

The Law of the Sum of the Two Faces of Bi-Monotone Brownian Motion Has Atoms: A Partial Answer to a Question of M. Gerhold

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Abstract

M. Gerhold asked for a concrete description of the vacuum distributions μ_t of the operators $\lambda_t^* + \lambda_t + \rho_t^* + \rho_t$ on the monotone Fock space, that is, of the sum of the two faces of bi-monotone Brownian motion (Oberwolfach Rep. 15 (2018)); the same question closes his paper on bi-monotone Brownian motion, where the even moments of μ_1 are identified as $\#\text{PP}_{\bowtie}(2n)/n!$ with the numbers 1, 4, 48, 928, 24448, ... of bi-monotone pair partitions, for which no explicit or recursive formula was known. We give a partial answer: we do not find a closed formula, and the question remains open; we prove structural facts. (1) The measures μ_t are the dilations by $\sqrt{t}$ of one measure $\mu = \mu_1$; this scaling is implicit in the source. (2) The compression of the operator to the orthogonal complement of the vacuum has norm at most $\sqrt{6}$, while the operator itself has norm $c > \sqrt{6}$. Hence the edge c of the support of μ is an isolated atom: $\mu = w(\delta_c + \delta_{-c}) + \mu_0$ with μ_0 carried by $[-\sqrt{6}, \sqrt{6}]$. This is proved without computation. Enclosures in exact rational arithmetic (computer-assisted) give $c = 2.585826606048204448\dots$ and $w = 0.251331654638419314\dots$, so the two atoms carry more than half of the mass. (3) The Boolean cumulants of μ are $k_{n+2} = 4\langle B^n 1 \rangle$ for an explicit linear operator B on polynomials in three variables and an explicit linear functional. This gives all moments exactly in polynomial time; the table of the source (eleven values) is extended to m_{306} . (4) Computer-assisted, exact: μ has mass in $(2.01148, 2.01149)$ and in $(2.001, 2.003)$, so μ_0 is not carried by $[-2, 2]$. Only observed numerically, and not proved: further spectral points accumulating at ± 2 that look like atoms, a nearly flat density of total mass about 0.49 on $[-2, 2]$, Jacobi parameters tending slowly to 1, and the absence of a linear recurrence with polynomial coefficients for the moments within the sizes searched. An explicit formula for μ , the nature of its part in $[-\sqrt{6}, \sqrt{6}]$, and a formula for the numbers $\#\text{PP}_{\bowtie}(2n)$ remain open. This is an unrefereed note.

1 Introduction

This note gives a partial answer to a question of M. Gerhold about the law of the sum of the two faces of bi-monotone Brownian motion. The question asks for a concrete description of a family of probability measures. We prove structural facts: they are the dilations of one measure; the edge of its support is an isolated atom; the two atoms at the edges carry more than half of the mass; the moments can be computed exactly in polynomial time; and there is more mass just above the point 2. We do not find a closed formula, and the question itself stays open.

1.1 The problem

For $n \geq 0$ let $\Delta_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_1 < \dots < t_n\}$; the set Δ_0 consists of the empty tuple Λ . Let Δ be the disjoint union of the sets Δ_n , with Lebesgue measure on each of them. The *monotone*

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Fock space is $\Gamma = L^2(\Delta) = \bigoplus_{n \geq 0} L^2(\Delta_n)$ [Mur97, Lu97], [Ger17, Definition 3]. Its *vacuum vector* Ω is the function which is 1 at Λ and 0 elsewhere, and the *vacuum state* is $a \mapsto \langle \Omega, a\Omega \rangle$. For $f \in L^2(\mathbb{R})$, [Ger17, Definition 4] defines four bounded operators on Γ : for $g \in \Gamma$ and $t = (t_1, \dots, t_n) \in \Delta_n$ with $n \geq 1$,

$$\begin{aligned} (\lambda^*(f)g)(t) &= f(t_1)g(t_2, \dots, t_n), & (\lambda(f)g)(t) &= \int_{-\infty}^{t_1} \overline{f(\tau)}g(\tau, t_1, \dots, t_n) d\tau, \\ (\rho^*(f)g)(t) &= g(t_1, \dots, t_{n-1})f(t_n), & (\rho(f)g)(t) &= \int_{t_n}^{\infty} \overline{f(\tau)}g(t_1, \dots, t_n, \tau) d\tau, \end{aligned} \tag{1}$$

and $(\lambda^*(f)g)(\Lambda) = (\rho^*(f)g)(\Lambda) = 0$, $(\lambda(f)g)(\Lambda) = (\rho(f)g)(\Lambda) = \int_{\mathbb{R}} \overline{f(\tau)}g(\tau) d\tau$; the last formula is the reading of the definition for $n = 0$ which makes $\lambda(f)$ and $\rho(f)$ the adjoints of $\lambda^*(f)$ and $\rho^*(f)$, as the source states (see Lemma 2.1). In words: if $t_1 < \dots < t_n$ are the positions of n particles, then $\lambda^*(f)$ creates a particle below all others, $\lambda(f)$ removes the lowest particle, $\rho^*(f)$ creates a particle above all others, and $\rho(f)$ removes the highest one. For $t > 0$ put $\lambda_t = \lambda(1_{[0,t]})$, $\rho_t = \rho(1_{[0,t]})$ and

$$b_t = \lambda_t^* + \lambda_t + \rho_t^* + \rho_t.$$

This is a bounded self-adjoint operator. Its *vacuum distribution* μ_t is the unique compactly supported probability measure on $\mathbb{R}$ with $\int x^n d\mu_t(x) = \langle \Omega, b_t^n \Omega \rangle$ for all $n \geq 0$.

The pair $(\lambda_t^* + \lambda_t, \rho_t^* + \rho_t)$ is the *bi-monotone Brownian motion* of Gerhold [Ger17, Ger23]; its two components, the *faces*, are the monotone and the antimonotone Brownian motion. In the Oberwolfach report [Ger18] the operators b_t are introduced, and it is said that they are not arcsine distributed, in contrast to each face alone. The abstract ends with the statement that it is “an open problem to give a concrete description of the corresponding probability measures” [Ger18, p. 1355]. The paper [Ger17] ends in the same way, for $t = 1$: “It would be nice to have an explicit formula of this probability measure.” This is the question we study. It is record OWR-16163-011 of the UnsolvedMath collection [ula26], where it is listed as open.

What the source knows. We refer to [Ger17] in the numbering of its arXiv version.

(i) Each face alone has the arcsine law: the vacuum distribution of $\lambda_t^* + \lambda_t$, and of $\rho_t^* + \rho_t$, is the arcsine law of variance t , which has the density $\pi^{-1}(2t - x^2)^{-1/2}$ on $(-\sqrt{2t}, \sqrt{2t})$ [Mur97, Lu97].

(ii) The pairs of faces over consecutive intervals are bi-monotonely independent, and the increments are stationary [Ger17, Theorems 4.1 and 4.2].

(iii) A central limit theorem for bi-monotone independence [Ger17, Theorem 6.5] gives the mixed moments of the two faces [Ger17, Theorem 7.1], and in particular

$$\int x^{2n} d\mu_1(x) = \frac{\#\text{PP}_{\bowtie}(2n)}{n!} \quad (n \geq 0) \tag{2}$$

[Ger17, Corollary 7.2] (there the formula is printed with n in place of $2n$). Here $\text{PP}_{\bowtie}(2n)$ is the set of bi-monotone pair partitions of $\{1, \dots, 2n\}$ with arbitrary pattern [Ger17, Definition 8]: a partition π of $\{1, \dots, 2n\}$ into n pairs, a total order on its blocks and a map $\delta: \{1, \dots, 2n\} \rightarrow \{\ell, r\}$ such that, whenever $a < b < c$, the points a, c lie in a block V and b lies in another block W , one has $V > W$ if $\delta(b) = r$ and $V < W$ if $\delta(b) = \ell$. The paper lists

$$\begin{aligned} \#\text{PP}_{\bowtie}(2n) &= 1, 4, 48, 928, 24448, 811776, 32460032, 1516774912, \\ &81064953344, 4876115246080, 325959895390976 \end{aligned} \tag{3}$$

for $n = 0, \dots, 10$ and states that no explicit or recursive formula for these numbers is known to its author.

Two normalisations should be kept in mind. The variance of μ_t is $4t$, not t (Lemma 2.2). And μ_t only sees the functions supported in $[0, t]$, which form an invariant subspace containing Ω ; all statements below about the norm and the spectrum of b_t refer to the restriction of b_t to this subspace.

1.2 Results

Let $\mathcal{H} = L^2(\Delta \cap [0, 1]^*)$ be the space of the functions in Γ which are supported in $[0, 1]$, let b be the restriction of b_1 to $\mathcal{H}$, let $\mu = \mu_1$, and let $m_n = \int x^n d\mu$.

Theorem 1.1 (one measure up to dilation). *For every $t > 0$ and every Borel set $A \subset \mathbb{R}$, $\mu_t(A) = \mu(A/\sqrt{t})$. The measure μ is symmetric, has variance $m_2 = 4$, and is carried by $[-4, 4]$.*

The scaling is implicit in the source, where it is used in the proof of [Ger17, Theorem 7.1]; we state it only because it reduces the question to one measure, and we make no claim of novelty for it.

For the main result let Q be the orthogonal projection of $\mathcal{H}$ onto $\Omega^\perp$, and let b_Q be the compression QbQ , regarded as an operator on $\Omega^\perp$. The *Boolean cumulants* $k_1, k_2, \dots$ of μ are defined by $m_N = \sum_{j=1}^N k_j m_{N-j}$ for $N \geq 1$ [SW97]; the odd ones vanish.

Theorem 1.2 (the edge of the support is an isolated atom). (a) $\|b_Q\| \leq \sqrt{6}$.

(b) *There are numbers $c > \sqrt{6}$ and $w \in (0, \frac{1}{3})$ such that*

$$\mu = w(\delta_c + \delta_{-c}) + \mu_0,$$

where μ_0 is a symmetric positive measure of total mass $1 - 2w$ with $\text{supp } \mu_0 \subset [-\sqrt{6}, \sqrt{6}]$. Moreover $c = \max \text{supp } \mu = \|b\|$, the number c is a simple eigenvalue of b , and the spectrum of b meets $(\sqrt{6}, \infty)$ only in c .

(c) *The function $K(z) = \sum_{n \geq 1} k_{2n} z^{1-2n}$ extends to an analytic function on $\mathbb{C} \setminus [-\sqrt{6}, \sqrt{6}]$, all k_{2n} are positive, c is the only solution of $z = K(z)$ in $(\sqrt{6}, \infty)$, and*

$$w = \frac{1}{1 - K'(c)} = \left(1 + \sum_{n \geq 1} (2n - 1) k_{2n} c^{-2n}\right)^{-1}.$$

(d) (computer-assisted, exact rational arithmetic) *From the moments $m_0, \dots, m_{44}$, computed directly from the definition of the four operators,*

$$2.58579 < c < 2.58604, \quad 0.25001 < w < 0.25313.$$

From the moments $m_0, \dots, m_{306}$, computed with Theorem 1.3,

$$2.58582660604820444863151523603 < c < 2.58582660604820444863151523668, \\ 0.251331654638419314530505656 < w < 0.251331654638419314530505699.$$

In particular $2w > \frac{1}{2}$: the two atoms $\pm c$ carry more than half of the mass of μ .

(e) *For every $t > 0$, $\mu_t = w(\delta_{c\sqrt{t}} + \delta_{-c\sqrt{t}}) + \mu_{0,t}$, where $\mu_{0,t}(A) = \mu_0(A/\sqrt{t})$ is carried by $[-\sqrt{6t}, \sqrt{6t}]$, and the norm of b_t on the functions supported in $[0, t]$ is $c\sqrt{t}$.*

Parts (a), (b), (c) and (e) are proved without any computation: the proof of (a) rests on an explicit positive function and on the sign of one quintic polynomial, which is read off from a factorisation of its derivative (Proposition 5.3), and the proof of (b) uses, besides (a), only the numbers $m_2 = 4$, $m_4 = 24$ and $m_6 = \frac{464}{3}$ of Lemma 2.2. Part (d) depends on programs, which evaluate the two-sided bounds of Lemma 9.1 at rational points in exact rational arithmetic; see Section 9. Figure 1 shows what is proved about μ .

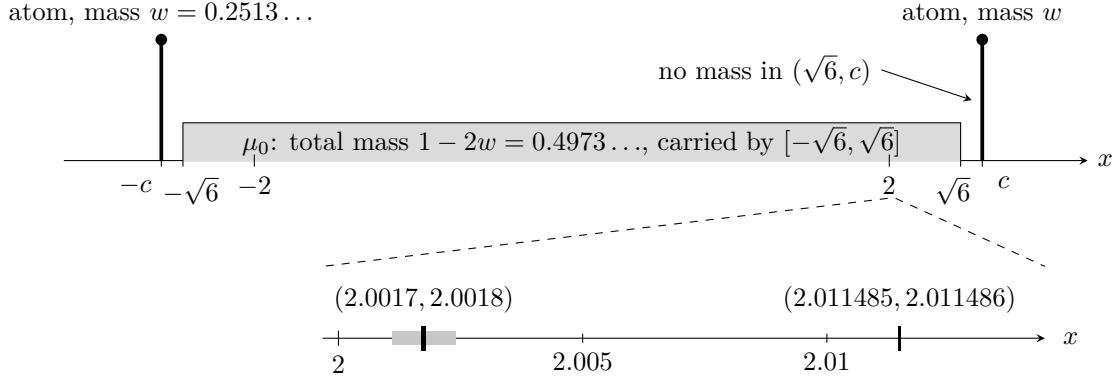


Figure 1: What is proved about μ (schematic; the heights of the atoms and the shaded bar are not to scale with each other, and the bar does not represent a density). The atoms at $\pm c$, $c = 2.5858\dots$, have mass $w = 0.2513\dots$ each; the rest of the mass lies in $[-\sqrt{6}, \sqrt{6}]$ (Theorem 1.2). Below: the interval $[2, 2.014]$ enlarged. The measure μ gives positive mass to each of the two black windows and to their mirror images (Theorem 1.4); the grey bar is the window $(2.0011, 2.0024)$.

