that is p_n with a_{ij} replaced by $-a_{ji}$. Restricting Proposition 3.2 to $W_{\mathbb{k}}$, which is invariant under $G(A)$ and $G(-A^T)$, and taking determinants, we get

$$p_n(-A^T) \cdot p_n(A) = (-1)^{nd} (\det A)^{2d}. \quad (11)$$

The ring $\mathbb{Q}[a_{ij}]$ is factorial and $\det A$ is irreducible (Lemma 5.1). So $p_n(A) = c (\det A)^k$ with a constant $c \in \mathbb{Q}^\times$ and an integer $k \geq 0$. Then $p_n(-A^T) = c (-1)^{nk} (\det A)^k$, and (11) gives $k = d$ and $c^2 = 1$. By Lemma 5.2, $T(I)$ is unipotent; hence so are its restriction to K and the map which it induces on V/K . Therefore $R_n(I) = 1$ and $p_n(I) = 1$, so that $c = 1$. This proves

$$p_n(A) = (\det A)^{C_n},$$

which is (b), and with (10) the factorisation in (c). Finally $\det T$ is homogeneous of degree nB_n by Section 1.1, and it is not zero because $\det T(I) = 1$; so its factor R_n is homogeneous, of degree $nB_n - nC_n$. $\square$

Remark 5.3. (i) The exponent $k = d$ also follows from degrees (in the non-crossing basis the entry (P, Q) of the induced map has the degree $n + |P| - |Q|$), and the integrality of R_n from Gauss's lemma.

(ii) The proof uses characteristic zero, in the exponential series and in the proof of Proposition 4.3. The result is an identity in $\mathbb{Z}[a_{ij}]$ and holds therefore over every commutative ring. Nothing is assumed about A , and no division by $\det A$ occurs.

(iii) For the divisibility asked for in Problem 1.1 the upper bound $\dim W \leq C_n$ is not needed: without it the proof gives $\det T = (\det A)^{\dim W} \det(T(A)|_K)$, and $\dim W \geq C_n$ by Proposition 4.2.

6 The tree pairing of Kenyon and Wilson, and a second proof

For $P, Q \in \Pi_n$ let $\Gamma(P, Q)$ be the bipartite multigraph whose vertices are the blocks of P and the blocks of Q , with one edge for every $i \in [n]$, which joins the block of P containing i with the block of Q containing i . Put

$$M_{P,Q} := \begin{cases} 1 & \text{if } \Gamma(P, Q) \text{ is a tree,} \\ 0 & \text{otherwise.} \end{cases}$$

$M = (M_{P,Q})$ is a symmetric matrix of size B_n . The graph $\Gamma(P, Q)$ has $|P| + |Q|$ vertices and n edges, and it is connected if and only if the join $P \vee Q$ is the partition $\{[n]\}$ with one block. Therefore

$$M_{P,Q} = 1 \iff P \vee Q = \{[n]\} \text{ and } |P| + |Q| = n + 1. \quad (12)$$

This is the bilinear form $\langle \cdot, \cdot \rangle_t$ of [KW11]: for planar partitions it is defined by meanders in [KW11, (3)] and characterised as in (12) after [KW11, (4)], and in [KW11, §2.3] it is extended to all partitions by this characterisation. Its matrix on NC_n is the matrix $\mathcal{M}_t$ of [KW11]. Let $\text{rad } M := \{v \in V : vM = 0\}$.

6.1 The pairing as an integral

Proposition 6.1. (a) For all $P, Q \in \Pi_n$, $\int_{[n-1]} \Phi(P) \Phi(Q) = M_{P,Q}$.

(b) $\text{rad } M = K$. In particular $\text{rank } M = C_n$.

(c) K is spanned by the vectors (9), and the integers $\pi_{Q,P}$ of Corollary 4.4 are the entries $\mathcal{P}_{P,Q}^{(t)}$ of the projection matrix of [KW11, §1.2].

Proof. (a) Let F_P and F_Q be unions of spanning trees of the blocks of P and of Q , on the vertex set $[n]$. Then $\Phi(P)\Phi(Q) = \prod_{e \in F_P} f_e \prod_{e \in F_Q} f_e$. If $\Gamma(P, Q)$ is a tree, then F_P and F_Q have no common edge, since a common edge ij would give two parallel edges of $\Gamma(P, Q)$; the graph $F_P \cup F_Q$ is connected and has $2n - |P| - |Q| = n - 1$ edges by (12). So it is a spanning tree, $\Phi(P)\Phi(Q) = f_{[n]}$, and $\int_{[n-1]} f_{[n]} = f_{\{n\}} = 1$ by Lemma 2.2. If $\Gamma(P, Q)$ is not a tree, then $\Phi(P)\Phi(Q) = 0$ if F_P and F_Q have a common edge e , because $f_e^2 = 0$, or if $F_P \cup F_Q$ has a cycle. Otherwise $F_P \cup F_Q$ is a forest with $2n - |P| - |Q|$ edges which is not a tree, by (12); so it has at least two components, and $\Phi(P)\Phi(Q) = \prod_k f_{C_k}$. A component which does not contain n is integrated out completely and gives the factor 0, by (11) and Lemma 2.2.

