

The Scaling-Rotation Chain Metric for Positive-Definite Two by Two Matrices

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Abstract

The scaling-rotation distance between positive-definite matrices minimizes the distance between their eigendecomposition fibers. Its associated chain metric also allows changes of eigenframe at repeated eigenvalues. For real two by two matrices we compute that chain metric explicitly. Writing m for the normalized logarithmic determinant, r, s for the normalized logarithmic spectral gaps, and α for the acute angle between the top eigenlines, its square is

$$(m_Y - m_X)^2 + \min\{(r - s)^2 + k\alpha^2, (r + s)^2\}.$$

We characterize all minimizing paths and determine exactly when a minimal smooth scaling-rotation curve remains globally shortest. The criterion includes an essential orthogonal-eigenline exception. We also prove determinant splitting and a sufficient spectral-gap condition in arbitrary dimension. These calculations use the established Groisser–Jung–Schwartzman framework. They address the two-dimensional case of Jung’s Oberwolfach question, not its general-dimensional classification; no absolute priority claim is made.

1 The smooth and piecewise comparison

Jung’s Oberwolfach contribution [1, pp. 172–173] asks when minimal smooth scaling-rotation (MSSR) curves remain shortest after allowing piecewise-smooth curves with changes of lift inside a fiber. The question is recorded as OWR-15988-004 in UnsolvedMath v1.6.0 [6]. We use the primary source’s smooth-versus-piecewise comparison. The source concerns general dimension, including a three-dimensional imaging application. Here we solve the explicit two-dimensional comparison and give a sufficient condition in higher dimensions.

The framework, the smooth-distance formula in dimension two, and the distinction between smooth and cornered minimizers are due to Groisser, Jung, Schwartzman and collaborators [2, 3]. The chain metric is not introduced here. Their separately listed manuscript *A scaling-rotation metric on the space of symmetric positive-definite matrices, I: Existence* is a relevant prior-art lead [4, 5]. Its text was not available for comparison in our bounded source review, so possible overlap with the calculation below is not excluded. Our contribution here is a self-contained explicit calculation, with no assertion of historical priority.

Fix $k > 0$. On $M_p = \text{SO}(p) \times \mathbb{R}^p$ put

$$\|(U\Omega, \dot{\ell})\|^2 = \frac{k}{2}\|\Omega\|_F^2 + \|\dot{\ell}\|_2^2, \quad F(U, \ell) = U \operatorname{diag}(e^{\ell_i})U^T. \quad (1)$$

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This is the logarithmic-eigenvalue form of [2, (3.7)]. For $X, Y \in \text{Sym}^+(p)$ define

$$d_{\text{SR}}(X, Y) = \min_{u \in F^{-1}(X), v \in F^{-1}(Y)} d_{M_p}(u, v), \quad (2)$$

$$\rho(X, Y) = \inf_{X=X_0, \dots, X_N=Y} \sum_{j=1}^N d_{\text{SR}}(X_{j-1}, X_j), \quad N < \infty. \quad (3)$$

The fibers are compact: their logarithmic spectra range over finitely many permutations, and their eigenframes form compact stabilizer orbits. Thus the minimum in (2) is attained. Completeness of M_p supplies minimizing geodesics. Replacing every edge of a finite chain by such a geodesic shows that (3) is the infimum of lengths of finitely many projected upstairs pieces, joined by free changes within a fiber. Conversely each lifted piece costs at least its endpoint fiber distance. Arbitrary piecewise C^1 lifts give the same infimum. Length throughout means this lift length, not length for a different metric on $\text{Sym}^+(p)$.

An MSSR curve is the projection of a minimizing upstairs geodesic between the endpoint fibers; its scaling-rotation length is d_{SR} . Hence the requested global-minimality comparison is precisely $d_{\text{SR}}(X, Y) = \rho(X, Y)$.