Theorem 1.3 (a polynomial model for the Boolean cumulants). *For an integer $a \geq 0$ put $m_a(u) = \binom{a}{a/2} (u/2)^{a/2}$ if a is even and $m_a(u) = 0$ if a is odd; for $u \geq 0$ these are the moments of the arcsine law of variance u . On the polynomial ring $\mathcal{P} = \mathbb{Q}[x_1, x_3, s]$ define linear maps M, U, R, D, B and a linear functional $\langle \cdot \rangle$ by*

$$\begin{aligned} Mf &= (x_1 + x_3) f, \\ U(x_1^a x_3^b s^c) &= \sum_{\substack{j+k=b-1 \\ j,k \geq 0}} \left(\int_0^s u^c m_a(u) m_k(1-u) du \right) x_3^j, \\ (Rf)(x_1, x_3, s) &= f(x_3, x_1, 1-s), \quad D = RUR, \quad B = M + U + D, \\ \langle x_1^a x_3^b s^c \rangle &= \int_0^1 s^c m_a(s) m_b(1-s) ds. \end{aligned}$$

Then

$$k_{n+2} = 4 \langle B^n 1 \rangle \quad \text{for all } n \geq 0.$$

More precisely, there is a linear map $\Pi: \mathcal{P} \rightarrow \Omega^\perp \subset \mathcal{H}$ with $b\Omega = 2\Pi(1)$ and

$$b\Pi(f) = \Pi(Bf) + 2\langle f \rangle \Omega \quad (f \in \mathcal{P}). \quad (4)$$

Consequently $m_0, \dots, m_N$ can be computed exactly with $O(N^5)$ arithmetic operations on rational numbers whose numerators and denominators have $O(N^2)$ binary digits.

The variable s is the position of the oldest particle present, which we call the *pivot*; x_1 and x_3 are spectral variables for the particles below and above it (Section 8, Figure 2). The exact program of this note computed $k_2, \dots, k_{306}$ and $m_0, \dots, m_{306}$ in this way; Table 1 lists the first values. By (2) the integers $n! m_{2n}$ continue the table (3) of the source: for $n = 11, \dots, 16$ they are

$$\begin{aligned} &23971543733582336, \quad 1923281730204625408, \quad 167173129086919363584, \\ &15648863483303987358208, \quad 1569520108422792156830720, \quad 167912198237160007914904576. \end{aligned}$$

These six values, and the values up to $n = 22$, were also obtained directly from the definition of the operators, without Theorem 1.3 (Section 2). The complete lists are in the reproducibility package (Section 12).

n	m_{2n}	k_{2n}	$n! m_{2n} = \#\text{PP}_{\bowtie}(2n)$	$n! k_{2n}$
0	1	—	1	—
1	4	4	4	4
2	24	8	48	16
3	464/3	80/3	928	160
4	3056/3	304/3	24448	2432
5	33824/5	6112/15	811776	48896
6	2028752/45	25328/15	32460032	1215744
7	94798432/315	448864/63	1516774912	35909120
8	633319948/315	212924/7	81064953344	1226442240
9	7618930072/567	53017256/405	4876115246080	47503461376
10	181897263053/2025	892403651/1575	325959895390976	2056098011904
11	93638842709306/155925	128002786598/51975	23971543733582336	98306140107264

Table 1: Exact moments and Boolean cumulants of μ . The fourth column for $n \leq 10$ is the table (3) of [Ger17]. The Jacobi parameters of μ begin with 4, 2, $\frac{4}{3}$, $\frac{7}{6}$, $\frac{11}{10}$, $\frac{414}{385}$, $\frac{16868}{15939}$ (Section 11).

Theorem 1.4 (more of the measure just above 2; computer-assisted). *Each of the following intervals has positive μ -measure:*

- (a) (2.01148, 2.01149) and (2.001, 2.003);
- (b) (2.011485, 2.011486) and (2.0017, 2.0018).

The same holds for the mirror images of these intervals. In particular the measure μ_0 of Theorem 1.2 is not carried by $[-2, 2]$, and the spectrum of b meets each of these intervals.

Part (a) uses the moments $m_0, \dots, m_{304}$ of the exact program of this note. Part (b) uses the moments $m_0, \dots, m_{556}$ computed by the second of the independent verification runs described in Section 12, with its own implementation of Theorem 1.3; this part is due to that run, and its program and output are in the package. In both parts the proof is the exact evaluation of one rational number, a localizing form of the truncated moment problem in the sense of [CF00] (Section 10).

In addition we show that the two norms in Theorem 1.2 are determined by the two measures alone: $\|b\| = \max \text{supp } \mu$, and $\|b_Q\| = \max \text{supp } \nu$ for the spectral measure ν of b_Q at $b_Q/2$ (Proposition 7.1). Hence $\|b_Q\| = \lim_n (k_{2n+2}/k_{2n})^{1/2}$, and $2.1019907 < \|b_Q\| \leq \sqrt{6}$ (Corollary 7.2).

1.3 The mechanism, and related work

The mechanism is classical. With respect to the decomposition $\mathcal{H} = \mathbb{C}\Omega \oplus \Omega^\perp$ the Cauchy transform of μ is given by a Schur complement, $G(z) = 1/(z - K(z))$ with $K(z) = \langle b\Omega, (z - b_Q)^{-1}b\Omega \rangle$, and K is the generating function of the Boolean cumulants of Speicher and Woroudi [SW97]. A zero of $z - K(z)$ outside the spectrum of the compression is an eigenvalue of the full operator, as in the theory of rank-one perturbations [Sim95]. The simplest example is the family of measures with Jacobi parameters $(\theta, 1, 1, \dots)$, for which K is θ times the Cauchy transform of the semicircle law of variance 1: an elementary computation shows that for $\theta > 2$ they have two atoms, at $\pm\theta/\sqrt{\theta-1}$, of mass $\frac{\theta-2}{2(\theta-1)}$ each. See Cohen and Trenholme [CT84] for orthogonal polynomials with constant recursion coefficients, and Bożejko and Wysoczański [BW01] for the transformation of measures which multiplies K by a constant; the case $\theta = 4$ is a first approximation of μ (Remark 8.3). What is new in this note is not the mechanism, but the proof that it applies to b , namely the bound $\|b_Q\| \leq \sqrt{6} < (m_6/m_4)^{1/2}$, the polynomial model, and the rigorous numbers.

Credits. The monotone Fock space and the arcsine law of monotone Brownian motion are due to Muraki [Mur97] and Lu [Lu97]. Bi-monotone independence, the realisation on the monotone Fock space, the central limit theorem, the moment formula (2), the table (3) and the question are due to Gerhold [Ger17, Ger23, Ger18]; independence for pairs of faces goes back to Voiculescu’s bi-freeness [Voi14]. Gu, Hasebe and Skoufranis [GHS20] define two notions of bi-monotonic independence. Their “type II” is the notion of the source, found independently, as they say; they do not pursue it. Their central limit theorem concerns “type I”, which is a different object: for unit variances and full correlation the two faces of their limit have the joint law of (X, X) with X arcsine distributed, so the law of the sum is $2 \times$ arcsine, with sixth moment 160 instead of $\frac{464}{3}$. For partial sums of position operators on the monotone Fock space over $\ell^2(\mathbb{N})$, a monotonically independent family, Crismale and Lu [CL20] show that the norm is the right endpoint of the support of the vacuum distribution, which is a discrete measure there; Corollary 7.2(a) is a statement of the same kind for b .

What we found, and what we could not read. In the sources we could read we found no description of μ , of its support, of the norm of b or of atoms, and no recursion for the numbers $\#PP_{\bowtie}(2n)$, which are not in the OEIS [OEI26]. The thesis of Varšo [Var21], to which the outlook of [Ger26] points, applies a necessary test for positivity to symmetric two-faced universal products (partition-induced products and q -deformed families): its Section 6.4 reports computations, for these products, of the moments up to order 10 and of the Hankel determinants $\Delta_0, \dots, \Delta_5$ of the sum of the two faces under the standard Gaussian functional. The bi-monotone product is not symmetric and is not among them; it occurs there only as an example of a two-faced product and as the model for a product of pointed representations, and the law studied here, its moments, support or atoms do not occur. One source could not be consulted, because automated access was refused: the published version [Ger23] of [Ger17]; the arXiv version, of which only one version exists, was read in full. No statement of this note about novelty depends on the published version, and a search that finds nothing is not a proof of novelty; see “Scope and priority” in Section 12.

1.4 What stays open

The question of Gerhold stays open: we have no closed formula for μ , for its Cauchy transform, or for the numbers $\#PP_{\bowtie}(2n)$, and we do not know the nature of μ_0 . The numerical picture of Section 11 suggests that μ_0 consists of atoms $\pm c_k$ with c_k decreasing to 2 and of an absolutely continuous part on $[-2, 2]$ with a nearly flat density; none of this is proved, beyond the two windows of Theorem 1.4, in which we do not know that μ is atomic. We do not know whether the generating function of the moments is D-finite, nor whether c is algebraic. In the polynomial model the question becomes one linear functional equation for a function of two variables, which we could not solve (Remarks 8.4 and 8.5).

2 The operators on the Fock space over $[0, 1]$, and the first moments

Inner products are linear in the second argument. For $n \geq 0$ let $S_n = \{t \in \mathbb{R}^n : 0 \leq t_1 < \dots < t_n \leq 1\}$; its volume is $1/n!$. Then $\mathcal{H} = \bigoplus_{n \geq 0} L^2(S_n)$, and we write $g = (g_n)_{n \geq 0}$ for the components of $g \in \mathcal{H}$; the number $g_0 = g(\Lambda)$ is the vacuum component, and g_n is the n -particle component. By $\mathbf{1}_n$ we denote the function which is 1 on S_n and 0 on the other components; thus $\mathbf{1}_0 = \Omega$ and $\|\mathbf{1}_n\|^2 = 1/n!$. From now on

$$\lambda^* = \lambda^*(\mathbf{1}_{[0,1]}), \quad \lambda = \lambda(\mathbf{1}_{[0,1]}), \quad \rho^* = \rho^*(\mathbf{1}_{[0,1]}), \quad \rho = \rho(\mathbf{1}_{[0,1]}), \quad b = \lambda^* + \lambda + \rho^* + \rho.$$

Lemma 2.1. *The four operators leave $\mathcal{H}$ invariant. For $g \in \mathcal{H}$, $n \geq 1$ and $t = (t_1, \dots, t_n) \in S_n$,*

$$\begin{aligned} (\lambda^*g)(t) &= g(t_2, \dots, t_n), & (\lambda g)(t) &= \int_0^{t_1} g(\tau, t_1, \dots, t_n) d\tau, \\ (\rho^*g)(t) &= g(t_1, \dots, t_{n-1}), & (\rho g)(t) &= \int_{t_n}^1 g(t_1, \dots, t_n, \tau) d\tau, \end{aligned} \quad (5)$$

and $(\lambda^*g)(\Lambda) = (\rho^*g)(\Lambda) = 0$, $(\lambda g)(\Lambda) = (\rho g)(\Lambda) = \int_0^1 g(\tau) d\tau$. The operator λ is the adjoint of λ^* , and ρ is the adjoint of ρ^* . Moreover

$$\begin{aligned} \|\lambda^*g\|^2 &= |g(\Lambda)|^2 + \sum_{n \geq 1} \int_{S_n} t_1 |g(t)|^2 dt \leq \|g\|^2, \\ \|\rho^*g\|^2 &= |g(\Lambda)|^2 + \sum_{n \geq 1} \int_{S_n} (1 - t_n) |g(t)|^2 dt \leq \|g\|^2. \end{aligned} \quad (6)$$

Hence b is a bounded self-adjoint operator on $\mathcal{H}$ with $\|b\| \leq 4$. All four operators map non-negative functions to non-negative functions.

Proof. The formulas (5) are (1) for $f = 1_{[0,1]}$ and g supported in $[0, 1]$, and they show that the images are again supported in $[0, 1]$. For $g, h \in \mathcal{H}$,

$$\langle h, \lambda^*g \rangle = \sum_{n \geq 1} \int_{S_n} \overline{h(t)} g(t_2, \dots, t_n) dt = \sum_{m \geq 0} \int_{S_m} \overline{\left(\int_0^{s_1} h(\tau, s) d\tau \right)} g(s) ds,$$

where for $m = 0$ the inner integral runs over $[0, 1]$. This is $\langle \lambda h, g \rangle$; in particular the value of λh at Λ is forced by adjointness. In the same way $\|\lambda^*g\|^2 = \sum_{n \geq 1} \int_{S_n} |g(t_2, \dots, t_n)|^2 dt$ is the first expression in (6): below a configuration with lowest point t_1 a new particle has room t_1 , and below the empty configuration it has room 1. The statements for ρ^* and ρ follow with the unitary reflection $(Jg)(t_1, \dots, t_n) = g(1 - t_n, \dots, 1 - t_1)$, for which $J\lambda^*J = \rho^*$ and $J\lambda J = \rho$. $\square$

Lemma 2.2. (a) *Let V be the unitary operator on $\mathcal{H}$ which multiplies the n -particle component by $(-1)^n$. Then $VbV = -b$ and $V\Omega = \Omega$. Hence $m_n = 0$ for odd n , and μ is symmetric.*

(b) $b\Omega = 2 \cdot \mathbf{1}_1$, $b\mathbf{1}_1 = 2\Omega + 2 \cdot \mathbf{1}_2$, $b\mathbf{1}_2 = \mathbf{1}_1 + 2 \cdot \mathbf{1}_3$.