(b) By (a), the coordinates of vM are the numbers $\int_{[n-1]} \Phi(v)\Phi(Q)$; hence $K \subseteq \text{rad } M$. The restriction of M to NC_n is the matrix $\mathcal{M}_t$, which is non-singular [KW11, §2.2], [DFGG97, (5.14)–(5.18)]. So $\text{rank } M \geq C_n = \dim V/K$, which forces $\text{rad } M = K$.

(c) The vectors (9) lie in K by Lemma 4.5; let K' be their span. By the first statement of [KW11, Theorem 1.2], every e_Q is congruent modulo K' to a combination of the e_P , $P \in \text{NC}_n$, namely to $\sum_P \mathcal{P}_{P,Q}^{(t)} e_P$. Hence $\dim V/K' \leq C_n = \dim V/K$ and $K' = K$; and since the classes of the e_P are a basis of V/K , the coefficients are the $\pi_{Q,P}$. $\square$

According to [KW11, end of §2.3] the entries of $\mathcal{P}^{(t)}$ are 0 or ± 1 for $n \leq 7$, and other integers appear for $n \geq 8$; in our computations the largest $|\pi_{Q,P}|$ is 1 for $4 \leq n \leq 7$ and 2 for $n = 8$.

Remark 6.2 (without the meander determinant). The equality $\text{rad } M = K$ also follows from Section 4, and this proves again that $\mathcal{M}_t$ is non-singular (but does not give its determinant). Define $\beta(g, h) := \int_{[n-1]} gh$ for $g, h \in \Lambda'$. The component of top degree $2(n-1)$ of Λ' is spanned by $\prod_t \bar{\chi}_t \chi_t = f_{[n]}$, and by (I4) and Lemma 2.2 the number $\beta(g, h)$ is the coefficient of $f_{[n]}$ in gh . Hence β is a non-degenerate bilinear form on Λ' : if $g \neq 0$ and θ is a basis monomial of smallest degree which occurs in g , then $g\theta'$ is a non-zero multiple of $f_{[n]}$ for the complementary monomial θ' . The derivations E and F of Proposition 4.3 preserve degrees and annihilate $f_{[n]}$; so the coefficient of $f_{[n]}$ in Eu and in Fu is 0 for all $u \in \Lambda'$, and therefore $\beta(Xg, h) + \beta(g, Xh) = 0$ for $X = E, F$, hence also for $X = H = EF - FE$. By Weyl's theorem [Hum72, §6.3], applied after extension of the scalars to the complex numbers, the module Λ' of the Lie algebra spanned by E, F, H is completely reducible; hence $\Lambda' = W \oplus N$ with $N := E\Lambda' + F\Lambda' + H\Lambda'$, because W is the space of invariants by Proposition 4.3. For $w \in W$ we get $\beta(w, Xu) = -\beta(Xw, u) = 0$, so $\beta(W, N) = 0$, and β is non-degenerate on W . Now let $v \in \text{rad } M$. By Proposition 6.1(a), $\beta(\Phi(v), W) = 0$; so $\Phi(v) = 0$, that is $v \in K$.

6.2 Two identities for $T(A)$ and M

The rest of this section does not use Grassmann variables. It was found by the second of the independent verification runs (see the paragraph “Verification”).

Lemma 6.3 (gluing). *Let U and U' be finite sets with $U \cap U' = Y$. Let F be a forest on U and F' a forest on U' without a common edge, such that every component of F and of F' contains a vertex of Y ; let Q' and Q'' be the partitions which they induce on Y . Then $F \cup F'$ is a spanning tree of $U \cup U'$ if and only if $M_{Q', Q''} = 1$.*

Proof. F has $|U| - |Q'|$ edges and F' has $|U'| - |Q''|$ edges, and $U \cup U'$ has $|U| + |U'| - n$ vertices. Every vertex is joined, inside its forest, to a vertex of Y , and two vertices of Y are joined in $F \cup F'$ if and only if they lie in the same block of $Q' \vee Q''$. So $F \cup F'$ is connected if and only if $Q' \vee Q'' = \{Y\}$, and a connected graph is a tree if and only if the number of its edges is the number of its vertices minus one, that is $|Q'| + |Q''| = n + 1$. Compare with (12). $\square$

For $Q \in \Pi_n$ let F_Q^Y be a union of spanning trees of the blocks of Q , drawn in Y ; F_P is drawn in X as before.

Lemma 6.4. $(T(A)M)_{P,Q} = \sum_S w(S)$, where the sum is over all sets S of edges $x_i y_j$ such that $F_P \cup S \cup F_Q^Y$ is a spanning tree of $X \cup Y$.

