

On a Question of Becher and Slaman: For Kurtz Random x the Discrepancy of $(2^n x)$ Is Not $O((\log N)/N)$

Alper Ferudun*

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Abstract

In Oberwolfach Report 1/2018, Becher and Slaman ask whether some Martin-Löf random real x has discrepancy $D_N((2^n x)_{n \geq 1}) = O((\log N)/N)$ for all large N . Read as asking for constants C, N_0 with $ND_N \leq C \log N$ for all $N \geq N_0$, the question has a negative answer, also for Schnorr, computably and Kurtz random x . More precisely, if x is Kurtz random and g is computable with $\limsup_N g(N)/\sqrt{N \log \log N} < 1/\sqrt{2}$, for instance any computable $g(N) = o(\sqrt{N \log \log N})$, then $ND_N > g(N)$ for infinitely many N ; in particular $\limsup_N ND_N/\sqrt{N \log \log N} \geq 1/\sqrt{2}$. We give two short proofs of the negative answer. The first combines Koksma's inequality for $e^{2\pi i t}$, the law of the iterated logarithm of Erdős and Gál for lacunary exponential sums, and the fact that a null effectively closed set contains no Kurtz random real; it gives the constant $1/(2\pi)$ in place of $1/\sqrt{2}$. The second replaces these analytic tools by the observation that $\{2^n x\} < \frac{1}{2}$ exactly when the $(n+1)$ st binary digit of x is 0, so that ND_N is at least the deviation from $N/2$ of a count of zeros; a binomial estimate then gives the negative answer, and Khinchin's law of the iterated logarithm gives the constant $1/\sqrt{2}$. For Martin-Löf random x the negative answer follows at once from this digit count and Vovk's law of the iterated logarithm for Martin-Löf random sequences; the note records a short observation from known results, which may be folklore. The question is due to Becher and Slaman, and the tools are classical. We do not treat the variant with “for infinitely many N ”, nor Korobov's question on the existence of a normal number with discrepancy $O((\log N)/N)$, which the source connects with the question, nor the other questions of the problem session. This is an unrefereed note.

1 Introduction

1.1 The question

For a real number t let $\{t\} = t - \lfloor t \rfloor$. Following [BSJ⁺18, p. 34], the discrepancy of the first N terms of a sequence $(x_n)_{n \geq 1}$ of real numbers is

$$D_N((x_n)_{n \geq 1}) = \sup_{0 \leq a < b < 1} \left| \frac{1}{N} \# \{n : 1 \leq n \leq N, \{x_n\} \in [a, b)\} - (b - a) \right|. \quad (1)$$

For $x \in \mathbb{R}$ we write $y_n = y_n(x) = \{2^n x\}$ and $D_N(x) = D_N((2^n x)_{n \geq 1})$. In problem (1) of the open-problem section of Oberwolfach Report 1/2018 [BSJ⁺18], Becher and Slaman ask (Question 1) whether some Martin-Löf random real x has the property that “for every N large enough” $D_N(x)$ is $O((\log N)/N)$. We read the question as follows:

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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(Q) *Are there a Martin-Löf random real x and constants $C > 0$ and $N_0 \geq 1$ such that $ND_N(x) \leq C \log N$ for all $N \geq N_0$?*

The phrase “for every N large enough” and the O -symbol express the same eventual bound; a bound required only for infinitely many N would be a different question (Remark 6.2). The base of the logarithm does not matter; neither does indexing the sequence from $n = 0$ (this changes ND_N by at most 1), nor the use of the star discrepancy D_N^* of Section 2, since $D_N^* \leq D_N \leq 2D_N^*$. Martin-Löf randomness refers to Lebesgue measure, as in Becher’s abstract on p. 17 of the same report; the definitions are recalled in Section 3. Since $D_N(x)$ depends only on $\{x\}$, we may assume that $x \in [0, 1)$.

The source adds that an answer to Question 1 would also answer Korobov’s question [Kor56] on the existence of a normal number with this discrepancy. This holds for a positive answer; the negative answer below leaves Korobov’s question open.