2 An explicit two-dimensional distance

Let $a \geq b$ and $c \geq d$ be the ordered logarithmic eigenvalues of $X, Y \in \text{Sym}^+(2)$, and put

$$m_X = \frac{a+b}{\sqrt{2}}, \quad r = \frac{a-b}{\sqrt{2}}, \quad m_Y = \frac{c+d}{\sqrt{2}}, \quad s = \frac{c-d}{\sqrt{2}}, \quad h = m_Y - m_X. \quad (4)$$

For $r, s > 0$, let $\alpha \in [0, \pi/2]$ be the acute angle between the unoriented top eigenlines. Set

$$\begin{aligned} A^2 &= h^2 + (r-s)^2 + k\alpha^2, \\ B^2 &= h^2 + (r+s)^2 + k(\pi/2 - \alpha)^2, \quad C^2 = h^2 + (r+s)^2. \end{aligned} \quad (5)$$

The known four-lift calculation [2, Section 4] gives

$$d_{\text{SR}}(X, Y) = \min\{A, B\}. \quad (6)$$

Indeed, the two ordered eigenframes have least angular difference α and squared scaling cost $h^2 + (r-s)^2$; the reversed eigenframes have least angular difference $\pi/2 - \alpha$ and squared scaling cost $h^2 + (r+s)^2$.

Theorem 1. *If $r, s > 0$, then*

$$\rho(X, Y)^2 = h^2 + \min\{(r-s)^2 + k\alpha^2, (r+s)^2\}. \quad (7)$$

If $r = 0$ or $s = 0$, then

$$\rho(X, Y) = d_{\text{SR}}(X, Y) = \sqrt{h^2 + (r-s)^2}. \quad (8)$$

The infimum is attained in all cases.

Proof. The regular stratum has coordinates $(m, r, \theta) \in \mathbb{R} \times (0, \infty) \times (\mathbb{R}/\pi\mathbb{Z})$, where θ specifies the top eigenline. Since $\ell_1 = (m+r)/\sqrt{2}$ and $\ell_2 = (m-r)/\sqrt{2}$, the lift metric is

$$dm^2 + dr^2 + k d\theta^2. \quad (9)$$

The straight interpolation in m, r , combined with a shortest projective angular arc, stays regular and costs A . Thus $\rho \leq A$.

There is also a competitor of cost C . Define

$$t_* = \frac{r}{r+s}, \quad m_* = \frac{sm_X + rm_Y}{r+s}, \quad Z = e^{m_*/\sqrt{2}}I. \quad (10)$$

Interpolate logarithmically from X to Z in X 's fixed eigenframe, change frame at the scalar matrix Z , and interpolate to Y in Y 's frame. The leg lengths are $rC/(r+s)$ and $sC/(r+s)$. Hence $\rho \leq \min(A, C)$.

For the reverse bound, replace an arbitrary finite chain by projected minimizing upstairs geodesics. Its projection is a continuous curve. If it never meets a scalar matrix, the ordered coordinates m, r are continuous, as is its top eigenline in $\mathbb{R}/\pi\mathbb{Z}$. Unwrap that angle continuously on the parameter interval. At a regular joining point, a change of lift is only a discrete signed permutation, so it cannot jump the ordered coordinates. By (9), the Euclidean length of $(m, r, \sqrt{k}\theta)$ equals the sum of upstairs lengths. The absolute total angular increment is at least α , and the Euclidean endpoint-distance bound gives length at least A .

If the projection meets a scalar matrix at time t_0 , set $v(t) = r(t)$ before t_0 and $v(t) = -r(t)$ afterward. This is continuous. On each lifted geodesic the logarithmic eigenvalues are affine, so their ordered mean and gap are piecewise affine; the gap may also vanish on an interval. Almost everywhere, $\sqrt{m'^2 + v'^2}$ is at most the upstairs speed. The planar path (m, v) joins (m_X, r) to $(m_Y, -s)$ and therefore has length at least C . Scalar intersections inside an edge are included; they need not be chain vertices. Every chain consequently costs at least $\min(A, C)$, proving (7).