Proof. $(T(A)M)_{P,Q} = \sum_{Q'} t_{P,Q'} M_{Q',Q}$ is the sum of $w(S)$ over the transitions S from P to some Q' with $M_{Q',Q} = 1$. By Lemma 6.3, applied to $F = F_P \cup S$ on $U = X \cup Y$ and $F' = F_Q^Y$ on $U' = Y$, these are the transitions S for which $F_P \cup S \cup F_Q^Y$ is a spanning tree. Conversely, if $F_P \cup S \cup F_Q^Y$ is a spanning tree, then $F_P \cup S$ is a forest each of whose components contains a vertex of Y : a component inside X would be a component of the whole graph, because F_Q^Y has no edge at X . So S is a transition from P to the partition which it induces on Y . $\square$

Proposition 6.5. For every $n \times n$ matrix A with entries in a commutative ring,

$$T(A) M = M T(A^\top)^\top, \quad (13)$$

$$T(A) M T(-A)^\top = (-1)^n (\det A)^2 M. \quad (14)$$

Proof. *Proof of (13).* The condition of Lemma 6.4 is symmetric: the map $x_i y_j \mapsto x_j y_i$ sends the sets S for which $F_P \cup S \cup F_Q^Y$ is a spanning tree to the sets S' for which $F_Q \cup S' \cup F_P^Y$ is a spanning tree, and the weight of S for the matrix A is the weight of S' for $A^\top$. Hence $(T(A)M)_{P,Q} = (T(A^\top)M)_{Q,P}$, which is (13) because M is symmetric.

Proof of (14). *Step 1: a graph.* Fix P and Q , and let B be a second matrix. Let $\mathcal{H} = \mathcal{H}_{A,B}(P, Q)$ be the multigraph with the vertices $y_1, \dots, y_n$, the blocks of P and the blocks of Q , and with the following edges: for all i, j an edge of weight a_{ij} between y_j and the block of P containing i , and an edge of weight b_{ij} between y_j and the block of Q containing i . We claim that $(T(A) M T(B)^\top)_{P,Q}$ is the sum of the weights of the spanning trees of $\mathcal{H}$, the weight of a tree being the product of the weights of its edges. Indeed, take a copy $X' = \{x'_1, \dots, x'_n\}$ of X which carries Q , with edges $x'_i y_j$ of weight b_{ij} , and let F'_Q be a union of spanning trees of the blocks of Q in X' . The entry is the sum of $w_A(S) w_B(S')$ over the pairs of a transition S from P to some Q' and a transition S' from Q to some Q'' , formed with the edges $x'_i y_j$, such that $M_{Q',Q''} = 1$. By Lemma 6.3 and the argument in the proof of Lemma 6.4, these are the pairs (S, S') for which $F_P \cup S \cup S' \cup F'_Q$ is a spanning tree of $X \cup Y \cup X'$. Contracting the trees of F_P and F'_Q maps these spanning trees bijectively onto the spanning trees of $\mathcal{H}$, and it preserves the weights.

Step 2: the matrix-tree theorem. Now let $B = -A$, and let O be the set of the blocks of P and of Q , the “outer” vertices of $\mathcal{H}$. For $o \in O$ let $\mathbf{1}_o \in \{0, 1\}^n$ be the indicator row vector of the block, and let $\varepsilon_o = 1$ for the blocks of P and $\varepsilon_o = -1$ for the blocks of Q . The total weight of the edges between o and y_j is $c_{o,j} := \varepsilon_o (\mathbf{1}_o A)_j$. The total weight at o is $d_o := \varepsilon_o r(o)$, where $r(o) := \sum_{i \in o} r_i$ and $r_i := \sum_j a_{ij}$ are the row sums of A . The total weight at y_j is $\sum_i a_{ij} - \sum_i a_{ij} = 0$. So the weighted Laplacian matrix of $\mathcal{H}$, for the order (O, Y) of the vertices, is

$$L = \begin{pmatrix} \Delta & -C \\ -C^\top & 0 \end{pmatrix}, \quad \Delta = \text{diag}(d_o)_{o \in O}, \quad C = (c_{o,j})_{o \in O, j \in [n]}.$$

By the matrix-tree theorem (see for instance [Cha82]; parallel edges may be replaced by one edge with the sum of the weights), the sum N_A of the weights of the spanning trees of $\mathcal{H}$ is the determinant of L with the row and the column of one vertex $o_0 \in O$ deleted. Changing the signs of the last n rows and columns we get

$$N_A = \det \begin{pmatrix} \hat{\Delta} & \hat{C} \\ \hat{C}^\top & 0 \end{pmatrix},$$

where the hat means that the row, respectively the row and the column, of o_0 is deleted.