1.2 Results

Theorem 1.1. *Let x be a Kurtz random real; for instance, let x be Martin-Löf random, computably random or Schnorr random. Then for every $C > 0$ there are infinitely many N with $ND_N(x) > C \log N$. In particular, the answer to (Q) is negative.*

A sequence $(g(N))_{N \geq 1}$ of real numbers is *computable* if some algorithm computes from N and k a rational number within 2^{-k} of $g(N)$. From now on $\log$ is the natural logarithm; for (Q) the base does not matter, but the constants $1/\sqrt{2}$, $1/(2\pi)$ and $\sqrt{84}/9$ below refer to it. We write $\sqrt{N \log \log N}$ only for $N \geq 3$, where $\log \log N > 0$.

Theorem 1.2. *Let x be Kurtz random, and let $(g(N))_{N \geq 1}$ be a computable sequence of real numbers with $\limsup_{N \rightarrow \infty} g(N)/\sqrt{N \log \log N} < 1/\sqrt{2}$. Then $ND_N(x) > g(N)$ for infinitely many N . This applies to every computable g with $g(N) = o(\sqrt{N \log \log N})$, for example to $g(N) = K(\log N)^A$ and $g(N) = K\sqrt{N}$ with rational $K, A > 0$, and to $g(N) = c\sqrt{N \log \log N}$ with rational $c < 1/\sqrt{2}$, for instance $c = \frac{1}{7}$. Consequently*

$$\limsup_{N \rightarrow \infty} \frac{ND_N(x)}{\sqrt{N \log \log N}} \geq \frac{1}{\sqrt{2}}.$$

Theorem 1.2 does not say that $ND_N(x)/\sqrt{N \log \log N}$ is unbounded; this would be false, since almost every x is Kurtz random and for almost every x the upper limit equals $\sqrt{84}/9 = 1.018\dots$ by Fukuyama [Fuk08, Corollary], sharpening Gál and Gál [GG64] and Philipp [Phi75] (as quoted in [ABSS17, Section 1]). Kurtz randomness gives no upper bound at all (Remark 6.3).

1.3 Two proofs, and what is known

The first proof (Section 4) combines Koksma’s inequality $|E_N(x)| \leq 2\pi ND_N(x)$, where $E_N(x) = \sum_{n \leq N} e^{2\pi i 2^n x}$, with the law of the iterated logarithm of Erdős and Gál [EG55], by which $\limsup_N |E_N(x)|/\sqrt{N \log \log N} = 1$ for almost every x . For rational K the set of x with $|E_N(x)| \leq K \log N$ for all $N \geq N_0$ is a null Π_1^0 class (an effectively closed set), and no Kurtz random real lies in such a class; this gives Theorem 1.1, and Theorem 1.2 with $1/(2\pi)$ in place of $1/\sqrt{2}$. The second proof (Section 5) uses that $\{2^n x\} < \frac{1}{2}$ if and only if the $(n+1)$ st binary digit of x is 0, so that $ND_N(x)$ is at least the deviation from $N/2$ of the number of zeros among the digits in the places $2, \dots, N+1$ (Lemma 2.2); an elementary binomial estimate then gives Theorem 1.1, and Khinchin’s law of the iterated logarithm [Khi24] gives Theorem 1.2. For Martin-Löf random x even less is needed: by Vovk’s law of the iterated logarithm for Martin-Löf random sequences [Vov87] (see also [Pan14, Theorem 3.5]) the binary digits of x obey the law of the iterated logarithm, and Lemma 2.2 gives the bound of Theorem 1.2 at once (Corollary 5.3). So, for the question as posed,

the negative answer is a short observation from known results, and it may be folklore; we found no printed statement of it (Section 7).