(c) $m_2 = 4$, $m_4 = 24$, $m_6 = \frac{464}{3}$, and $m_6/m_4 = \frac{58}{9} > 6$.

Proof. (a) Each of the four operators changes the number of particles by one. Therefore $m_n = \langle V\Omega, b^n V\Omega \rangle = (-1)^n m_n$. A compactly supported measure is determined by its moments, so μ is invariant under $x \mapsto -x$.

(b) Both creation operators map $\mathbf{1}_n$ to $\mathbf{1}_{n+1}$. By (5), $\lambda\Omega = \rho\Omega = 0$, $(\lambda\mathbf{1}_1)(\Lambda) = (\rho\mathbf{1}_1)(\Lambda) = \int_0^1 d\tau = 1$, $(\lambda\mathbf{1}_2)(t_1) = t_1$ and $(\rho\mathbf{1}_2)(t_1) = 1 - t_1$.

(c) By (b), $b^2\Omega = 4\Omega + 4 \cdot \mathbf{1}_2$ and $b^3\Omega = 8 \cdot \mathbf{1}_1 + 4(\mathbf{1}_1 + 2 \cdot \mathbf{1}_3) = 12 \cdot \mathbf{1}_1 + 8 \cdot \mathbf{1}_3$. Since b is self-adjoint, $m_{2k} = \|b^k\Omega\|^2$, so that $m_2 = 4$, $m_4 = 16 + \frac{16}{2} = 24$ and $m_6 = 144 + \frac{64}{6} = \frac{464}{3}$. $\square$

These values are $4/1!$, $48/2!$ and $928/3!$, in agreement with (2) and (3). The computation can be continued mechanically: $b^N\Omega$ is a finite family of polynomials on the simplices S_n , and (5) maps monomials to combinations of monomials. The exact program of this note obtains $m_0, \dots, m_{44}$ in this way, “directly from the definition”, as we shall say; for $n \leq 10$ the numbers $n!m_{2n}$ so obtained are those of (3).

3 Scaling: proof of Theorem 1.1

For $t > 0$ let $\mathcal{H}_t = L^2(\Delta \cap [0, t]^*)$ be the space of the functions in Γ supported in $[0, t]$. As in Lemma 2.1, $\mathcal{H}_t$ contains Ω and is invariant under $\lambda_t^*, \lambda_t, \rho_t^*, \rho_t$, so μ_t is the vacuum distribution of the restriction of b_t to $\mathcal{H}_t$. Define

$$U_t: \mathcal{H}_t \rightarrow \mathcal{H}, \quad (U_t g)(s_1, \dots, s_n) = t^{n/2} g(ts_1, \dots, ts_n).$$

By the substitution $u = ts$ on each simplex, U_t is unitary, and $U_t \Omega = \Omega$. For $g \in \mathcal{H}_t$, $n \geq 1$ and $s \in S_n$,

$$(U_t \lambda_t^* g)(s) = t^{n/2} g(ts_2, \dots, ts_n) = \sqrt{t} (\lambda^* U_t g)(s),$$

and both sides vanish at Λ . Hence $U_t \lambda_t^* = \sqrt{t} \lambda^* U_t$, and in the same way $U_t \rho_t^* = \sqrt{t} \rho^* U_t$. On $\mathcal{H}_t$ the operator λ_t is the adjoint of λ_t^* ; taking adjoints, and using that U_t is unitary, we get $U_t \lambda_t = \sqrt{t} \lambda U_t$ and $U_t \rho_t = \sqrt{t} \rho U_t$. Therefore

$$U_t b_t U_t^{-1} = \sqrt{t} b, \quad (7)$$

and $\langle \Omega, b_t^n \Omega \rangle = t^{n/2} m_n$ for all n . So μ_t is the image of μ under $x \mapsto \sqrt{t} x$. The symmetry of μ and $m_2 = 4$ are in Lemma 2.2, and μ is carried by $[-4, 4]$ because $\|b\| \leq 4$. $\square$

Remark 3.1. The two faces are strongly correlated: $(\lambda^* + \lambda)\Omega = (\rho^* + \rho)\Omega = \mathbf{1}_1$. Hence the difference $\lambda_t^* + \lambda_t - \rho_t^* - \rho_t$ annihilates the vacuum and has the vacuum distribution δ_0 , while the sum has variance $4t$. We only study the sum, which is what the question asks for.

4 The Schur complement at the vacuum, and Boolean cumulants

Let P be the orthogonal projection onto $\mathbb{C}\Omega$ and $Q = 1 - P$. Put

$$\xi = b\Omega = 2 \cdot \mathbf{1}_1, \quad \|\xi\|^2 = 4, \quad b_Q = QbQ|_{\Omega^\perp}.$$

Since $\langle \Omega, b\Omega \rangle = 0$ and $Pb\psi = \langle b\Omega, \psi \rangle \Omega$, the operator b has the block form

$$b = \begin{pmatrix} 0 & \langle \xi, \cdot \rangle \\ \xi & b_Q \end{pmatrix} \quad \text{with respect to } \mathcal{H} = \mathbb{C}\Omega \oplus \Omega^\perp. \quad (8)$$

Let ν be the spectral measure of b_Q at the unit vector $\xi/2$; it is a probability measure on $[-\|b_Q\|, \|b_Q\|]$. Let

$$G(z) = \int \frac{d\mu(x)}{z-x} = \langle \Omega, (z-b)^{-1} \Omega \rangle, \quad K(z) = \langle \xi, (z-b_Q)^{-1} \xi \rangle = 4 \int \frac{d\nu(x)}{z-x}, \quad F(z) = z - K(z).$$

Lemma 4.1. (a) Let $z \in \mathbb{C} \setminus \sigma(b_Q)$. Then $z - b$ is invertible if and only if $F(z) \neq 0$, and in this case $\langle \Omega, (z-b)^{-1} \Omega \rangle = 1/F(z)$. If $F(z) = 0$, then the kernel of $z - b$ is spanned by the vector $\Omega + (z - b_Q)^{-1} \xi$.

(b) $G(z) = 1/(z - K(z))$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

(c) The Boolean cumulants of μ are $k_1 = 0$ and

$$k_{n+2} = \langle \xi, b_Q^n \xi \rangle = 4 \int x^n d\nu(x) \quad (n \geq 0).$$

The measure ν is symmetric and different from δ_0 . Hence $k_n = 0$ for odd n , $k_{2n} > 0$ for all $n \geq 1$, and $K(z) = \sum_{n \geq 1} k_{2n} z^{1-2n}$ for $|z| > \|b_Q\|$.

Proof. (a) Write $\varphi = \alpha \Omega + \psi$ and $\eta = \beta \Omega + \theta$ with $\psi, \theta \perp \Omega$. By (8) the equation $(z - b)\varphi = \eta$ is the system

$$z\alpha - \langle \xi, \psi \rangle = \beta, \quad (z - b_Q)\psi - \alpha \xi = \theta.$$

The second equation is equivalent to $\psi = (z - b_Q)^{-1}(\theta + \alpha \xi)$, and with this the first becomes $F(z) \alpha = \beta + \langle \xi, (z - b_Q)^{-1} \theta \rangle$. If $F(z) \neq 0$, there is exactly one solution φ for every η , so the bounded operator $z - b$ is bijective, hence invertible, and for $\eta = \Omega$ we get $\alpha = 1/F(z)$. If $F(z) = 0$ and $\eta = 0$, the solutions are $\alpha (\Omega + (z - b_Q)^{-1} \xi)$.

(b) The spectra of b and b_Q are real. For $z \notin \mathbb{R}$ the operator $z - b$ is invertible, so $F(z) \neq 0$ and $G(z) = 1/F(z)$ by (a).

(c) For $|z| > \|b_Q\|$ the Neumann series gives $K(z) = \sum_{n \geq 0} \langle \xi, b_Q^n \xi \rangle z^{-n-1}$, and for $|z| > \|b\|$ we have $G(z) = \sum_{N \geq 0} m_N z^{-N-1}$. Comparing coefficients in $G(z)(z - K(z)) = 1$ gives $m_N = \sum_{j=2}^N \langle \xi, b_Q^{j-2} \xi \rangle m_{N-j}$ for $N \geq 1$. These relations determine the Boolean cumulants recursively, since k_N occurs in the N -th relation with the coefficient $m_0 = 1$. The operator V of Lemma 2.2 commutes with Q , so $Vb_QV = -b_Q$, and $V\xi = -\xi$; hence $\int x^n d\nu = (-1)^n \int x^n d\nu$, and ν is symmetric. Finally $b_Q \xi = 2Qb\mathbf{1}_1 = 4 \cdot \mathbf{1}_2$ by Lemma 2.2(b), so $k_4 = \|b_Q \xi\|^2 = 8 > 0$. Thus $\nu \neq \delta_0$, and therefore $\int x^{2n} d\nu > 0$ for all n . $\square$

By Lemma 2.2(c) and the definition, $k_2 = m_2 = 4$, $k_4 = m_4 - k_2^2 = 8$ and $k_6 = m_6 - 2k_2k_4 - k_2^3 = \frac{80}{3}$, in agreement with $k_4 = \|b_Q \xi\|^2$.

5 A positive supersolution and the Schur test

In this section we prove $\|b_Q\| \leq \sqrt{6}$. For a family $h = (h_n)_{n \geq 0}$ of non-negative measurable functions h_n on S_n with $h_0 = 0$ put, for $n \geq 1$ and $t \in S_n$,

$$(Th)_n(t) = h_{n-1}(t_2, \dots, t_n) + h_{n-1}(t_1, \dots, t_{n-1}) + \int_0^{t_1} h_{n+1}(\tau, t) d\tau + \int_{t_n}^1 h_{n+1}(t, \tau) d\tau. \quad (9)$$

This is the action of b on h according to (5), with the vacuum component removed.

Fix a number $\gamma > 0$ and a continuous function $\varphi: [0, 1] \rightarrow (0, \infty)$, and let $q(r) = (\gamma + 2r)^{-1/2}$ for $r \geq 0$. For $n \geq 1$ and $t \in S_n$ define

$$P_j(t) = \prod_{i \neq j} q(|t_i - t_j|) \quad (1 \leq j \leq n), \quad h_n(t) = \sum_{j=1}^n \varphi(t_j) P_j(t), \quad (10)$$

and $h_0 = 0$. Thus $h_1 = \varphi$, and every h_n is positive, continuous and bounded.

Lemma 5.1 (supersolution). *Assume that for all $s \in [0, 1]$*

$$\int_0^s \varphi(\tau) q(s - \tau) d\tau \leq \sqrt{\gamma} \varphi(s) \quad \text{and} \quad \int_s^1 \varphi(\tau) q(\tau - s) d\tau \leq \sqrt{\gamma} \varphi(s). \quad (11)$$

Then $(Th)_n(t) \leq 2\sqrt{1 + \gamma} h_n(t)$ for all $n \geq 1$ and all $t \in S_n$.

Proof. Let $Q(r) = \int_0^r q = \sqrt{\gamma + 2r} - \sqrt{\gamma}$, so that $1/q(r) - Q(r) = \sqrt{\gamma}$ for all $r \geq 0$. Consider first the “lower half” $L = h_{n-1}(t_2, \dots, t_n) + \int_0^{t_1} h_{n+1}(\tau, t) d\tau$ of (9).

(i) Removing t_1 from the configuration removes exactly the factor $q(t_j - t_1)$ from $P_j(t)$ for $j \geq 2$. Hence $h_{n-1}(t_2, \dots, t_n) = \sum_{j \geq 2} \varphi(t_j) P_j(t) / q(t_j - t_1)$; for $n = 1$ both sides are 0.

(ii) For $0 \leq \tau < t_1$,

$$h_{n+1}(\tau, t_1, \dots, t_n) = \varphi(\tau) \prod_{i=1}^n q(t_i - \tau) + \sum_{j=1}^n \varphi(t_j) P_j(t) q(t_j - \tau),$$

and $\int_0^{t_1} q(t_j - \tau) d\tau = Q(t_j) - Q(t_j - t_1)$.

(iii) Adding (i) and the integral of (ii) over $\tau \in [0, t_1]$, the coefficient of $\varphi(t_j)P_j(t)$ is $1/q(t_j - t_1) - Q(t_j - t_1) + Q(t_j) = \sqrt{\gamma} + Q(t_j) = \sqrt{\gamma + 2t_j}$ for $j \geq 2$, and $Q(t_1) = \sqrt{\gamma + 2t_1} - \sqrt{\gamma}$ for $j = 1$. Therefore

$$L = \sum_{j=1}^n \varphi(t_j) P_j(t) \sqrt{\gamma + 2t_j} - \sqrt{\gamma} \varphi(t_1) P_1(t) + \int_0^{t_1} \varphi(\tau) \prod_{i=1}^n q(t_i - \tau) d\tau.$$

(iv) The function q is decreasing, and $t_i - \tau \geq t_i - t_1 \geq 0$ for $\tau \leq t_1$. Hence $\prod_{i=1}^n q(t_i - \tau) \leq q(t_1 - \tau) \prod_{i \geq 2} q(t_i - t_1) = q(t_1 - \tau) P_1(t)$, and by the first inequality in (11) at $s = t_1$ the last integral is at most $\sqrt{\gamma} \varphi(t_1) P_1(t)$. Thus $L \leq \sum_j \varphi(t_j) P_j(t) \sqrt{\gamma + 2t_j}$.