Step 3: a bordered determinant. For a diagonal matrix $\hat{\Delta} = \text{diag}(d_1, \dots, d_m)$ and an $m \times n$ matrix $\hat{C}$,

$$\det \begin{pmatrix} \hat{\Delta} & \hat{C} \\ \hat{C}^\top & 0 \end{pmatrix} = (-1)^n \sum_{S \subseteq [m], |S|=n} \left(\prod_{o \notin S} d_o \right) \det(\hat{C}_S)^2, \quad (15)$$

where $\hat{C}_S$ consists of the rows of $\hat{C}$ in S . Both sides are polynomials in the entries; if the d_o are invertible, the left side is $\det \hat{\Delta} \cdot \det(-\hat{C}^\top \hat{\Delta}^{-1} \hat{C})$ by the Schur complement, and the Cauchy–Binet formula gives the right side. In our case $\hat{C}_S = E_S J_S A$, where J_S is the $n \times n$ matrix with the rows $\mathbf{1}_o$ ($o \in S$) and $E_S = \text{diag}(\varepsilon_o)_{o \in S}$, so that $\det(\hat{C}_S)^2 = \det(J_S)^2 (\det A)^2$. Therefore

$$N_A = (-1)^n (\det A)^2 \cdot \vartheta_{P,Q}(r_1, \dots, r_n), \quad \vartheta_{P,Q}(r) := \sum_S \det(J_S)^2 \prod_{o \in O \setminus (S \cup \{o_0\})} \varepsilon_o r(o), \quad (16)$$

where S runs through the n -subsets of $O \setminus \{o_0\}$. Here $\vartheta_{P,Q}$ is a polynomial in n variables with integer coefficients which depends only on P, Q and o_0 , and in (16) it is evaluated at the row sums of A .

Step 4: evaluation of $\vartheta_{P,Q}$. Apply (16) to the diagonal matrix $D = \text{diag}(r_1, \dots, r_n)$ with indeterminates r_i ; its row sums are the r_i . This gives $N_D = (-1)^n (r_1 \cdots r_n)^2 \vartheta_{P,Q}(r)$. We compute N_D directly. In $\mathcal{H}_{D,-D}(P, Q)$ the edges of non-zero weight are, for each i , an edge e_i^+ of weight r_i from y_i to the block of P containing i and an edge e_i^- of weight $-r_i$ from y_i to the block of Q containing i . Consider the spanning trees formed by these edges in which some y_i is a leaf. Exchanging the edge at y_i , for the smallest such i , with the other edge at y_i is an involution on these trees which preserves the set of the y_j that are leaves and reverses the sign of the weight; so their weights cancel. A spanning tree in which no y_i is a leaf contains all the $2n$ edges $e_i^\pm$. The graph with all these edges is $\Gamma(P, Q)$ with every edge subdivided by a vertex y_i ; it is a tree if and only if $M_{P,Q} = 1$, and then its weight is $\prod_i r_i (-r_i) = (-1)^n (r_1 \cdots r_n)^2$. Hence $N_D = (-1)^n (r_1 \cdots r_n)^2 M_{P,Q}$, and therefore $\vartheta_{P,Q} = M_{P,Q}$ is constant. By (16), $N_A = (-1)^n (\det A)^2 M_{P,Q}$ for every A , which by Step 1 is (14). $\square$

6.3 Second proof of Theorem 1.2

Let $\mu := \text{rank } M$ and $K' := \text{rad } M$, a subspace of dimension $B_n - \mu$ which does not depend on A . By (13), $vT(A)M = vMT(A^\top)^\top = 0$ for $v \in K'$; so K' is invariant under $T(A)$. Let $\bar{T}(A)$ be the matrix of the map induced by $T(A)$ on V/K' in a fixed basis, and let $\bar{M}$ be the Gram matrix of the form induced by M , which is symmetric and invertible. From (14), $\bar{T}(A) \bar{M} \bar{T}(-A)^\top = (-1)^n (\det A)^2 \bar{M}$, and for $p(A) := \det \bar{T}(A)$ we get

$$p(A) \cdot p(-A) = (-1)^{n\mu} (\det A)^{2\mu}.$$

As in Section 5, the irreducibility of $\det A$ gives $p(A) = c (\det A)^k$; then $p(-A) = c (-1)^{nk} (\det A)^k$, so $k = \mu$ and $c^2 = 1$; and $c = 1$ by Lemma 5.2. With a basis of V adapted to K' this yields

$$\det T(A) = (\det A)^\mu \cdot \det(T(A)|_{K'}). \quad (17)$$

It remains to determine μ , and here we quote [KW11, DFGG97]. The restriction of M to NC_n is non-singular [KW11, §2.2], [DFGG97, (5.14)–(5.18)]; so $\mu \geq C_n$, and (17) already gives the divisibility asked for in Problem 1.1. By [KW11, Theorem 2.6] every partition is equivalent, for the pairing with all partitions, to an integer combination of planar partitions; so every e_Q is congruent modulo K' to a combination of the e_P , $P \in \text{NC}_n$, and $\mu \leq C_n$. Hence $\mu = C_n$, the vectors $e_Q - \sum_P \mathcal{P}_{P,Q}^{(t)} e_P$ for crossing Q form an integral basis of K' , and (17) is the factorisation of Theorem 1.2, with $K' = K$ by Proposition 6.1. $\square$

Remark 6.6. Combining (13) and (14) one obtains $(T(A)T(-A^\top) - (-1)^n(\det A)^2)M = 0$, which is Proposition 3.2 read on V/K . By Proposition 6.1(a) the second proof is not a different mechanism; but it needs no exterior algebra and no count of dimensions, and it quotes [KW11, DFGG97] for rank M .

