A Counterexample to a Conjecture of Sportiello on Coloured Permutations in Convex Shapes

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September 28, 2026

Abstract

At the 2018 Oberwolfach workshop on enumerative combinatorics, A. Sportiello compared two families of objects attached to a digitally convex shape λ (a convex polyomino): the permutations whose graph lies in λ and that avoid a 123-pattern whose two corner cells also lie in λ, counted by A_λ, and the red–blue coloured permutations in λ that avoid two coloured versions of the 12-pattern, counted by B_λ. For the full square these numbers are the Catalan number and the central binomial coefficient, and he conjectured that $B_\lambda \geq A_\lambda$ for every digitally convex shape. We show that the conjecture is false. For the band $\lambda = \{(x, y) \in [7]^2 : -3 \leq y - x \leq 4\}$ one has $A_\lambda = 536$ and $B_\lambda = 515$. An exhaustive computation shows that every digitally convex shape of side at most 6 satisfies the inequality, and that among the 5,693,968 shapes of side 7 exactly this band and its transpose violate it. For bands of side 10 the ratio B_λ/A_λ drops to about 0.18. This is an unrefereed note.

1 The conjecture

We follow Sportiello's problem-session contribution [BMDKN18, pp. 1455–1457], which defines the objects by pictures; here they are written in coordinates.

Shapes. A cell is a pair $(x, y) \in \mathbb{Z}^2$, where x is the column, counted from the left, and y the row, counted from the top. A finite set λ of cells has *side n* if it fits in an $n \times n$ square but not in an $(n-1) \times (n-1)$ square; we then place it in $[n]^2 = \{1, \ldots, n\}^2$. It is *digitally convex* if any two of its cells are joined, inside λ, by a path of unit steps that uses at most two of the four directions N, S, E, W. Equivalently, λ is a convex polyomino; this is also the definition used in [CS16].

Permutations in a shape. A permutation $\sigma \in S_n$ is identified with the set of cells $(i, \sigma(i))$, $1 \leq i \leq n$. Let S_λ be the set of $\sigma \in S_n$ with $(i, \sigma(i)) \in \lambda$ for all i.

The two families. Let A_λ be the number of $\sigma \in S_\lambda$ for which there are no indices $i_1 < i_2 < i_3$ with

$$\sigma(i_1) < \sigma(i_2) < \sigma(i_3), \qquad (i_3, \sigma(i_1)) \in \lambda, \qquad (i_1, \sigma(i_3)) \in \lambda.$$

Let B_λ be the number of pairs (σ, c) with $\sigma \in S_\lambda$ and $c\colon [n] \to \{\text{red}, \text{blue}\}$ such that there is no red pair $i_1 < i_2$ with $\sigma(i_1) < \sigma(i_2)$ and $(i_1, \sigma(i_2)) \in \lambda$, and no blue pair $i_1 < i_2$ with $\sigma(i_1) < \sigma(i_2)$ and $(i_2, \sigma(i_1)) \in \lambda$.

In the pictures of [BMDKN18], the first condition is a 3×3 pattern with its points on the diagonal and "crosses" in the two other corners, and the second consists of two 2×2 patterns, one red and one blue, each with one cross. An occurrence of a pattern counts only if every cross lands in a cell of λ.

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Conjecture 1.1 (Sportiello [BMDKN18, p. 1456])**.** *For every digitally convex shape λ, $B_\lambda \geq A_\lambda$.*

For the full square $[n]^2$ every cross lies in λ. Then A_λ is the number C_n of 123-avoiding permutations, and B_λ counts permutations together with a colouring in which both colour classes are decreasing. Such a pair is determined by the set of positions and the set of values of the red points, which have the same size, so $B_\lambda = \sum_k \binom{n}{k}^2 = \binom{2n}{n} = (n+1)A_\lambda$. The conjecture asks whether the inequality survives when the square is replaced by a convex shape.

2 The counterexample

Theorem 2.1. *Let $\lambda = \{(x,y) \in [7]^2 : -3 \leq y - x \leq 4\}$, that is,*

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with rows listed from the top. Then λ is a digitally convex shape of side 7, $|S_\lambda| = 1066$, $A_\lambda = 536$ and $B_\lambda = 515$. In particular Conjecture 1.1 is false.

Proof. Every row and every column of λ is an interval, λ is connected, and it meets every row and column of $[7]^2$, so λ is a convex polyomino of side 7. Both A_λ and B_λ are finite counts, obtained by complete enumeration.

- The set S_λ consists of the permutations with $-3 \leq \sigma(i) - i \leq 4$ for all i; it has 1066 elements.
- For each of them we test the 35 triples $i_1 < i_2 < i_3$ and find $A_\lambda = 536$.
- For each of them we test all 2^7 colourings and find $B_\lambda = 515$.