(v) The “upper half” of (9) consists of the second and the fourth term. In the same way, with the second inequality in (11) at $s = t_n$, it is at most $\sum_j \varphi(t_j) P_j(t) \sqrt{\gamma + 2(1 - t_j)}$.

(vi) By concavity of the square root, $\sqrt{\gamma + 2s} + \sqrt{\gamma + 2(1 - s)} \leq 2\sqrt{\gamma + 1}$ for $s \in [0, 1]$. $\square$

Lemma 5.2 (Schur test). *Let $h_0 = 0$, let $h_n: S_n \rightarrow (0, \infty)$ be measurable for $n \geq 1$, and let $\kappa > 0$ be such that $(Th)_n \leq \kappa h_n$ almost everywhere on S_n for every $n \geq 1$. Then $|\langle \psi, b\psi \rangle| \leq \kappa \|\psi\|^2$ for all $\psi \in \Omega^\perp$, and consequently $\|b_Q\| \leq \kappa$.*

Proof. For $\omega' \in S_{n+1}$ let $\omega'_- \in S_n$ and $\omega'_+ \in S_n$ be the configurations obtained by removing the lowest, respectively the highest, point of ω' . Let $\psi \in \Omega^\perp$, so that $\psi_0 = 0$. Since λ and ρ are the adjoints of λ^* and ρ^* ,

$$\langle \psi, b\psi \rangle = 2 \operatorname{Re} \langle \psi, (\lambda^* + \rho^*)\psi \rangle = 2 \operatorname{Re} \sum_{n \geq 1} \int_{S_{n+1}} \overline{\psi_{n+1}(\omega')} (\psi_n(\omega'_-) + \psi_n(\omega'_+)) d\omega'.$$

For $\omega \in \{\omega'_-, \omega'_+\}$ we use

$$2 |\psi_{n+1}(\omega')| |\psi_n(\omega)| \leq |\psi_{n+1}(\omega')|^2 \frac{h_n(\omega)}{h_{n+1}(\omega')} + |\psi_n(\omega)|^2 \frac{h_{n+1}(\omega')}{h_n(\omega)}.$$

All terms are non-negative, so sums and integrals may be interchanged. The first terms on the right give

$$\sum_{m \geq 2} \int_{S_m} |\psi_m(\omega)|^2 \frac{h_{m-1}(\omega_-) + h_{m-1}(\omega_+)}{h_m(\omega)} d\omega.$$

In the second terms we write $\omega' = (\tau, \omega)$ or $\omega' = (\omega, \tau)$ and integrate over τ first; they give

$$\sum_{m \geq 1} \int_{S_m} |\psi_m(\omega)|^2 \frac{1}{h_m(\omega)} \left(\int_0^{\omega_1} h_{m+1}(\tau, \omega) d\tau + \int_{\omega_m}^1 h_{m+1}(\omega, \tau) d\tau \right) d\omega.$$

Because $h_0 = 0$, the sum of the two expressions is $\sum_{m \geq 1} \int_{S_m} |\psi_m|^2 (Th)_m / h_m \leq \kappa \|\psi\|^2$. The last statement holds because b_Q is self-adjoint and $\langle \psi, b_Q \psi \rangle = \langle \psi, b\psi \rangle$ for $\psi \in \Omega^\perp$. $\square$

No integrability or boundedness of h is needed in Lemma 5.2.

Proposition 5.3. (a) (Certificate A) For $\gamma = \frac{2}{3}$ and $\varphi \equiv 1$ the condition (11) holds. Hence $\|b_Q\| \leq 2\sqrt{5/3} = 2.5819\dots$

(b) (Certificate B) For $\gamma = \frac{1}{2}$ and $\varphi(s) = 1 + \frac{3}{5}(2s-1)^2$ the condition (11) holds. Hence $\|b_Q\| \leq \sqrt{6} = 2.4494\dots$

Proof. In both cases $\varphi(1-s) = \varphi(s)$, so the substitution $\tau \mapsto 1-\tau$ turns the second inequality in (11) at s into the first one at $1-s$. It suffices to prove the first. The bounds then follow from Lemmas 5.1 and 5.2 with $\kappa = 2\sqrt{1+\gamma}$.

(a) Here $\int_0^s q(s-\tau) d\tau = \sqrt{\gamma+2s} - \sqrt{\gamma}$, and this is at most $\sqrt{\gamma}$ if and only if $s \leq \frac{3}{2}\gamma = 1$.

(b) For $\gamma = \frac{1}{2}$ we have $q(r)/\sqrt{\gamma} = 2(1+4r)^{-1/2}$. Divide the first inequality in (11) by $\sqrt{\gamma}$, put $r = s - \tau$, and substitute $y = \sqrt{1+4r}$, so that $2(1+4r)^{-1/2} dr = dy$. With $Y = \sqrt{1+4s} \in [1, \sqrt{5}]$ the inequality becomes $E(Y) \geq 0$, where

$$E(Y) = \varphi\left(\frac{Y^2-1}{4}\right) - \int_1^Y \varphi\left(\frac{Y^2-y^2}{4}\right) dy.$$

Now $\varphi(\frac{Y^2-1}{4}) = 1 + \frac{3}{20}(Y^2-3)^2$ and $\varphi(\frac{Y^2-y^2}{4}) = 1 + \frac{3}{20}(A-y^2)^2$ with $A = Y^2-2$, and

$$\int_1^Y (A-y^2)^2 dy = A^2(Y-1) - \frac{2A}{3}(Y^3-1) + \frac{1}{5}(Y^5-1) = \frac{8}{15}Y^5 - Y^4 - \frac{8}{3}Y^3 + \frac{14}{3}Y^2 + 4Y - \frac{83}{15}.$$

Therefore

$$E(Y) = -\frac{2}{25}Y^5 + \frac{3}{10}Y^4 + \frac{2}{5}Y^3 - \frac{8}{5}Y^2 - \frac{8}{5}Y + \frac{209}{50}, \quad E'(Y) = -\frac{2}{5}(Y-2)(Y^3-Y^2-5Y-2). \quad (12)$$

For $0 < Y \leq \sqrt{5}$ the cubic factor is negative, because $Y^3 - Y^2 - 5Y - 2 = Y(Y^2 - 5) - (Y^2 + 2)$. So E decreases on $[1, 2]$ and increases on $[2, \sqrt{5}]$, and

$$\min_{[1, \sqrt{5}]} E = E(2) = -\frac{64}{25} + \frac{24}{5} + \frac{16}{5} - \frac{32}{5} - \frac{16}{5} + \frac{209}{50} = \frac{1}{50} > 0. \quad \square$$

Remark 5.4. (1) The constant in Lemma 5.1 is attained in the limit. If all n points of t approach one point $s \in (0, 1)$, the formula in step (iii) gives $L/h_n \rightarrow \sqrt{\gamma+2s} - \frac{1}{n}(\sqrt{\gamma} - \int_0^s \frac{\varphi(\tau)}{\varphi(s)} q(s-\tau) (\sqrt{\gamma} q(s-\tau))^{n-1} d\tau)$, and the integral is $O(1/n)$. With the upper half, the limit of $(Th)_n/h_n$ is $\sqrt{\gamma+2s} + \sqrt{\gamma+2(1-s)} - 2\sqrt{\gamma}/n + O(n^{-2})$, and for $s = \frac{1}{2}$ this tends to $2\sqrt{1+\gamma}$ as $n \rightarrow \infty$. So for this family h the constant cannot be improved, and a smaller bound needs a smaller γ , which (11) limits. The package contains floating-point tests of Lemma 5.1 on random configurations; they are not part of the proof, and no margin should be read from the largest ratios they report.

(2) *One face.* The same test applies to $\lambda^* + \lambda$ alone, now on all of $\mathcal{H}$: for $h_n(t) = \prod_i (2-2t_i)^{-1/2}$, $n \geq 0$, one finds $h_{n-1}(t_2, \dots, t_n) + \int_0^{t_1} h_{n+1}(\tau, t) d\tau = \sqrt{2} h_n(t)$ for $n \geq 1$ and $\int_0^1 h_1 = \sqrt{2} h_0$, because $\int_0^{t_1} (2-2\tau)^{-1/2} d\tau = \sqrt{2} - \sqrt{2-2t_1}$. This gives $\|\lambda^* + \lambda\| \leq \sqrt{2}$, the edge of the arcsine law of variance 1, so the test is sharp there. Here h is unbounded. For one face there is no gap between the operator and its compression: decomposing $\Omega^\perp$ by the position s of the highest particle, which λ can only remove when it is the last one, one sees that $Q(\lambda^* + \lambda)Q$ is a direct integral over $s \in [0, 1]$ of copies of $\sqrt{s}(\lambda^* + \lambda)$; its norm is again $\sqrt{2}$. This is consistent with the absence of atoms in the arcsine law.

(3) *Where h comes from.* In the language of Section 8, h_n is a sum over the possible pivots t_j : the pivot has weight $\varphi(t_j)$, and every other particle is weighted by a function of its distance to the pivot.

6 The atom: proof of Theorem 1.2(a), (b), (c), (e)

Part (a) is Proposition 5.3(b). Let $\kappa = \sqrt{6}$.

Step 1. By (a) the spectrum of b_Q , and with it the support of ν , lies in $[-\kappa, \kappa]$. So $K(z) = 4 \int d\nu(x)/(z-x)$ is analytic on $\mathbb{C} \setminus [-\kappa, \kappa]$, and by Lemma 4.1(c) it equals $\sum_{n \geq 1} k_{2n} z^{1-2n}$ for $|z| > \kappa$, with all $k_{2n} > 0$.

Step 2. On (κ, ∞) the function $F(z) = z - K(z)$ is real, $F'(z) = 1 + 4 \int d\nu(x)/(z-x)^2 > 1$, and $F(z) \rightarrow +\infty$ as $z \rightarrow +\infty$. Hence F has at most one zero in (κ, ∞) .

Step 3. A real number $z > \kappa$ does not belong to $\sigma(b_Q)$. By Lemma 4.1(a), such a z belongs to $\sigma(b)$ if and only if $F(z) = 0$.

Step 4: the zero exists. Let $c = \max \text{supp } \mu$. Since μ is symmetric, it is carried by $[-c, c]$, and therefore $m_6 = \int x^2 \cdot x^4 d\mu \leq c^2 m_4$. By Lemma 2.2(c), $c^2 \geq m_6/m_4 = \frac{58}{9} > 6 = \kappa^2$. The measure μ is the spectral measure of b at Ω , so $\text{supp } \mu \subset \sigma(b)$; thus $c \in \sigma(b) \cap (\kappa, \infty)$. By Steps 2 and 3, c is the unique zero of F in (κ, ∞) and $\sigma(b) \cap (\kappa, \infty) = \{c\}$. Since $VbV = -b$ (Lemma 2.2), also $\sigma(b) \cap (-\infty, -\kappa) = \{-c\}$. Hence

$$\sigma(b) \subset [-\kappa, \kappa] \cup \{-c, c\}, \quad \|b\| = c = \max \text{supp } \mu.$$

Step 5: c is an atom of μ , not only an eigenvalue of b . By Lemma 4.1(a) the eigenspace of b for the eigenvalue c is spanned by $v = \Omega + (c - b_Q)^{-1}\xi$; in particular the eigenvalue is simple. Since $(c - b_Q)^{-1}\xi \perp \Omega$,

$$\langle \Omega, v \rangle = 1, \quad \|v\|^2 = 1 + \|(c - b_Q)^{-1}\xi\|^2 = 1 + 4 \int \frac{d\nu(x)}{(c-x)^2} = F'(c).$$

The spectral projection of b for the set $\{c\}$ is the orthogonal projection onto the eigenspace $\mathbb{C}v$. Therefore

$$w := \mu(\{c\}) = \frac{|\langle \Omega, v \rangle|^2}{\|v\|^2} = \frac{1}{F'(c)} = \frac{1}{1 - K'(c)} = \left(1 + \sum_{n \geq 1} (2n-1) k_{2n} c^{-2n}\right)^{-1} > 0;$$

the series converges because $c > \kappa$. (Equivalently, $G = 1/F$ is meromorphic on $\mathbb{C} \setminus [-\kappa, \kappa]$ with simple poles exactly at $\pm c$, and w is the residue of G at c .) By symmetry $\mu(\{-c\}) = w$.

Step 6. Put $\mu_0 = \mu - w(\delta_c + \delta_{-c})$. By Step 4 the support of μ lies in $[-\kappa, \kappa] \cup \{\pm c\}$, so μ_0 is the restriction of μ to $[-\kappa, \kappa]$: a symmetric positive measure of mass $1 - 2w$. Finally $2wc^2 \leq \int x^2 d\mu = 4$ gives $w \leq 2/c^2 < \frac{1}{3}$. This proves (b) and (c).

Part (e) follows from (b) and (7). □

7 Positivity: the norms are determined by μ and ν

The measures μ and ν only see the cyclic subspaces generated by Ω under b and by ξ under b_Q . A priori the norms of b and b_Q could be larger than the largest points of the supports of μ and ν . Positivity excludes this.