7 Small n , exactness of the exponent, and the cofactor

7.1 The entries of $T(A)$

For a set $B \subseteq [n]$ and $j \in [n]$ let $b_B(j) := \sum_{i \in B} a_{ij}$.

Lemma 7.1 (contraction). *For $P, Q \in \Pi_n$,*

$$t_{P,Q} = \sum_{(N_B)} \prod_{B \in P} \prod_{j \in N_B} b_B(j),$$

where the sum is over all families $(N_B)_{B \in P}$ of non-empty subsets of $[n]$ such that the bipartite graph on the blocks of P and the vertices of Y , with the edges $\{B, y_j\}$ for $j \in N_B$, is a forest whose components induce Q on Y .

Proof. Let S be a transition from P to Q . Two edges of S from the same block B to the same vertex y_j would close a cycle with the tree on B . So S is given by the sets N_B of those j for which S has an edge from B to y_j , together with the choice of the endpoint $i \in B$ of this edge for every B and $j \in N_B$. Contracting the trees of the blocks maps $F_P \cup S$ onto the bipartite graph of the lemma. Therefore $F_P \cup S$ is a forest if and only if this graph is one, and the two graphs induce the same partition on Y ; a component of $F_P \cup S$ without a vertex of Y is a block B with $N_B = \emptyset$. The sum of $w(S)$ over the choices of the endpoints is $\prod_B \prod_{j \in N_B} b_B(j)$. $\square$

Adding the blocks of $P = \{B_1, \dots, B_k\}$ one at a time, we get from Lemma 7.1 the recursion

$$e_P T(A) = e_{\hat{0}} U(b_{B_1}) \cdots U(b_{B_k}), \quad e_\pi U(b) := \sum_{\emptyset \neq S \subseteq \pi} \prod_{C \in S} \left(\sum_{j \in C} b(j) \right) e_{\pi[S]}, \quad (18)$$

where $\hat{0}$ is the partition into singletons, π runs through the partitions of Y , and $\pi[S]$ is π with the blocks in S merged into one block: the new block of P is joined to exactly one vertex in each of the components in S . For $n \geq 5$ the programs compute $T(A)$ by (18) or by an equivalent recursion over the vertices (Section 8).

7.2 Proof of Corollary 1.3

(a) For $n \leq 3$ we have $B_n = C_n$, hence $K = 0$ and $R_n = 1$.

(b) Let $n = 4$. By Proposition 4.3, $\dim K = 1$, and by Lemma 4.5 the vector $\kappa = \kappa(\{1\}, \{2\}, \{3\}, \{4\}; \emptyset)$ of (3) lies in K . So $\kappa T(A) = R_4(A) \kappa$, and comparing the coordinates at $Q_0 := 12|34$ we get $R_4(A) = \sum_P \kappa_P t_{P,Q_0}$, where the sum is over the seven partitions $P = \{B, B^c\}$ with two blocks, and $\kappa_P = (-1)^{|B|}$. For such P the forest of Lemma 7.1 has two components, which contain $\{y_1, y_2\}$ and $\{y_3, y_4\}$; each of them contains exactly one of the two blocks, which is joined to both of its vertices. Hence $t_{P,Q_0} = b_B(1)b_B(2)b_{B^c}(3)b_{B^c}(4) + b_{B^c}(1)b_{B^c}(2)b_B(3)b_B(4)$, and

$$R_4(A) = \sum_{\emptyset \neq B \subsetneq [4]} (-1)^{|B|} b_B(1)b_B(2)b_{B^c}(3)b_{B^c}(4) = \sum_{i_1, i_2, i_3, i_4} c(i) a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4},$$

where $c(i) := \sum_B (-1)^{|B|}$ over the sets B which contain i_1, i_2 and do not contain i_3, i_4 . There is no such B if $\{i_1, i_2\} \cap \{i_3, i_4\} \neq \emptyset$. Otherwise B is the union of $\{i_1, i_2\}$ and of an arbitrary subset of the set F of the indices different from $i_1, \dots, i_4$, so that $c(i) = (-1)^{|\{i_1, i_2\}|} \sum_{F' \subseteq F} (-1)^{|F'|}$; this is 0 unless $F = \emptyset$, that is unless $i_1, \dots, i_4$ are distinct, and then $c(i) = 1$. Hence $\bar{R}_4(A) = \text{per } A$, and $\det T = (\det A)^{14} \text{per } A$ by Theorem 1.2. $\square$

7.3 Exactness of the exponent

Proof of Proposition 1.4. For $n = 4$ we have $R_4 = \text{per } A$, which is not divisible by $\det A$: both polynomials are homogeneous of degree 4, and $\text{per } A$ is not a constant multiple of $\det A$.