If an endpoint is scalar, sorting logarithmic spectra is nonexpansive in Euclidean norm. Summing their endpoint-distance bounds along any chain gives the lower bound in (8). A fixed-frame interpolation using the nonscalar endpoint's frame attains it. If both are scalar, any fixed frame does so. $\square$

In particular (7)–(8) are positive for distinct matrices. The chain construction supplies the triangle inequality, so this also verifies directly that ρ is a metric in dimension two.

Corollary 2. *Every MSSR curve from X to Y is globally shortest among the allowed piecewise curves if and only if an endpoint is scalar, or both endpoints are regular and*

$$\alpha = \frac{\pi}{2} \quad \text{or} \quad k\alpha^2 \leq 4rs = 2(a-b)(c-d). \quad (11)$$

Proof. For $\alpha < \pi/2$, we have $B > C$. Thus $\min(A, B) = \min(A, C)$ exactly when $A \leq C$, equivalent to $k\alpha^2 \leq 4rs$. If $\alpha = \pi/2$, then $B = C$, so equality holds regardless of that inequality. Apply (8) for scalar endpoints. All MSSR curves have the same length d_{SR} . $\square$

3 Minimizing paths and a strict gap

Call the straight regular competitor in the proof an *angular path*, and the two fixed-frame legs through (10) the *scalar path*.

Proposition 3. *For regular endpoints, up to increasing reparameterization and stationary pauses, all minimizing projected paths in the finite-piece class are as follows. If $A < C$, only angular paths minimize. If $C < A$, only the scalar path minimizes. If $A = C$, both types minimize. There is one angular image for $\alpha < \pi/2$ and two for $\alpha = \pi/2$; the scalar image is unique. Thus equality gives two images, or three at orthogonality. With a scalar endpoint the unique minimizing image is fixed-frame logarithmic interpolation.*

Proof. Equality in the Euclidean length bound for a regular path forces the unwrapped $(m, r, \sqrt{k}\theta)$ to be a straight segment with shortest angular increment. A projective circle has two shortest arcs precisely for orthogonal lines.

For a scalar-hitting path, equality forces (m, v) to be straight. Its radial endpoint coordinates have opposite nonzero signs, so it meets the scalar axis exactly once, apart from stationary pauses. Equality in the speed bound forces zero angular motion on each regular side. The straight segment fixes m_* , and the endpoints fix the incoming and outgoing eigenframes. Additional effective scalar visits or rotating legs cannot occur. These length arguments also apply to piecewise C^1 lifts, since the ordered gap and its signed reflection are absolutely continuous. For a scalar endpoint, equality in the spectral bound fixes the straight logarithmic interpolation and forces zero angular speed wherever the spectrum is simple. Coincident endpoints give the constant path. $\square$

The scalar path is smooth when $\alpha = \pi/2$: it is one fixed-frame upstairs geodesic whose eigenvalues cross. For $0 < \alpha < \pi/2$ it has a genuine corner. To see this, let H_X, H_Y be the trace-free involutions positive on the respective top eigenlines. At the collision the radial derivatives are proportional to $-H_X$ and H_Y with the same mean component. Their constant-speed tangents agree exactly when $H_Y = -H_X$, which is orthogonality. This explains the exception in (11).

For an explicit strict gap, let $k = 1$, $0 < \varepsilon < \pi/8$, and

$$X = \text{diag}(e^{\varepsilon/\sqrt{2}}, e^{-\varepsilon/\sqrt{2}}), \quad Y = R_{\pi/4} X R_{\pi/4}^T.$$

Then $r = s = \varepsilon$, $h = 0$, and $\alpha = \pi/4$, so

$$d_{\text{SR}}(X, Y) = \pi/4, \quad \rho(X, Y) = 2\varepsilon.$$

The ratio is unbounded as $\varepsilon \downarrow 0$, even for determinant-one endpoints. This compares two different distances; it is not a failure of the triangle inequality for ρ .