Proposition 7.1. *Let A be a bounded self-adjoint operator on $L^2(X, m)$, for a σ -finite measure space (X, m) , which maps non-negative functions to non-negative functions. Let $\xi_0 \in L^2(X, m)$ be non-negative, and let r be the largest absolute value of a point in the support of the spectral measure of A at ξ_0 . Assume that the functions ψ for which there are $N \geq 0$ and constants $C_0, \dots, C_N \geq 0$ with $|\psi| \leq \sum_{k=0}^N C_k A^k \xi_0$ almost everywhere are dense in $L^2(X, m)$. Then $\|A\| = r$.*

Proof. The operator A maps real functions to real functions, and $|A\psi| \leq A|\psi|$ almost everywhere for every ψ : for real ψ this follows from $-|\psi| \leq \psi \leq |\psi|$, and in general from $|A\psi| = \sup_{\theta} \operatorname{Re}(e^{i\theta} A\psi) = \sup_{\theta} A \operatorname{Re}(e^{i\theta} \psi) \leq A|\psi|$ with θ in a countable dense subset of $[0, 2\pi]$. Let ψ be as in the hypothesis and $\eta = \sum_k C_k A^k \xi_0$. Then $|A^n \psi| \leq A^n |\psi| \leq A^n \eta$ for all n , and for the spectral measure σ_{ψ} of A at ψ we get

$$\int x^{2n} d\sigma_{\psi}(x) = \|A^n \psi\|^2 \leq \|A^n \eta\|^2 = \sum_{k,l} C_k C_l \langle \xi_0, A^{2n+k+l} \xi_0 \rangle \leq \left(\sum_k C_k r^k \right)^2 \|\xi_0\|^2 r^{2n}.$$

If σ_{ψ} gave a positive mass p to $\{|x| \geq r + \varepsilon\}$ for some $\varepsilon > 0$, the left-hand side would be at least $p(r + \varepsilon)^{2n}$, which is impossible for large n . So the spectral projection E of A for $\mathbb{R} \setminus [-r, r]$ annihilates a dense set of vectors. Hence $E = 0$, $\sigma(A) \subset [-r, r]$ and $\|A\| \leq r$. The inequality $r \leq \|A\|$ is clear. $\square$

Corollary 7.2. (a) $\|b\| = \max \operatorname{supp} \mu$.

(b) $\|b_Q\| = \max \operatorname{supp} \nu$. The sequence k_{2n+2}/k_{2n} , $n \geq 1$, is non-decreasing, and its limit is $\|b_Q\|^2$.

(c) $\|b_Q\|^2 \geq k_{10}/k_8 = \frac{382}{95} > 4$, and $2.1019907 < (k_{306}/k_{304})^{1/2} \leq \|b_Q\| \leq \sqrt{6}$.

Proof. (a) Apply Proposition 7.1 to $A = b$ on $\mathcal{H} = L^2(\bigsqcup_{n \geq 0} S_n)$ and $\xi_0 = \Omega$. By Lemma 2.1 the four operators preserve non-negativity, so $b^n \Omega \geq (\lambda^*)^n \Omega = \mathbf{1}_n$. Every bounded function ψ with finitely many non-zero components satisfies $|\psi| \leq \sum_n \|\psi_n\|_{\infty} b^n \Omega$, and these functions are dense. By the symmetry of μ , $r = \max \operatorname{supp} \mu$.

(b) Apply the proposition to $A = b_Q$ on $\Omega^{\perp} = L^2(\bigsqcup_{n \geq 1} S_n)$ and $\xi_0 = \xi = 2 \cdot \mathbf{1}_1$; now $b_Q^{n-1} \xi \geq (\lambda^*)^{n-1} \xi = 2 \cdot \mathbf{1}_n$ for $n \geq 1$. This gives $\|b_Q\| = \max \operatorname{supp} \nu =: r$. Let $s_n = \int x^{2n} d\nu = k_{2n+2}/4$. By the Cauchy–Schwarz inequality $s_n^2 \leq s_{n-1} s_{n+1}$, so s_{n+1}/s_n is non-decreasing. It is bounded by r^2 , and its limit is $\lim_n s_n^{1/n} = r^2$.

(c) The first ratio is $(6112/15)/(304/3)$, from Table 1. The second is an exact rational number, computed with Theorem 1.3; the inequality $k_{306}/k_{304} > 2.1019907^2$ is checked in exact arithmetic. The upper bound is Theorem 1.2(a). $\square$

So $\|b_Q\| > 2$ follows already from the table (3) of the source. In Section 11 we quote the value $2.10199076 \dots$ for $\max \operatorname{supp} \nu = \|b_Q\|$. It is the largest node of Gauss quadrature rules for ν , computed in floating point; such a node is a lower bound for $\max \operatorname{supp} \nu$, and the value is stable over the range computed. It is a numerical value; what is proved is Corollary 7.2.

8 The pivot model: proof of Theorem 1.3

8.1 The two one-sided systems

A *polynomial vector* is an element of $\mathcal{H}$ with finitely many non-zero components, each of which is a polynomial in the positions of the particles; its coefficients may depend polynomially on a parameter s . For $s \in [0, 1]$ let $\mathcal{H}_{[0,s]}$ and $\mathcal{H}_{[s,1]}$ be the subspaces of functions supported in $[0, s]$ and in $[s, 1]$. On $\mathcal{H}_{[0,s]}$ let $\lambda_{(s)}^*$ and $\lambda_{(s)}$ be the operators (1) for $f = 1_{[0,s]}$: they create and remove the lowest particle, and $(\lambda_{(s)} g)(\Lambda) = \int_0^s g(\tau) d\tau$. On $\mathcal{H}_{[s,1]}$ let $\rho^{(s)*}$ and $\rho^{(s)}$ be the operators for $f = 1_{[s,1]}$: they create and remove the highest particle, and $(\rho^{(s)} g)(\Lambda) = \int_s^1 g(\tau) d\tau$. Put

$$X_s = \lambda_{(s)}^* + \lambda_{(s)} \quad \text{on } \mathcal{H}_{[0,s]}, \quad Y_s = \rho^{(s)*} + \rho^{(s)} \quad \text{on } \mathcal{H}_{[s,1]}.$$

Let $g_u(w) = (w^2 - 2u)^{-1/2} = \sum_{k \geq 0} m_{2k}(u) w^{-2k-1}$, a formal power series in w^{-1} whose coefficients are polynomials in u .

Lemma 8.1 (product formula). *Define formal power series in w^{-1} by*

$$E_s(w)(t_1, \dots, t_n) = g_s(w) \prod_{i=1}^n g_{t_i}(w) \quad \text{on } \mathcal{H}_{[0,s]}, \quad E'_s(w)(t_1, \dots, t_n) = g_{1-s}(w) \prod_{i=1}^n g_{1-t_i}(w) \quad \text{on } \mathcal{H}_{[s,1]}.$$

Their coefficients are polynomial vectors, and for all $a, b \geq 0$

$$X_s^a \Omega = [w^{-a-1}] E_s(w), \quad Y_s^b \Omega = [w^{-b-1}] E'_s(w).$$

Consequently:

- (i) $\langle \Omega, X_s^a \Omega \rangle = m_a(s)$ and $\langle \Omega, Y_s^b \Omega \rangle = m_b(1-s)$;
- (ii) for $s \leq s'$ and a configuration σ in $(s', 1]$, $(Y_s^b \Omega)(s', \sigma) = \sum_{j+k=b-1} m_k(1-s) (Y_{s'}^j \Omega)(\sigma)$;
- (iii) for $s' \leq s$ and a configuration τ in $[0, s')$, $(X_s^a \Omega)(\tau, s') = \sum_{j+k=a-1} m_k(s) (X_{s'}^j \Omega)(\tau)$.

The sums run over $j, k \geq 0$ and are empty for $b = 0$, respectively $a = 0$.

Proof. The n -particle component of $E_s(w)$ is $O(w^{-n-1})$, so the coefficient e_a of w^{-a-1} is a polynomial vector with at most a particles, and $e_0 = \Omega$. The operators act coefficientwise. We claim that $X_s E_s(w) = w E_s(w) - \Omega$; comparing coefficients, this says $X_s e_a = e_{a+1}$, hence $e_a = X_s^a \Omega$. Coefficientwise $\partial_u(w - (w^2 - 2u)^{1/2}) = g_u(w)$, and $w - (w^2 - 2u)^{1/2}$ vanishes at $u = 0$, so $\int_0^t g_u(w) du = w - 1/g_t(w)$. For $n \geq 1$ this gives

$$(\lambda_{(s)}^* E_s)(t) = E_s(t_2, \dots, t_n) = \frac{E_s(t)}{g_{t_1}(w)}, \quad (\lambda_{(s)} E_s)(t) = E_s(t) \int_0^{t_1} g_\tau(w) d\tau = E_s(t) \left(w - \frac{1}{g_{t_1}(w)} \right),$$

with sum $w E_s(t)$. At Λ we get $(\lambda_{(s)} E_s)(\Lambda) = g_s(w) \int_0^s g_\tau(w) d\tau = w g_s(w) - 1$ and $(\lambda_{(s)}^* E_s)(\Lambda) = 0$. This proves the claim. The computation for Y_s is the mirror image, with $\int_{t_n}^1 g_{1-\tau}(w) d\tau = w - 1/g_{1-t_n}(w)$.

(i) The vacuum components of $E_s(w)$ and $E'_s(w)$ are $g_s(w)$ and $g_{1-s}(w)$. (ii) By definition $E'_s(w)(s', \sigma) = g_{1-s}(w) \cdot E'_{s'}(w)(\sigma)$; take the coefficient of w^{-b-1} . (iii) is the mirror image: $E_s(w)(\tau, s') = g_s(w) \cdot E_{s'}(w)(\tau)$. $\square$

For $s = 1$, part (i) is the arcsine law of one face.

8.2 The lift and the intertwining relation

A *marked configuration* $(\tau; s; \sigma)$ consists of a point $s \in [0, 1]$, a configuration τ in $[0, s)$ and a configuration σ in $(s, 1]$. For a function F of marked configurations define $\pi(F) \in \mathcal{H}$ by $\pi(F)(\Lambda) = 0$ and

$$\pi(F)(t_1, \dots, t_n) = \sum_{j=1}^n F(t_{<j}; t_j; t_{>j}), \quad t_{<j} = (t_1, \dots, t_{j-1}), \quad t_{>j} = (t_{j+1}, \dots, t_n).$$

We use functions of the form $F(\tau; s; \sigma) = c(s) A_s(\tau) A'_s(\sigma)$, where c is a polynomial and $A_s \in \mathcal{H}_{[0,s]}$, $A'_s \in \mathcal{H}_{[s,1]}$ are polynomial vectors depending polynomially on s .

Lemma 8.2. For $F(\tau; s; \sigma) = c(s)A_s(\tau)A'_s(\sigma)$ as above,

$$b\pi(F) = \pi(c(X_s A_s)A'_s) + \pi(cA_s(Y_s A'_s)) + \pi(U \bullet F) + \pi(D \bullet F) + 2\left(\int_0^1 c(u)A_u(\Lambda)A'_u(\Lambda)du\right)\Omega, \quad (13)$$

where

$$(U \bullet F)(\tau; s; \sigma) = 1_{\{\tau=\Lambda\}} \int_0^s c(u)A_u(\Lambda)A'_u(s, \sigma)du,$$

$$(D \bullet F)(\tau; s; \sigma) = 1_{\{\sigma=\Lambda\}} \int_s^1 c(u)A_u(\tau, s)A'_u(\Lambda)du.$$

Proof. Let $n \geq 1$ and $t \in S_n$. By (5),

$$(\lambda^* \pi(F))(t) = \pi(F)(t_2, \dots, t_n) = \sum_{j \geq 2} c(t_j)A_{t_j}(t_2, \dots, t_{j-1})A'_{t_j}(t_{>j}).$$

Since $(\lambda_{(s)}^* A_s)(t_{<j})$ is $A_s(t_2, \dots, t_{j-1})$ for $j \geq 2$ and 0 for $j = 1$, this is $\pi(c(\lambda_{(s)}^* A_s)A'_s)(t)$. Next, $(\lambda \pi(F))(t) = \int_0^{t_1} \pi(F)(\tau, t) d\tau$. In $\pi(F)(\tau, t)$ the marked point is either one of the t_j or the new point τ :

$$\pi(F)(\tau, t) = \sum_{j=1}^n c(t_j)A_{t_j}(\tau, t_{<j})A'_{t_j}(t_{>j}) + c(\tau)A_\tau(\Lambda)A'_\tau(t).$$

The integral of the j -th term over $\tau \in [0, t_1]$ is $c(t_j)(\lambda_{(t_j)} A_{t_j})(t_{<j})A'_{t_j}(t_{>j})$; for $j = 1$ this uses that the vacuum component of $\lambda_{(s)} A_s$ is $\int_0^s A_s(\tau) d\tau$. The integral of the last term is $(U \bullet F)(\Lambda; t_1; t_{>1})$, which equals $\pi(U \bullet F)(t)$ because $U \bullet F$ vanishes unless its lower configuration is empty. At the vacuum, $(\lambda^* \pi(F))(\Lambda) = 0$ and $(\lambda \pi(F))(\Lambda) = \int_0^1 \pi(F)(u) du = \int_0^1 c(u)A_u(\Lambda)A'_u(\Lambda) du$. Hence

$$(\lambda^* + \lambda) \pi(F) = \pi(c(X_s A_s)A'_s) + \pi(U \bullet F) + \left(\int_0^1 c(u)A_u(\Lambda)A'_u(\Lambda) du\right)\Omega.$$

The mirror computation gives the same for $\rho^* + \rho$, with Y_s acting on A'_s and $D \bullet F$ in place of $U \bullet F$: in $\pi(F)(t, \tau)$ with $\tau > t_n$ the term with marked point τ is $c(\tau)A_\tau(t)A'_\tau(\Lambda)$, and its integral over $\tau \in [t_n, 1]$ is $(D \bullet F)(t_{<n}; t_n; \Lambda)$. $\square$

For $a, b, c \geq 0$ let $F_{a,b,c}(\tau; s; \sigma) = s^c(X_s^a \Omega)(\tau)(Y_s^b \Omega)(\sigma)$, and define the linear map $\Pi: \mathcal{P} \rightarrow \Omega^\perp$ by

$$\Pi(x_1^a x_3^b s^c) = \pi(F_{a,b,c}), \quad \text{that is} \quad \Pi(x_1^a x_3^b s^c)(t) = \sum_{j=1}^n t_j^c (X_{t_j}^a \Omega)(t_{<j}) (Y_{t_j}^b \Omega)(t_{>j}).$$

For a polynomial $f = \sum f_{abc} x_1^a x_3^b s^c$ we write $F_f = \sum f_{abc} F_{a,b,c}$, so that $\Pi(f) = \pi(F_f)$. For example $\Pi(1) = \mathbf{1}_1$ and $\Pi(x_1) = \Pi(x_3) = \mathbf{1}_2$; the map Π is not injective.