In general, suppose that $\det A$ divides R_n in $\mathbb{Q}[a_{ij}]$. By Gauss's lemma $R_n = \det A \cdot g$ with $g \in \mathbb{Z}[a_{ij}]$. Then $v_p(\det T(A_0)) = C_n + v_p(R_n(A_0)) \geq C_n + 1$ for every integer matrix A_0 with $|\det A_0| = p$ prime, where v_p is the p -adic valuation. So one integer matrix A_0 with $|\det A_0| = p$ and $v_p(\det T(A_0)) = C_n$ shows that $\det A$ does not divide R_n . Table 1 lists such matrices for $4 \leq n \leq 8$, with three methods for the valuation.

Method D. $T(A_0)$ is computed over $\mathbb{Z}$ and its determinant by fraction-free elimination.

Method K. In the integral basis of Corollary 4.4 the program verifies that K is invariant under $T(A_0)$, computes exactly the determinant of the induced map on V/K and compares it with $(\det A_0)^{C_n}$, and computes $R_n(A_0) = \det(T(A_0)|_K)$ modulo p . If the residue is not 0, then $v_p(\det T(A_0)) = C_n$ by (10).

Method S (staged elimination; it uses no theory). $T(A_0)$ is computed modulo p^k by (18). In stage 0 one eliminates with pivots which are units modulo p , by row operations and exchanges of rows and columns, until the remaining block has no such entry. Then all its entries are divisible by p ; one divides it by p , so that it is known modulo p^{k-1} , and continues with stage 1, and so on. If the block is exhausted after the stages $0, \dots, j$ with $j \leq k-1$, and c_i pivots were found in stage i , then $v_p(\det T(A_0)) = \sum_i i c_i$. Indeed, the computation describes an exact elimination over the ring of p -adic integers, in which the remaining block after stage i is p times a matrix that is known modulo p^{k-i-1} ; so $\det T(A_0)$ is a p -adic unit times $\prod_i p^{i c_i}$. $\square$

For $n = 7$ the recorded outputs contain four further certificates (method S: $\det A_0 = -17, -19$, and $-2 \cdot 2309$ at $p = 2309$; method K: $\det A_0 = 1543$). The rows of Table 1 for $n \leq 7$ and the row with $\det A_0 = 607$ were computed at least twice, with independently written programs; for the two matrices with $n = 8$ treated by method S, rows of $T(A_0)$ modulo p^3 were compared with the enumeration of all forests. In all six certificates by method S the numbers of pivots in the stages 0, 1, 2 were $(B_n - C_n) + C_{n-1}$, $C_n - 2C_{n-1}$ and C_{n-1} ; so $T(A_0)$ has elementary divisors p^2 , and its rank modulo p does not give v_p .

7.4 The cofactor: observations and open questions

Let $P_k(A)$ be the matrix of size $\binom{n}{k}$ whose entries are the permanents of the $k \times k$ submatrices of A ; $\det P_k(A)$ is homogeneous of degree $k \binom{n}{k}$, and $P_n(A) = \text{per } A$. Thus $R_4 = \det P_4$. The following statements are tested and not proved.

- (O1) $n = 5$. On two random integer lines $A_0 + xA_1$ the polynomial $R_5(A_0 + xA_1) \in \mathbb{Q}[x]$, computed exactly, has degree 50 and two irreducible factors, of degrees 20 and 30, the degrees which the source reports for $n = 5$; the factor of degree 20 is $\det P_4(A_0 + xA_1)$ up to the sign. On a third line $\det P_4$ divides R_5 modulo the primes 10007, 20011 and 30011. At 209 random integer matrices $\det P_4(A)$ divides $R_5(A)$, whereas $\text{per } A$ does not divide $R_5(A)$ at 84 of 196 matrices.

n	$\det A_0$	method	result	program and recorded output
4	137	D	$v_{137} = 14 = C_4$	original: exactness_certificates
5	47	D	$v_{47} = 42 = C_5$	original: exactness_certificates
6	251	D	$v_{251} = 132 = C_6$	original: exactness_certificates
7	-15299	K	$R_7(A_0) \equiv 12090 \pmod{p}$	original: verify_quotient_n7
7	-15299	S, $k = 4$	pivots 580, 165, 132: $v_p = 429 = C_7$	run A: vA_test5_theorem_n7
8	607	S, $k = 3$	pivots 3139, 572, 429: $v_p = 1430 = C_8$	run A: vA_test8_n8_run1
8	-137	S, $k = 3$	pivots 3139, 572, 429: $v_p = 1430 = C_8$	run A: vA_test8_n8_run2
8	4513	K, modular	$R_8(A_0) \equiv 1480 \pmod{p}$	run B: b12_n8