4 Determinant splitting and a general sufficient condition

For arbitrary $p \geq 2$ set

$$m(X) = p^{-1/2} \log \det X, \quad \hat{X} = (\det X)^{-1/p} X.$$

Proposition 4. *Writing $h = m(Y) - m(X)$, we have*

$$d_{\text{SR}}(X, Y)^2 = h^2 + d_{\text{SR}}(\hat{X}, \hat{Y})^2, \quad \rho(X, Y)^2 = h^2 + \rho(\hat{X}, \hat{Y})^2. \quad (12)$$

Proof. The trace and trace-free parts of the logarithmic spectrum are orthogonal, and the trace is unchanged by eigenframe permutations. Taking the minimum over fibers proves the first identity. For the second, any chain has edge lengths $\sqrt{(\Delta m_j)^2 + l_j^2}$, where l_j is the centered fiber distance. The Euclidean triangle inequality gives the lower bound $\sqrt{h^2 + (\sum_j l_j)^2}$, and $\sum_j l_j \geq \rho(\hat{X}, \hat{Y})$. Conversely choose a centered chain of positive total length L and assign mean increments $\Delta m_j = h l_j / L$. Its full length is $\sqrt{h^2 + L^2}$. Let L approach the centered infimum. A zero-length centered chain is handled by a pure mean change; a zero infimum not attained follows by the same positive- L approximation. $\square$

For simple-spectrum X , define the distance of its logarithmic spectrum to the nearest collision wall by

$$\delta(X) = \frac{1}{\sqrt{2}} \min_{i < p} (\log \lambda_i(X) - \log \lambda_{i+1}(X)), \quad \lambda_1(X) > \cdots > \lambda_p(X).$$

Proposition 5. *For simple-spectrum X, Y ,*

$$\rho(\hat{X}, \hat{Y}) \geq \min\{d_{\text{SR}}(\hat{X}, \hat{Y}), \delta(X) + \delta(Y)\}. \quad (13)$$

In particular, if $d_{\text{SR}}(\hat{X}, \hat{Y}) \leq \delta(X) + \delta(Y)$, then $\rho(X, Y) = d_{\text{SR}}(X, Y)$.

Proof. In the ordered trace-free Weyl chamber, the distance to its walls is the smallest adjacent gap divided by $\sqrt{2}$. The closest-wall projection averages that adjacent pair, preserving order and trace. Sorting spectra is nonexpansive, so spectral path length does not exceed upstairs length.

Replace a centered chain by minimizing lifted geodesics. If the projection meets the repeated-eigenvalue stratum, the initial portion up to its first meeting costs at least $\delta(X)$ and the final portion after its last meeting costs at least $\delta(Y)$. If there is no meeting, every change of lift is a discrete even signed permutation. Its simultaneous action on the eigenframe and logarithms is an isometry. Apply such an isometry successively to glue the lifted pieces into one continuous upstairs path, with unchanged length and endpoints in the required fibers. Its length is at least $d_{\text{SR}}(\hat{X}, \hat{Y})$. These cases prove (13). The sufficient condition follows from $\rho \leq d_{\text{SR}}$ and Proposition 4. $\square$

This condition is not claimed necessary. Several non-scalar collision strata occur for $p \geq 3$, and the complete comparison in those dimensions remains outside the result proved here.

The accompanying standard-library Python checker supplies 20,719 rational and 19,503 numerical regression checks, including independent four-fiber enumeration, scalar endpoints, orthogonality, exact transition cases and a 75-vertex shortest-path graph. These finite checks support reproducibility, not the general proofs. This unrefereed manuscript was prepared with AI assistance in research, calculations, checking and writing; the author is responsible for its claims. No independent review or proof-assistant verification is represented.

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