The count was carried out by three independent programs, described in Section 4, which agree. □

The transpose $\{-4 \leq y - x \leq 3\}$ gives the same numbers. The failure grows with the side of the band; Table 1 lists, for each side, the band $\{a \leq y - x \leq b\}$ with the smallest ratio.

side n	band	$\lvert S_\lambda \rvert$	A_λ	B_λ	B_λ/A_λ
6	$-3 \leq y-x \leq 3$	230	150	220	1.467
7	$-3 \leq y-x \leq 4$	1066	536	515	0.961
8	$-4 \leq y-x \leq 4$	6902	2380	1360	0.571
9	$-4 \leq y-x \leq 5$	41506	9655	3241	0.336
10	$-5 \leq y-x \leq 5$	329462	46194	8484	0.184

Table 1: The band with the smallest ratio B_λ/A_λ among all bands of side n.

Proposition 2.2. *Every digitally convex shape of side at most 6 satisfies $B_\lambda \geq A_\lambda$; there are 349,320 such shapes, and among those with $A_\lambda > 0$ the smallest value of B_λ/A_λ is 22/15, attained by the band $-3 \leq y - x \leq 3$ of side 6. Among the 5,693,968 digitally convex shapes of side 7, exactly two violate the inequality, namely the band of Theorem 2.1 and its transpose.*

This is again a finite computation. The shapes of side k are enumerated by their column intervals, and their numbers $1, 5, 68, 1110, 19010, 329126, 5693968$ for $k = 1, \ldots, 7$ agree with Gessel's count of convex polyominoes in a $k \times k$ box. For every shape the definition of digital convexity was tested directly. So the band of Theorem 2.1 is, up to symmetry, the unique smallest counterexample.

Remark 2.3. All crosses sit at the top-right or bottom-left corner of the pair or triple they belong to. So the inequality has content only for shapes cut at the top-right or bottom-left corner of their box. Suppose instead that λ is a convex polyomino whose column intervals $[t(x), b(x)]$ have both t and b non-increasing in x; this is a shape cut only at the top-left and bottom-right corners.

Let $i < j$ with $\sigma(i) < \sigma(j)$ and both points in λ. Then

$$t(j) \le t(i) \le \sigma(i) < \sigma(j) \le b(j) \qquad \text{and} \qquad t(i) \le \sigma(i) < \sigma(j) \le b(j) \le b(i),$$

so both corner cells $(j, \sigma(i))$ and $(i, \sigma(j))$ lie in λ. Hence, as for the square, A_λ counts the 123-avoiding $\sigma \in S_\lambda$, and B_λ counts the colourings in which both colour classes are decreasing.

A permutation avoids 123 if and only if it is the union of two decreasing subsequences. Every σ counted by A_λ therefore has an admissible colouring, and exchanging the two colours gives a second one. So $B_\lambda \ge 2A_\lambda$ for these shapes. The counterexamples are of the opposite type.

3 The reading of the source

The source defines the patterns by pictures, so the reading above needs a check. It reproduces the report's own examples.

- *The worked example* [BMDKN18, p. 1457]. The shape has column intervals $[2, 6]$, $[1, 7]$, $[1, 7]$, $[1, 8]$, $[1, 8]$, $[3, 8]$, $[3, 6]$, $[3, 6]$. With $\sigma = 4\,7\,2\,3\,1\,8\,6\,5$ we get $\sigma \in S_\lambda$ and σ is counted by A_λ. The printed colouring, red on positions $1, 2, 4$ and blue elsewhere, is counted by B_λ. The drawn occurrence of a 4×4 pattern puts its crosses exactly where the text says.
- *The Remark* [BMDKN18, p. 1456]. For the 4×4 square minus the cells $(1, 3), (1, 4), (2, 4)$, the identity is counted by A_λ but has no admissible colouring.

Two variants of the reading make no difference.

- Exchanging which colour carries which cross, as the corpus version of the problem does, is a bijection on colourings, so it does not change B_λ.
- Reading the patterns in a mirrored frame, with rows counted from the bottom, fails the checks above; it only mirrors the counterexample, which remains a counterexample.

The only reading we found under which the inequality holds for all shapes up to side 7 requires the crosses to lie *outside* λ. That reading contradicts the text and fails the worked example and the Remark.

4 Verification and scope

The following exact computations, with scripts in the source archive, were used.

1. A direct Python enumeration of S_λ with a literal check of the conditions. It covers Theorem 2.1, the worked example and the Remark, and every digitally convex shape of side at most 5.

2. An independent referee's Python code with a generic pattern-with-crosses routine, which reproduces $(|S_\lambda|, A_\lambda, B_\lambda) = (1066, 536, 515)$. Two C programs extend the census to all shapes of side 6 and 7 and compute Table 1. The C and Python codes agree on all shapes of side at most 5 and on 3000 random shapes of side 6.

3. Two short programs by the author recompute Theorem 2.1, its transpose, the square values for $n \le 5$, the worked example, and the rows $n = 6, 8, 9$ of Table 1.

Scope and priority. Theorem 2.1 answers Sportiello's question [BMDKN18, p. 1456] in the negative. It says nothing about possible corrected versions of the conjecture, for instance for other classes of shapes, and we do not propose one. Searches of arXiv, Crossref and OpenAlex in September 2026, including the works citing the report and Sportiello's later papers, found no discussion of the conjecture. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexample, to write and run the verification programs, to referee the reading of the source, and to help prepare this manuscript. The author has reviewed the definitions, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

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