Proof of Theorem 1.3. The intertwining relation. Apply Lemma 8.2 to $F = F_{a,b,c}$. The first two terms of (13) are $\pi(F_{a+1,b,c}) + \pi(F_{a,b+1,c}) = \Pi(M x_1^a x_3^b s^c)$. By Lemma 8.1(i) and (ii), the latter with (s, s') replaced by (u, s) ,

$$(U \bullet F)(\tau; s; \sigma) = 1_{\{\tau=\Lambda\}} \int_0^s u^c m_a(u) (Y_u^b \Omega)(s, \sigma) du$$

$$= 1_{\{\tau=\Lambda\}} \sum_{j+k=b-1} \left(\int_0^s u^c m_a(u) m_k(1-u) du \right) (Y_s^j \Omega)(\sigma).$$

Since $1_{\{\tau=\Lambda\}} = (X_s^0 \Omega)(\tau)$, this is the function F_f for the polynomial $f = U(x_1^a x_3^b s^c)$, so $\pi(U \bullet F) = \Pi(U(x_1^a x_3^b s^c))$. In the same way, by Lemma 8.1(i) and (iii),

$$(D \bullet F)(\tau; s; \sigma) = 1_{\{\sigma=\Lambda\}} \sum_{j+k=a-1} \left(\int_s^1 u^c m_b(1-u) m_k(u) du \right) (X_s^j \Omega)(\tau).$$

The polynomial $\sum_{j+k=a-1} \left(\int_s^1 u^c m_b(1-u) m_k(u) du \right) x_1^j$ is $RUR(x_1^a x_3^b s^c)$: indeed $R(x_1^a x_3^b s^c) = x_1^b x_3^a (1-s)^c$, then U gives $\sum_{j+k=a-1} \left(\int_0^s (1-u)^c m_b(u) m_k(1-u) du \right) x_3^j$, and applying R and substituting $u \mapsto 1-u$ gives the polynomial above. So $\pi(D \bullet F) = \Pi(D(x_1^a x_3^b s^c))$. Finally the coefficient of Ω in (13) is $2 \int_0^1 u^c m_a(u) m_b(1-u) du = 2 \langle x_1^a x_3^b s^c \rangle$. This proves (4) for monomials, hence for all $f \in \mathcal{P}$, and $b\Omega = 2 \cdot \mathbf{1}_1 = 2\Pi(1)$.

The cumulants. By induction on $n \geq 0$,

$$b^{n+1}\Omega = 2\Pi(B^n \mathbf{1}) + 4 \sum_{j=0}^{n-1} \langle B^j \mathbf{1} \rangle b^{n-1-j} \Omega :$$

for $n = 0$ this is $b\Omega = 2\Pi(1)$, and applying b and (4) gives the statement for $n+1$. The vectors $\Pi(f)$ are orthogonal to Ω . Taking the vacuum component we get $m_N = \sum_{i=2}^N 4 \langle B^{i-2} \mathbf{1} \rangle m_{N-i}$ for all $N \geq 1$. These are the relations which define the Boolean cumulants, and they determine them; hence $k_1 = 0$ and $k_i = 4 \langle B^{i-2} \mathbf{1} \rangle$ for $i \geq 2$.

Complexity. Give x_1 and x_3 the weight 1 and s the weight 2. The maps M , U and D raise the weight of a polynomial by at most one: for U , a monomial of $U(x_1^a x_3^b s^c)$ is a multiple of $x_3^j s^e$ with $j+k=b-1$ and $e \leq c + \frac{a}{2} + \frac{k}{2} + 1$, and R does not raise the weight. So $B^n \mathbf{1}$ is a combination of the $O(n^3)$ monomials of weight at most n . In $U(x_1^a x_3^b s^c)$ the exponents a and c enter only through $u^c m_a(u)$, a multiple of $u^{c+a/2}$. Summing first over a , one applies U to a polynomial of weight at most n with $O(n^4)$ operations: there are $O(n^2)$ pairs $(b, c + \frac{a}{2})$, $O(n)$ values of k , and $O(n)$ terms in the expansion of $(1-u)^{k/2}$. The maps M , R and the functional cost no more. Thus $k_2, \dots, k_N$ cost $O(N^5)$ operations, and the moments follow from the cumulants with $O(N^2)$ operations.

The size of the numbers. With $(1-u)^{k/2}$ and $(1-s)^c$ expanded, each of the maps M , U , R sends a monomial of weight at most n to at most $(n+1)^2$ monomials. Their coefficients are 1 for M , $\pm \binom{c}{i}$ for R , and for U the numbers $\pm \binom{a}{a/2} \binom{k}{k/2} \binom{k/2}{i}$ divided by $2^{(a+k)/2} (c + \frac{a}{2} + i + 1)$. The numerators are at most 4^n . The denominators divide $L_n = 2^n \text{lcm}(1, \dots, n+1)$, because $a+k \leq n$ and $c + \frac{a}{2} + i + 1 \leq n+1$, and $L_n \leq 2^{O(n)}$ by Chebyshev's estimate of $\text{lcm}(1, \dots, m)$. Hence, if f has weight at most n and Δf has integer coefficients with sum of absolute values A , then $\Delta L_n B f$ has integer coefficients with sum of absolute values at most $2^{O(n)} A$; for $D = RUR$ the three maps are applied in turn, and only U divides. The functional divides in the same way as U . By induction, all numbers which occur up to the n -th step, the values $\langle B^j \mathbf{1} \rangle$ included, have the form $p/(L_1 \cdots L_{n+1})$ with integers p of $O(n^2)$ binary digits, and $L_1 \cdots L_{n+1}$ has $O(n^2)$ binary digits. In the last step, $m_N = \sum_j k_j m_{N-j}$, all terms are non-negative and at most 4^N , and $n! m_{2n}$ is an integer by (2); so the numbers there are of this size as well. $\square$

8.3 Remarks on the model

Remark 8.3 (without U and D : a semicircle). For fixed s the functional $f \mapsto \int f d(\alpha_s \otimes \alpha_{1-s})$, with α_u the arcsine law of variance u , gives $x_1^a x_3^b$ the value $m_a(s) m_b(1-s)$. Integrating the product density over $s \in [0, 1]$ gives $\frac{1}{2} \int (\pi^2 \sqrt{(u-x_1^2)(2-x_3^2-u)})^{-1} du = \frac{1}{2\pi}$ for $x_1^2 + x_3^2 < 2$ (with $u = 2s$): under $\langle \cdot \rangle$ the pair (x_1, x_3) is uniformly distributed on the disc of radius $\sqrt{2}$. Hence $x_1 + x_3$ has the semicircle law of variance 1, and $\langle (x_1 + x_3)^{2n} \rangle$ is the n -th Catalan number. If U and D are dropped from B ,

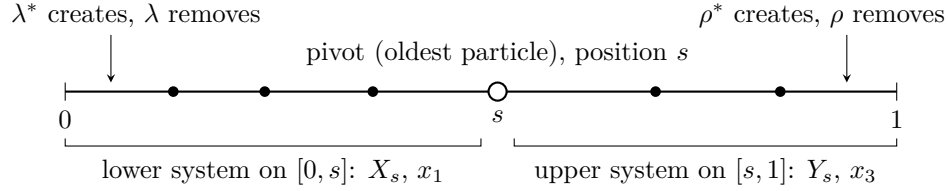


Figure 2: The pivot picture. A word in the four operators acts on the particles like on a double-ended queue ordered by position. The particles below the oldest one, the pivot, were created by λ^* after it and form a system on which only λ^*, λ act; those above it form a system on which only ρ^*, ρ act. A monomial $x_1^a x_3^b s^c$ stands for the state with lower system $X_s^a \Omega$ and upper system $Y_s^b \Omega$. The pivot can be removed by λ only when the lower system is empty (the map U), and by ρ only when the upper system is empty (the map D); then the next oldest particle becomes the pivot, and Lemma 8.1(ii), (iii) express the new state in the same form.

one therefore gets $K(z) = 4G_{\text{sc}}(z)$ with the Cauchy transform G_{sc} of the semicircle law, that is, the measure with Jacobi parameters $(4, 1, 1, \dots)$ of Section 1.3, with atoms at $\pm 4/\sqrt{3}$ of mass $\frac{1}{3}$. The terms U and D move the atoms to $\pm c = \pm 2.5858\dots$ and lower their mass to $0.2513\dots$; the true Jacobi parameters begin with $4, 2, \frac{4}{3}, \frac{7}{6}, \dots$.

Remark 8.4 (the shape of the resolvent). Since $RMR = M$, $RUR = D$ and $R^2 = 1$, the map B commutes with R , and $RB^n 1 = B^n 1$. Let $\Phi = \sum_{n \geq 0} z^{-n-1} B^n 1$, a formal power series in z^{-1} with coefficients in $\mathcal{P}$; then $K(z) = 4\langle \Phi \rangle$ and $(z - x_1 - x_3)\Phi = 1 + U\Phi + D\Phi$. Here $\mathcal{U} := U\Phi$ depends only on (x_3, s) , and $D\Phi = RU\Phi$. So

$$\Phi = \frac{1 + \mathcal{U}(x_3, s; z) + \mathcal{U}(x_1, 1 - s; z)}{z - x_1 - x_3}, \quad \mathcal{U} = \frac{s}{z^2} + \frac{s x_3}{z^3} + \frac{2s + \frac{3}{2}s^2 + s x_3^2}{z^4} + \dots,$$

with a single unknown series $\mathcal{U}$ in two variables, determined by the linear equation $\mathcal{U} = U\Phi$. A closed formula for μ would follow from a solution of this equation. We could not solve it.

Remark 8.5 (a solvable sector; tested, not proved). The polynomial $x_1 - x_3$ lies in the kernel of Π . On it the resolvent of B appears to be explicit:

$$\sum_{n \geq 0} z^{-n-1} B^n(x_1 - x_3) = \frac{\sqrt{(z - x_3)^2 - 2s} - \sqrt{(z - x_1)^2 - 2(1 - s)}}{z - x_1 - x_3}$$

as formal power series in z^{-1} . This identity was tested to order 12 by an exact program of the package; it is not proved here, and it is not used.

Remark 8.6 (integrality, and a faster formulation). (1) For all computed n the numbers $n!m_{2n}$ and $n!k_{2n}$ are integers (for $n \leq 153$ in the files of this note, for $n \leq 280$ in those of the second verification run). For $n!m_{2n}$ this is (2). For $n!k_{2n}$ it would follow from a decomposition formula which is stated, without proof, in the remark at the end of Section 5 of [Ger17]: by it, $n!k_{2n}$ is the number of irreducible bi-monotone pair partitions of $\{1, \dots, 2n\}$, which an enumeration by the third verification run confirms for $n \leq 8$. We do not use this.

(2) The second verification run reorganised the computation. Write $B^n 1 = \sum_{a,b} F_n[a, b](s) x_1^a x_3^b$ with $F_n[a, b] \in \mathbb{Q}[s]$, and let $\binom{2i}{i}$ be denoted by C_i . With

$$g_b(u) = \sum_{a \text{ even}} F_n[a, b](u) C_{a/2} (u/2)^{a/2}, \quad H_j(s) = \int_0^s \sum_{i \geq 0} g_{j+2i+1}(u) C_i \left(\frac{1-u}{2}\right)^i du,$$

one has $U(B^n 1) = \sum_j H_j(s) x_3^j$ and $D(B^n 1) = \sum_j H_j(1-s) x_1^j$, and $\langle B^n 1 \rangle$ is the integral over $[0, 1]$ of $\sum_{b \text{ even}} g_b(s) C_{b/2} (\frac{1-s}{2})^{b/2}$. This takes $O(n^2)$ operations in $\mathbb{Q}[s]$ per step. The program of that run computed $k_2, \dots, k_{560}$ in a few minutes; its values agree with those of the program of this note on the common range $k_2, \dots, k_{306}$.