The question was later restated in more open forms. In a talk of March 2021, Becher asked for the minimal discrepancy of $(2^n x)_{n \geq 1}$ for a random real x [Bec21, slide 20], and the report of a workshop of October 2023 states the belief that all Martin-Löf random reals have the discrepancy of almost all real numbers, which “remains to be proved” [BBG⁺23, p. 2]. Theorem 1.2 gives a lower bound of the almost-everywhere order $\sqrt{N \log \log N}$ for every Kurtz random real, with the constant $1/\sqrt{2}$ instead of $\sqrt{84}/9$; upper bounds and the exact constant are not treated. Work on algorithmic randomness and uniform distribution [Avi13, BG22] concerns uniform distribution, not its rate.

2 Two elementary inequalities

Fix $x \in [0, 1]$ and $N \geq 1$, and write $y_n = \{2^n x\}$. For $0 \leq t \leq 1$ let $A_N(t) = \#\{n : 1 \leq n \leq N, y_n < t\}$, and let $D_N^*(x) = \sup_{0 \leq t \leq 1} |A_N(t)/N - t|$ be the star discrepancy. Since the number of y_n in $[a, b)$ is $A_N(b) - A_N(a)$, we have $D_N(x) \leq 2D_N^*(x)$; we need the other inequality.

Lemma 2.1. $D_N^*(x) \leq D_N(x)$.

Proof. For $0 < t < 1$ the interval $[0, t)$ is admissible in (1), and $A_N(t)$ counts the y_n in it. For $t = 0$ and $t = 1$ the expression $|A_N(t)/N - t|$ vanishes, because all y_n lie in $[0, 1)$. $\square$

Let $x = \sum_{j \geq 1} b_j 2^{-j}$ with $b_j \in \{0, 1\}$ be the binary expansion of x that does not end with infinitely many digits 1, and put $Z_N(x) = \#\{j : 2 \leq j \leq N + 1, b_j = 0\}$.

Lemma 2.2 (digit count). *For $n \geq 1$ we have $y_n < \frac{1}{2}$ if and only if $b_{n+1} = 0$. Consequently*

$$ND_N(x) \geq ND_N^*(x) \geq \left| Z_N(x) - \frac{N}{2} \right|.$$

Proof. We have $y_n = \sum_{j \geq 1} b_{n+j} 2^{-j}$, and this expansion does not end with infinitely many digits 1. If $b_{n+1} = 1$ then $y_n \geq \frac{1}{2}$. If $b_{n+1} = 0$ then $y_n \leq \sum_{j \geq 2} b_{n+j} 2^{-j} \leq \frac{1}{2}$, with equality only if $b_{n+j} = 1$ for all $j \geq 2$, which is excluded. Hence $A_N(\frac{1}{2}) = Z_N(x)$ and $|Z_N(x)/N - \frac{1}{2}| \leq D_N^*(x)$; Lemma 2.1 completes the proof. $\square$

$$\text{Let } E_N(x) = \sum_{n=1}^N e^{2\pi i 2^n x} = \sum_{n=1}^N e^{2\pi i y_n}.$$

Lemma 2.3 (Koksma’s inequality [KN74] for $e^{2\pi i t}$). $|E_N(x)| \leq 2\pi ND_N^*(x) \leq 2\pi ND_N(x)$.

Proof. Let $f(t) = e^{2\pi i t}$. Then $f(1) = 1$, $\int_0^1 f = 0$, $|f'| = 2\pi$ and $\int_0^1 t f'(t) dt = f(1) - \int_0^1 f = 1$. For $y \in [0, 1)$ we have $f(y) = f(1) - \int_0^1 f'(t) \mathbf{1}[y \leq t] dt$. Summing over $n \leq N$, and using that $\#\{n \leq N : y_n \leq t\} = A_N(t)$ for all t outside the finite set $\{y_1, \dots, y_N\}$, we obtain

$$E_N(x) = N - \int_0^1 f'(t) A_N(t) dt = - \int_0^1 f'(t) (A_N(t) - Nt) dt,$$

since $N = \int_0^1 Nt f'(t) dt$. Hence $|E_N(x)| \leq \int_0^1 |f'| \cdot \sup_t |A_N(t) - Nt| = 2\pi ND_N^*(x)$, and Lemma 2.1 gives the second inequality. $\square$