$$\begin{pmatrix} 3 & 0 & -1 & 0 \\ 2 & -1 & -2 & 2 \\ -2 & 3 & -2 & 3 \\ 0 & -1 & 3 & 3 \end{pmatrix} \quad \begin{pmatrix} -2 & 1 & 3 & -1 & 3 \\ 0 & -1 & 0 & 1 & 3 \\ -2 & 0 & -1 & 0 & -1 \\ 0 & -2 & 0 & 1 & -1 \\ 1 & 2 & -2 & -1 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & -2 & -2 & 3 & 1 & 2 \\ 0 & 2 & -1 & -1 & 3 & -2 \\ -2 & -2 & -2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 2 & -2 & -1 \\ -2 & 3 & -2 & 0 & 0 & -2 \\ 3 & -1 & 3 & 1 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & -2 & -3 & -1 & 1 & -1 \\ -3 & 1 & -2 & -3 & 0 & 4 & 2 \\ 4 & 4 & 0 & -2 & 4 & 0 & -1 \\ 2 & -3 & 4 & 4 & 1 & 0 & -2 \\ 3 & -1 & -1 & 2 & 1 & 1 & 4 \\ 2 & 2 & -2 & 2 & -3 & -2 & 4 \\ -2 & -3 & 3 & 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 & -1 & 0 \\ -2 & -1 & 2 & -1 & -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 1 & 1 & 2 & -2 \\ -1 & -2 & 1 & 2 & 0 & 0 & -1 & -2 \\ -1 & -2 & 0 & 0 & 2 & 2 & 0 & 0 \\ -1 & 2 & 1 & -2 & -2 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 1 & -2 & -1 & -1 \\ -1 & -1 & -1 & 1 & 2 & 1 & -1 & 0 \end{pmatrix}$$

Table 1: Integer matrices A_0 with $|\det A_0| = p$ prime and $v_p(\det T(A_0)) = C_n$; below the table those of the first six rows (determinants 137, 47, 251, -15299, 607; the other two are in the recorded outputs). In the last row the invariance of K and the determinant on V/K are verified modulo two primes only; this row depends on Theorem 1.2.

- (O2) $n = 5$. In the basis n_Q of Corollary 4.4, the block of the matrix of $T(A)|_K$ which maps the five crossing partitions with two blocks to those with three blocks has the determinant $2 \det P_4(A)$, at 46 random integer matrices. At seven random integer matrices the characteristic polynomial of $T(A)|_K$ is irreducible; so the factorisation of R_5 does not come from an invariant subspace of K there.
- (O3) $n = 6$. On two random lines, modulo three primes each (10007, 20011, 30011 and 40009, 43003, 46021), the polynomial R_6 , of degree 426, is divisible by $\det P_4$, of degree 60, and by $\det A$, and it is coprime to $\det A$, $\det P_2$, $\det P_3$ and $\det P_5$; the remaining factor has degree 360. At 43 random integer matrices $\det A$ and $\det P_4(A)$ divide $R_6(A)$.

Open questions. Is R_n prime to $\det A$ for all $n \geq 4$ (proved for $n \leq 8$, Proposition 1.4)? Describe R_n for $n \geq 5$: is $R_5 = \pm \det P_4 \cdot R'_5$ with R'_5 irreducible of degree 30; do $\det P_4$ and $\det A$ divide R_6 ; which $\det P_k$ divide R_n in general?

8 Computations and reproducibility

Theorem 1.2, Corollary 1.3 and all statements of Sections 2 to 6 are proved without computation. Computations enter in two places: Proposition 1.4 for $5 \leq n \leq 8$ is proved by the exact computations of Table 1, which are computer-assisted proofs, and the statements (O1)–(O3) of Section 7.4 are

tests. All other computations are tests of the proofs. The arithmetic is exact throughout (integers, polynomials, residues modulo primes and prime powers).

The reproducibility package has five parts with programs and recorded outputs: `original/` (the first version of the results), `verification_run_A/`, `verification_run_B/`, `independent_run_2/` (the three verification runs) and `writing_stage/` (tests of the statements added when the note was written). Each part computes $T(A)$ from the definition, by enumeration of all edge sets or of all forests, and compares Lemma 7.1 or (18) with it; between them, the verification runs do this on all rows for $n \leq 6$, on up to 365 rows for $n = 7$, and on rows modulo p^3 for two matrices with $n = 8$. Together the parts test every lemma and proposition on small cases, Theorem 1.2 exactly at more than 500 integer matrices with $n \leq 7$, and the divisibility without any theory at more than 4000 integer matrices with $n \leq 6$ and on lines. The lists, with all numbers, are in the verification report and in the README of the package. Two programs of `original/` were corrected as the second verification run required: the test of the divisibility now counts its vacuous cases (one of 172 matrices has $|\det A| = 1$), and an exploratory program, whose first output for $n = 4$ contained artefacts of an interpolation from too few points, now uses enough points and twelve further check points; that output has been replaced.

Verification. The results were first obtained as Theorem 1.2 with exactness for $n \leq 7$, with the proofs of Sections 2 to 5 and the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs, and the first two did not see each other.

The first run compared the statement with the source and re-derived every step of the first proof; it found no gap, required one correction of the description of the source, and added Corollary 4.4 and the certificates for $n = 8$. The second run recomputed everything from the definition, examined the original programs (the two corrections above) and the literature, found the second proof (Lemmas 6.3 and 6.4, Proposition 6.5), and required that the note say what is announced in [CSS07] and contained in [KW11]. All these corrections were made.