9 Exact enclosures: proof of Theorem 1.2(d)

Lemma 9.1 (remainder bounds). *Let $\kappa_+ \geq \|b_Q\|$, let $J \geq 0$ be an integer and $z > \kappa_+$. Put*

$$K_J(z) = \sum_{j=0}^J k_{2j+2} z^{-2j-1}, \quad D_J(z) = 1 + \sum_{j=0}^J (2j+1) k_{2j+2} z^{-2j-2}, \quad t_J(z) = k_{2J+4} z^{-2J-2},$$

and $e_J(z) = t_J(z) \left(\frac{2J+3}{z(z-\kappa_+)} + \frac{\kappa_+}{z(z-\kappa_+)^2} \right)$. Then

$$K_J(z) + \frac{t_J(z)}{z + \kappa_+} \leq K(z) \leq K_J(z) + \frac{t_J(z)}{z - \kappa_+}, \quad |F'(z) - D_J(z)| \leq e_J(z).$$

Moreover F is increasing and F' is decreasing on (κ_+, ∞) .

Proof. For $|x| \leq \kappa_+$ we have $\frac{1}{z-x} = \sum_{i=0}^{2J+1} x^i z^{-i-1} + \frac{(x/z)^{2J+2}}{z-x}$. Integrate against 4ν . The odd moments of ν vanish and $4 \int x^{2j} d\nu = k_{2j+2}$ (Lemma 4.1), so $K(z) = K_J(z) + 4 \int x^{2J+2} z^{-2J-2} (z-x)^{-1} d\nu(x)$. The factor x^{2J+2} is non-negative and $\frac{1}{z+\kappa_+} \leq \frac{1}{z-x} \leq \frac{1}{z-\kappa_+}$ on the support of ν ; this gives the bounds for K . Similarly

$$\frac{1}{(z-x)^2} = \sum_{i=0}^{2J+1} (i+1) x^i z^{-i-2} + \frac{(2J+3) x^{2J+2} z^{-2J-3}}{z-x} + \frac{x^{2J+3} z^{-2J-3}}{(z-x)^2},$$

which is the derivative with respect to x of the identity $\frac{1}{z-x} = \sum_{i=0}^{2J+2} x^i z^{-i-1} + \frac{(x/z)^{2J+3}}{z-x}$. Integrating against 4ν and using $|x|^{2J+3} \leq \kappa_+ x^{2J+2}$ and $\frac{1}{z-x} \leq \frac{1}{z-\kappa_+}$ gives the bound for $F'(z) = 1 + 4 \int (z-x)^{-2} d\nu$. Finally $F' > 0$ and $F''(z) = -8 \int (z-x)^{-3} d\nu < 0$ on (κ_+, ∞) . $\square$

Proof of Theorem 1.2(d). Let $\kappa_+ = \frac{49}{20}$. Since $\kappa_+^2 = \frac{2401}{400} > 6$, we have $\kappa_+ > \sqrt{6} \geq \|b_Q\|$ by Theorem 1.2(a). By Lemma 9.1,

$$F_{\text{lo}}(z) := z - K_J(z) - \frac{t_J(z)}{z - \kappa_+} \leq F(z) \leq F_{\text{up}}(z) := z - K_J(z) - \frac{t_J(z)}{z + \kappa_+} \quad (z > \kappa_+).$$

Suppose that $z_- < z_+$ are rational numbers larger than κ_+ with $F_{\text{up}}(z_-) < 0 < F_{\text{lo}}(z_+)$ and $D_J(z_+) > e_J(z_+)$. Then $F(z_-) < 0 < F(z_+)$, so $z_- < c < z_+$, because c is the only zero of the increasing function F . Since F' is decreasing, $F'(z_+) < F'(c) < F'(z_-)$, and with $w = 1/F'(c)$,

$$z_- < c < z_+, \quad \frac{1}{D_J(z_-) + e_J(z_-)} < w < \frac{1}{D_J(z_+) - e_J(z_+)}.$$

For rational z all quantities here are rational numbers which are computed exactly from $k_2, \dots, k_{2J+4}$. From $m_0, \dots, m_N$ one obtains $k_2, \dots, k_N$ by the defining relations, so $J = (N-4)/2$ can be used: $J = 20$ for $N = 44$ and $J = 151$ for $N = 306$. The exact program of this note finds suitable $z_{\pm}$ by bisection and then verifies the inequalities $F_{\text{up}}(z_-) < 0 < F_{\text{lo}}(z_+)$ and evaluates the bounds in exact rational arithmetic. Its results are in Table 2; the intervals stated in Theorem 1.2(d) are these, rounded outward. $\square$

moments or cumulants used	J	enclosure of c	enclosure of w
$m_0, \dots, m_{44}$, from the definition	20	(2.585794744, 2.586036944)	(0.250011133, 0.253121332)
$m_0, \dots, m_{160}$, model, first run	78	width $8.8 \cdot 10^{-15}$	width $3.2 \cdot 10^{-13}$
$m_0, \dots, m_{200}$, model	98	width $2.2 \cdot 10^{-18}$	width $9.7 \cdot 10^{-17}$
$m_0, \dots, m_{306}$, model	151	width $6.4 \cdot 10^{-28}$	width $4.2 \cdot 10^{-26}$
$k_2, \dots, k_{560}$, model, second run	278	width $9.0 \cdot 10^{-51}$	width $1.0 \cdot 10^{-48}$

Table 2: Enclosures of c and w with $\kappa_+ = \frac{49}{20}$. Rows 1, 3 and 4: the program of this note. Row 2: the first verification run, with the cumulants of its own implementation of Theorem 1.3; its intervals are $2.585826606048203 < c < 2.585826606048213$ and $0.25133165463828 < w < 0.25133165463861$. Row 5: the second verification run. “Model” means that the moments come from Theorem 1.3. The intervals of rows 3 to 5 are nested and contain the numbers displayed below.

What each line depends on, and what was reproduced. The first line of Theorem 1.2(d) depends on Theorem 1.2(a), on the moments $m_0, \dots, m_{44}$ computed directly from the definition, and on the evaluation just described; it does not depend on Theorem 1.3. All three verification runs reproduced it with their own programs for the moments and for the bounds ($2.58579474 < c < 2.58603695$ and $0.25001113 < w < 0.25312134$ in the first run). The second line depends in addition on Theorem 1.3 and on the long run of the program of this note. The second and the third run reproduced both intervals from their own cumulants $k_2, \dots, k_{306}$; the first run reproduced the cumulants up to k_{160} , which confirms 14 significant digits of c and 12 of w (row 2). From its cumulants up to k_{560} the second run obtained

$$c = 2.5858266060482044486315152361168415078092428079979 \dots,$$

$$w = 0.25133165463841931453050567432858126056719804 \dots$$

The statement $2w > \frac{1}{2}$ is certified by the first line only with a small margin: $w > 0.2500111$ exceeds $\frac{1}{4}$ by $1.1 \cdot 10^{-5}$. This needs the constant $\sqrt{6}$: with the constant $2\sqrt{5/3}$ of Certificate A the same 44 moments give only $0.0717 < w < 0.6145$, and with $m_0, \dots, m_{42}$ and $\kappa_+ = \frac{49}{20}$ the lower bound is 0.2493. From the moments $m_0, \dots, m_{48}$, which the second run computed from the definition, it obtained $w > 0.25072$. With the moments of Theorem 1.3 the margin is large.

10 More of the measure above 2: proof of Theorem 1.4

Lemma 10.1. *Let $0 < a < b$ and $p(x) = (x^2 - a^2)(x^2 - b^2)$. If there is a real polynomial q with*

$$I(q) := \int p(x) q(x)^2 d\mu(x) < 0,$$

then $\mu((a, b)) = \mu((-b, -a)) > 0$. For an even polynomial $q(x) = \sum_i q_i x^{2i}$,

$$I(q) = \sum_{i,j} q_i q_j (m_{2i+2j+4} - (a^2 + b^2) m_{2i+2j+2} + a^2 b^2 m_{2i+2j}). \quad (14)$$

Proof. The polynomial p is negative exactly on $(a, b) \cup (-b, -a)$. If $\mu((a, b)) = 0$, then $\mu((-b, -a)) = 0$ by symmetry, so $p \geq 0$ μ -almost everywhere and $I(q) \geq 0$ for every real polynomial q . $\square$

In the language of the truncated moment problem, $q \mapsto I(q)$ is the localizing form of the moment sequence (m_n) for the polynomial p [CF00]: a measure carried by the set $\{p \geq 0\}$ has a positive semidefinite localizing form, so one polynomial q with $I(q) < 0$ certifies that μ is not carried by this

window (a, b)	K	moments used	computed by	$I(q)/\int q^2 d\mu$
(2.0114, 2.0116)	100	$m_0, \dots, m_{200}$	note; second run	$-1.55 \cdot 10^{-7}$
(2.01148, 2.01149)	152	$m_0, \dots, m_{304}$	note; second run	$-3.97 \cdot 10^{-10}$
(2.001, 2.003)	152	$m_0, \dots, m_{304}$	note; second run	$-7.31 \cdot 10^{-6}$
(2.011485, 2.011486)	278	$m_0, \dots, m_{556}$	second run	$-3.43 \cdot 10^{-12}$
(2.0017, 2.0018)	278	$m_0, \dots, m_{556}$	second run	$-3.63 \cdot 10^{-8}$
(2.0011, 2.0024)	278	$m_0, \dots, m_{556}$	second run	$-6.77 \cdot 10^{-6}$

Table 3: Window certificates. In each row $I(q) < 0$ for the polynomial q described in the text; the sign is decided on exact rational numbers, and the last column is a floating-point value of the quotient, for orientation. “Note; second run”: computed by the program of this note from its moments, and again by the program of the second verification run from its own moments, with the same value.

set. For rational a, b and rational coefficients q_i the number $I(q)$ is rational and is computed exactly from the moments by (14); if q has degree $K - 2$, the moments up to m_{2K} enter.

Only the sign of $I(q)$ matters; how q is found does not. The programs choose $q = (P_K - P_K(r))/(x^2 - r^2)$, where P_K is the monic orthogonal polynomial of even degree K for μ , computed exactly from the moments, and r^2 is a rational approximation of the zero of P_K , as a polynomial in x^2 , in the window (a^2, b^2) . The results are in Table 3.

Proof of Theorem 1.4. By Lemma 10.1 and Table 3, rows 2 and 3 for part (a) and rows 4 and 5 for part (b). All moments beyond m_{48} come from Theorem 1.3. The statement about μ_0 holds because the windows lie in $(2, \sqrt{6})$, and the statement about the spectrum because $\text{supp } \mu \subset \sigma(b)$. $\square$

Two attempts gave no conclusion, that is, $I(q) > 0$ for the polynomial q constructed in this way: the window (2.0011, 2.0024) with $K = 152$, and the window (2.0004, 2.0009) with $K = 278$.

11 Numerical observations and questions

Nothing in this section is proved, except where a theorem is quoted. The Jacobi parameters β_k of μ were computed exactly from the exact moments, $\beta_1, \dots, \beta_{153}$ by the program of this note and $\beta_1, \dots, \beta_{280}$ by the second verification run. The observations come from floating-point computations with them, with 60 to 270 digits.

We recall the notions [Gau04]. The monic orthogonal polynomials of the symmetric measure μ satisfy $P_{k+1}(x) = xP_k(x) - \beta_k P_{k-1}(x)$ with $\beta_k > 0$. The zeros of P_K are the eigenvalues of the $K \times K$ Jacobi matrix with zero diagonal and off-diagonal entries $\sqrt{\beta_1}, \dots, \sqrt{\beta_{K-1}}$; they are the nodes of the Gauss quadrature rule with K points and lie in the convex hull of $\text{supp } \mu$. The measure ν of Section 4 has the Jacobi parameters $\beta_2, \beta_3, \dots$, and $\beta_1 = k_2 = 4$.

Observation 1: Jacobi parameters. Exactly, $\beta_1, \dots, \beta_8 = 4, 2, \frac{4}{3}, \frac{7}{6}, \frac{11}{10}, \frac{414}{385}, \frac{16868}{15939}, \frac{102259729}{97766928}$. Further values are in Table 4. The β_k seem to decrease to 1, slowly. The quantity $k(\beta_k - 1)$ is about 0.142 at $k \approx 150$ and is still decreasing, to 0.138 at $k = 280$; a limit is not determined by the data.

Observation 2: Gauss nodes above 2. See Table 4. The second node has converged: $c_2 \approx 2.011485694658159583$ with weight $w_2 \approx 0.0031285045345$. Theorem 1.4 proves that μ gives positive mass to (2.011485, 2.011486); it does not prove that c_2 is an atom. The third node has settled near 2.0017597, with weight about $2.016 \cdot 10^{-4}$, inside the window (2.0017, 2.0018) of Theorem 1.4. A fourth node is near 2.00065 and is still moving, and a fifth has just passed 2 at $K = 280$.

Observation 3: the part in $[-2, 2]$. The total weight of the Gauss nodes in $[-2, 2]$ is between 0.4902 and 0.4907 for all K from 100 to 280; it increases slowly with K and drops when a node passes the point 2 (at $K = 154$ and at $K = 277$). A rough density, obtained from the continued fraction of G

k	10	20	50	100	150	200	240	280
β_k	1.030482	1.010177	1.003106	1.001467	1.000947	1.000702	1.000579	1.000493
$k(\beta_k - 1)$	0.3048	0.2035	0.1553	0.1467	0.1420	0.1404	0.1389	0.1381

node	$K = 100$	$K = 153$	$K = 200$	$K = 280$	weight at $K = 280$
second	2.0114856926	2.0114856947	2.0114856947	2.0114856947	0.0031285
third	2.001463	2.001744	2.001759	2.001760	0.000202
fourth	(1.99739)	(1.99998)	2.00048	2.00065	0.000058
fifth			(1.99914)	2.00002	0.000073
weight in $[-2, 2]$	0.49045	0.49066	0.49047	0.49041	

Table 4: Numerical data, not proved. Above: Jacobi parameters of μ (exact rationals, rounded). Below: Gauss nodes larger than 2 and their weights, apart from the largest node, which agrees with c and w to all digits shown in Theorem 1.2(d) for every $K \geq 100$; a node in parentheses is still below 2. The last row is the total weight of the nodes in $[-2, 2]$.

cut after K levels and closed with the tail $\beta \equiv 1$, is 0.132 at $x = 0$, 0.130 at 0.5, 0.125 at 1, 0.119 at 1.5, 0.110 at 1.75 and 0.101 at 1.9, for all K between 100 and 280. It is nearly flat and does not seem to vanish at ± 2 .

Observation 4: the compression. The largest Gauss node of ν is 2.101990760466021919875, with weight 0.0998095879805, for all K from 100 to 280. By Corollary 7.2 this is a numerical value for $\|b_Q\| = \max \text{supp } \nu$, and it suggests that ν has an atom there.

Observation 5: truncations. The first verification run computed eigenvalues of piecewise constant Galerkin truncations of b , of dimension up to 10^7 , which use neither moments nor any of the results above. The largest eigenvalue is 2.58548 and increases with the dimension (extrapolated: 2.58581); the vacuum weight of its eigenvector is 0.25171 (extrapolated: 0.25135); the largest eigenvalue of the truncated compression is 2.0955 (extrapolated: 2.100), and no second eigenvalue of a truncated b exceeds 1.98. This agrees with Theorem 1.2 and Corollary 7.2, at this lower accuracy.

Observation 6: searches that found nothing. The sequences m_{2n} , $n! m_{2n}$, $m_{2n}/(2n)!$, k_{2n} , $n! k_{2n}$ and some variants satisfy no linear recurrence of order r with polynomial coefficients of degree at most d for $(r, d) = (4, 27)$, $(8, 14)$ and other pairs with $r \leq 14$, on 154 terms (program of this note), and for $(r, d) = (2, 89)$, $(4, 52)$, $(8, 28)$, $(14, 16)$, $(24, 8)$ and other pairs with $r \leq 24$, on 281 terms (second run). For these sizes the statement is exact (full column rank modulo a prime); it says nothing beyond them. No integer polynomial of degree at most 10 with coefficients up to 10^4 was found which vanishes at c , c^2 , w or $1/w$, and no integer relation with coefficients up to 100 between c , c^2 or w and the numbers $1, \pi, \pi^2, \sqrt{2}, \sqrt{3}, \log 2, e^\pi, e^{-\pi}, e^{\pi/2}$; the second run searched in addition with e , Euler's constant and Catalan's constant, with coefficients up to 1000 and 44 digits.

Questions. (1) Find a closed formula for μ , or for its Cauchy transform; this is the question of Gerhold. (2) Is it true that

$$\mu = \sum_{k \geq 1} w_k (\delta_{c_k} + \delta_{-c_k}) + \varrho(x) dx$$

with $c = c_1 > c_2 > c_3 > \dots$ decreasing to 2 and a continuous density ϱ on $[-2, 2]$? The data suggest $c_2 \approx 2.0114857$, $c_3 \approx 2.00176$, $\sum_k 2w_k \approx 0.5096$ and $\varrho(0) \approx 0.132$. Proved are only the first atom, the gap $(\sqrt{6}, c)$, and the two windows of Theorem 1.4. (3) Does $k(\beta_k - 1)$ converge, and to which limit? For a perturbation of the free Jacobi matrix of Coulomb type, decaying like $1/k$, one expects infinitely many eigenvalues accumulating at the edge; as a heuristic, the ansatz $c_k - 2 \approx \alpha^2/(k - k_0)^2$ fitted to c_2, c_3 gives $\alpha \approx 0.069$, $k_0 \approx 1.36$ and $c_4 \approx 2.0007$. (4) Is the generating function of the moments D-finite? Is c algebraic? Is there a recursion for $\#PP_{\bowtie}(2n)$? (5) Solve the equation of Remark 8.4,

and prove the identity of Remark 8.5.

12 Computations and reproducibility

What depends on computation. Theorem 1.1, Theorem 1.2(a), (b), (c), (e), Theorem 1.3, Proposition 7.1 and Corollary 7.2(a), (b) are proved without computation. Theorem 1.2(d), the second lower bound in Corollary 7.2(c) and Theorem 1.4 are computer-assisted: each is the exact evaluation of finitely many rational numbers, in exact rational arithmetic, from exact moments. The moments $m_0, \dots, m_{44}$ are computed directly from the definition of the operators; all higher moments rest on Theorem 1.3. Section 11 contains numerical observations only.

The reproducibility package has five parts.

The programs of this note (folder `original/`), with their recorded outputs. The program `moments_fock.py` computes $m_0, \dots, m_{44}$ from the definition, and `moments_sphere.py` implements Theorem 1.3. Complete runs of the latter exist up to m_{200} . The run which produced $k_2, \dots, k_{306}$ was stopped at a time limit, after 28 minutes; it had printed every cumulant as soon as it was known, and the file of the moments up to m_{306} was rebuilt from this record. Its first 101 lines are identical with the complete run to m_{200} , and the second verification run recomputed all values up to k_{306} . The programs `enclose_atom.py` and `window_certificate.py` produce Theorem 1.2(d) and Theorem 1.4(a); `certificate_B.py` checks the positivity of (12) a second time, by Sturm's theorem. The complete lists are in `original/outputs/moments_sphere_upto_N306.txt` (up to m_{306}) and `verification_run_B/outputs/c_fast_N560.txt` (cumulants up to k_{560}).

The programs of the first two independent verification runs, with their recorded outputs (folders `verification_run_A/` and `verification_run_B/`), and *those of the third run*, which examined the final text (folder `independent_run_2/`); see the next paragraph.

A program written with this note (`writing_stage/check_paper.py`). It recomputes, from the recorded files, the numbers printed in this note, among them Table 1, the quintic (12) and its minimum, the inequalities of Corollary 7.2(c) and the intervals of Theorem 1.2(d).

Re-runs. When this note was written, the programs of the package with a running time below ten minutes were run again, one process at a time; among them are the fast implementation of Theorem 1.3 by the second run (six minutes for $k_2, \dots, k_{560}$), the enclosures of Table 2 and the window certificates of Table 3 (seven minutes each for $K = 278$). The outputs agree with the recorded ones, and the third verification run repeated the quick re-runs from the archive with the same result. Not run again: the run of the program of this note to m_{306} , the long run of the first verification run to k_{160} , its truncations of large dimension, and the counts of the second run for $2n = 18, 20$. The package explains how the programs of this note run with the standard rational type of Python only.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed. Each wrote its own programs, those for the moments before reading the programs of the note. The first two examined the first written version of the results, and neither saw the report of the other; the third examined the text of this note.

The first run compared the statement with the two sources and re-derived, line by line, Lemmas 2.1, 2.2 and 4.1, the proof of Theorem 1.1, Lemmas 5.1 and 5.2, both certificates, the argument of Section 6, Lemmas 8.1 and 8.2 with the proof of Theorem 1.3, and Lemmas 9.1 and 10.1. It found no gap. It required two corrections in the preliminaries, which are made here: the identity (6) had been stated without the vacuum term, and the reason for $k_{2n} > 0$ in Lemma 4.1(c) had been omitted. It contributed the proof of Proposition 5.3(b) through the factorisation of E' , the observation that $m_6/m_4 > 6$ suffices in Step 4 of Section 6, and Proposition 7.1. Its programs gave: the moments

$m_0, \dots, m_{44}$ from the definition, identical with those of the note; the numbers $\#PP_{\bowtie}(2n)$ for $n \leq 6$ by enumeration; cumulants from its own implementation of Theorem 1.3, equal to those from the definition up to k_{44} and to those of the note up to k_{160} ; a symbolic test of Lemma 8.1 and of (4) on 39 monomials, which fails, as it must, when D is removed; the truncations of Observation 5; and rows 1 and 2 of Table 2.

The second run recomputed all numbers. It obtained the moments by three routes: by counting the partitions in the definition of the source ($n \leq 10$, the complete table (3)), from the definition of the operators $(m_0, \dots, m_{48})$, and from Theorem 1.3 with two implementations (to k_{46} and to k_{560}). They agree with each other, with the table of the source, and with all 154 moments and 153 cumulants of the note. It tested (4) on 455 monomials, reproduced the enclosures of Theorem 1.2(d) and rows 1 to 3 of Table 3 with its own programs and moments, certified the positivity of the quintic (12) in two further ways, added row 5 of Table 2 and rows 4 to 6 of Table 3, and extended the Jacobi parameters to $k = 280$. It read the proofs of Lemmas 4.1, 5.1, 5.2, 8.1 and 8.2 and of Theorem 1.3 and found no error. It required six corrections, which are made here: 2.10199076... is presented as a numerical value for $\max \text{supp } \nu$, which Proposition 7.1 identifies with $\|b_Q\|$; the OEIS was queried directly; the thesis [Var21], which that run could not obtain, has since been read; the credits of Section 1.3 were added; no margin is quoted for the numerical tests of Lemma 5.1; and the limit of $k(\beta_k - 1)$ is described as not determined.

The third run read the two sources again and compared the definitions, the numbering and the quotations with this text. It checked all proofs of this note line by line, among them those of Proposition 7.1 and Corollary 7.2, which no run other than the first had seen; it found no error, and one proof which was incomplete, that of the last statement of Theorem 1.3. It examined what had been added when this note was written: the formulas for the measures with Jacobi parameters $(\theta, 1, 1, \dots)$ in Section 1.3 and Remark 8.3 (confirmed); the statement on the size of the numbers in Theorem 1.3 (correct; its proof is now written out); Remark 8.6(1) (confirmed, also by the enumeration mentioned there); and what is said about [Var21], of which it read Sections 6.4 and 6.5 in a copy provided by the author of this note, about [GHS20] and about [BW01] (confirmed). With its own programs it recomputed the moments from the definition of the operators $(m_0, \dots, m_{44})$ and the cumulants from two implementations of Theorem 1.3 (to k_{122} and to k_{562} ; equal to all values in the files of the package), the numbers $\#PP_{\bowtie}(2n)$ for $n \leq 8$ by enumeration, every entry of Tables 1 to 4, the intervals of Theorem 1.2(d), the first from its own moments from the definition, Corollary 7.2(c), and the numbers of Section 11 except the truncations of Observation 5 and the last search of Observation 6. For the two windows of Section 10 without conclusion it found other polynomials q with $I(q) < 0$; they are in the package and are not used here. It required four corrections, which are made: the proof of the size statement in Theorem 1.3; the range in Observation 3, which had been given as found for the four values of K of Table 4; the number m_2 in the sentence after Theorem 1.2; and the reference [CL20]. The program `check_paper.py` confirms the printed numbers.

Scope and priority. The note gives a partial answer to the question of [Ger18, Ger17]: it does not describe μ by a formula. It proves that the measures μ_t are dilations of one measure, a fact implicit in the source; that μ has two atoms at the edges of its support which carry more than half of its mass; that the rest of μ lies in $[-\sqrt{6}, \sqrt{6}]$ and is not carried by $[-2, 2]$; and that the moments are given by an explicit polynomial model. A closed formula, the nature of the rest of μ , and a formula for $\#PP_{\bowtie}(2n)$ remain open. No priority is claimed for Theorem 1.1, for the mechanism of Section 4, or for anything contained in the sources quoted in Section 1.3. For the other results we say only that we did not find them in the literature we could read.

What was read: [Ger17] in the arXiv version, of which there is only one, in full; its Definitions 3, 4 and 8, Theorems 4.1, 4.2, 6.5 and 7.1 and Corollary 7.2 were compared with our statement by all

three verification runs. The abstract [Ger18], in the file of the report on the publisher’s site. Of [GHS20], in the arXiv version, the passages on the two types in the introduction, the subsection on type II and the example on the central limit distribution; what we say about its law for full correlation is a computation from the formula of that example. Of [Ger26], in the arXiv version, the outlook section. Of [BW01], in the copy on Numdam, the definition of the t -transformation and its effect on the Boolean cumulants; of [CL20], the abstract. The works [Mur97, Lu97, SW97, CT84, Sim95, Voi14, CF00, Gau04] are cited for standard facts or for context; they were not read for this note. The bibliographic data of all references were checked with Crossref, zbMATH or arXiv. Of the thesis [Var21], Sections 6.4 and 6.5 were read in full, and the rest was searched for the words bimonotone, Brownian, Hankel, central limit and Gerhold. Not consulted, because automated access was refused: the published version [Ger23]. The record [ula26] states, with a literature check dated 21 August 2026, that the published version does not contain the requested description; we could not confirm this.

The searches were made on 10 October 2026: queries to arXiv (for bi-monotone, bimonotone, bi-monotonic and monotone Fock space, also together with eigenvalue, atom and norm, for two-faced central limit theorems, and for the papers of the author of the question), to zbMATH (the same terms; 24 entries for “monotone Fock space”), to Crossref, the lists of works citing [Ger17] and [GHS20] in OpenAlex (seven and nine records: titles, available abstracts, and for several of them keyword searches in the arXiv sources), eight web searches, and direct queries of the OEIS [OEI26] for the sequence (3), for four of its subsequences, and for the sequence 4, 16, 160, 2432, 48896 of the numbers $n! k_{2n}$. The third verification run repeated the queries to arXiv, zbMATH, Crossref, OpenAlex and the OEIS. Nothing was found on the law of the sum, its support, its norm, atoms, a Cauchy transform, a recursion or a generating function. MathSciNet was not used, and the texts of the citing works were not read beyond keyword searches. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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