The third run examined the final text: the two pages of the source; all proofs line by line, first those added when the note was written (Proposition 6.1, Remark 6.2, Lemma 7.1, the proofs of Lemma 4.5 and of the formula for $n = 4$ without a computer) and the second proof, then the first proof with all signs; and every sentence about [CSS07, KW11, DFGG97], against the TeX sources of the arXiv versions. It found no wrong statement and no gap, and it required eleven corrections of the text and of the package, which were made; the main one is the disclosure of the draft in the source of [CSS07] (Section 1.3(ii)), and one concerns a proof (a reason added in Remark 6.2). With programs written from the definition (`independent_run_2/`) it enumerated all forests for $n \leq 5$, found Theorem 1.2 to hold exactly at 340 integer matrices with $n \leq 6$ (125 with prime determinant, 88 singular ones of all ranks), tested every statement of Sections 2 to 4, 6 and 7 on small cases, and recomputed the rows of Table 1 for $n \leq 7$ and for $\det A_0 = 607$, with the same residue and the same numbers of pivots. It also reproduced the recorded outputs of the quick part of the package.

No run compared the journal versions of the cited papers or checked the proofs of the statements quoted from [KW11, DFGG97]; (O1)–(O3) are tests. A reading of the literature by a person with access to the journals and a question to the proposers, which the second run recommended before a formal publication, have not taken place. No proof except that of Proposition 1.4 for $n \geq 5$ depends on a computation.

Scope and priority. The problem is Problem 1 of Teufl and Wagner in the problem session of Oberwolfach Report 23/2018 [TW18, pp. 1457–1458]. The report states as known: $\det T = (\det A)^{B_n}$ for $n \leq 3$; $\det T = (\det A)^{14} \text{ per } A$ for $n = 4$; as an experimental observation, the factorisation $(\det A)^{42} P_1 P_2$ for $n = 5$; and experimental evidence for the divisibility for $n = 6$. The wording of the record OWR-16164-020 of [ula26] mixes in the title of the next problem of the report (Section 1.1).

The credits are those of Section 1.3: the Grassmann-algebra representation of forests is due to Caracciolo, Jacobsen, Saleur, Sokal and Sportiello [CJS⁺04, CSS07]; the statement of Section 4 is announced without proof in [CSS07], a draft of a proof is in the source file of that preprint, and Section 4 is not claimed as new; the subspace K , the dimension C_n and the planar basis of V/K and the integrality of the projection are due to Kenyon and Wilson [KW11, Theorems 1.2 and 2.6, Rule 1], with the meander determinant of Di Francesco, Golinelli and Guitter [DFGG97]. What the note adds is the determinant of the map induced by $T(A)$ on V/K , hence the divisibility for every n , the factorisation over $\mathbb{Z}[a_{ij}]$, the proof of the formula for $n = 4$, and the exactness of the exponent for $n \leq 8$; the cofactor R_n is not determined for $n \geq 5$. The divisibility is close to this published structure theory and may be known to specialists. No priority is claimed.

What was read: the two pages of [TW18] in the file of the report; in the TeX sources of the arXiv versions, of [CSS07] (version 2) Section 4, Section 5 up to Theorem 5.4 and Section 7, of [KW11] (version 4) §1.2 and §2.1–2.3, and of [DFGG97] §5.2 and §5.3 up to (5.18); the journal versions were not compared. The paper [CJS⁺04] is cited on the authority of [CSS07]. The later paper [CSS17] of the same authors was not accessible to us; its abstract and its list of references were seen, and they do not mention the announced paper. Not read beyond the abstract: [Lam18, CGS21, BGKT23] and, not used in this note, the papers of Lam and Pylyavskyy (arXiv:1103.3475), Bychkov, Gorbounov, Guterman and Kazakov (arXiv:2406.03021), Gao, Lam and Xu (arXiv:2208.12798), Jekel (arXiv:1601.00247), Kogan (arXiv:2610.07539), Jacobsen, Salas and Sokal (arXiv:cond-mat/0401026) and Brankov, Poghosyan, Priezzhev and Ruelle (arXiv:1405.0436), and four further items which are listed in the verification report. A preprint version of [TW10] and the text of Kogan’s preprint were searched for the terms of the problem by the second verification run, without result; neither was read.

The searches were made on 11 October 2026, at the four stages of the work: queries to the arXiv interface, zbMATH, Crossref and OpenAlex (for the proposers, the terms of the problem and of the method, the works which cite [TW10], the report and [CSS07], and the announced paper), and nine web searches. No paper, preprint or note answering Problem 1.1 was found, and no statement of the determinant of $T(A)$ on the quotient. For the announced paper only a seminar announcement of January 2009 with nearly the same title and slides of 2007 listing the work as in progress were found; four later papers of the group on arXiv on spanning forests and hyperforests (0802.1506, 0906.4503, 0903.4432, 1608.02916) and the doctoral thesis [Bed09], whose Section 3.2 reproduces Section 4 of [CSS07], contain neither the statement nor a proof. Full texts were not searched: an answer in a paper whose abstract does not mention the problem, or a private communication to the proposers, would not have been seen. The proposers were not contacted. A search that finds nothing is no proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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