3 Effectively closed null sets

Let λ be Lebesgue measure. A Σ_1^0 class is a set $U = [0, 1] \cap \bigcup_{i \in \mathbb{N}} I_i$, where (I_i) is a computable sequence of open intervals with rational endpoints; a Π_1^0 class (effectively closed set) is the complement $[0, 1] \setminus U$ of a Σ_1^0 class. A sequence of such classes is *uniformly* Σ_1^0 or Π_1^0 if the intervals are computable from the index. The intersection of a uniformly Π_1^0 sequence of classes is a Π_1^0 class (the complements can be enumerated together), and a finite union of closed intervals with rational endpoints is a Π_1^0 class, uniformly in the list of endpoints.

A real $x \in [0, 1]$ is *Martin-Löf random* if $x \notin \bigcap_k V_k$ for every uniformly Σ_1^0 sequence $(V_k)_{k \geq 1}$ with $\lambda(V_k) \leq 2^{-k}$, and *Kurtz random* (or weakly 1-random) if x lies in no Π_1^0 class of measure 0. A real $x \in \mathbb{R}$ is called random in one of these senses if $\{x\}$ is. Martin-Löf random reals are computably random, computably random reals are Schnorr random, and Schnorr random reals are Kurtz random [DH10, Nie09] ([Avi13, Section 2] states the implications from Martin-Löf to Schnorr and from Schnorr to Kurtz randomness); for (Q) only the first and last notions matter, and we prove the implication between them.

Lemma 3.1. (a) *No Martin-Löf random real lies in a Π_1^0 class of measure 0; that is, every Martin-Löf random real is Kurtz random.* (b) *No computable real is Kurtz random; in particular, a Kurtz random real is not a dyadic rational.*

Proof. (a) Let $P = [0, 1] \setminus \bigcup_{i \geq 1} I_i$ be a Π_1^0 class with $\lambda(P) = 0$. The measures $\lambda([0, 1] \cap (I_1 \cup \dots \cup I_m))$ are rational numbers computable from m , and they increase to 1. So for every k we can compute the least m_k with $\lambda([0, 1] \cap (I_1 \cup \dots \cup I_{m_k})) > 1 - 2^{-k}$. The set $F_k = [0, 1] \setminus (I_1 \cup \dots \cup I_{m_k})$ contains P , has measure less than 2^{-k} , and is the union of at most $m_k + 1$ closed intervals (possibly points) with rational endpoints computable from k . Let V_k be $[0, 1]$ intersected with the union of the open intervals of length $|J| + 2^{-k}/(m_k + 1)$ with the same centres as these intervals J . Then (V_k) is uniformly Σ_1^0 , $P \subseteq V_k$ and $\lambda(V_k) < 2^{-k} + 2^{-k}$. Hence $(V_{k+1})_{k \geq 1}$ is a Martin-Löf test with $P \subseteq \bigcap_k V_{k+1}$.

(b) If $x \in [0, 1]$ is computable, choose rationals q_k computably with $|x - q_k| \leq 2^{-k}$. Then $\{x\} = [0, 1] \cap \bigcap_k [q_k - 2^{-k}, q_k + 2^{-k}]$ is a Π_1^0 class of measure 0. $\square$

4 First proof: Koksma's inequality and the theorem of Erdős and Gál

Theorem 4.1 (Erdős and Gál [EG55, Theorem in the Introduction]). *Let $n_1 < n_2 < \dots$ be positive numbers with $n_{\nu+1}/n_\nu \geq q > 1$ for all ν . Then for almost all x*

$$\limsup_{N \rightarrow \infty} \frac{|\sum_{\nu=1}^N \exp(2\pi i n_\nu x)|}{\sqrt{N \log \log N}} = 1.$$

We use only the inequality “ ≥ 1 ”, for $n_\nu = 2^\nu$ and $q = 2$; then the sum is $E_N(x)$.

Lemma 4.2. *Let $(g(N))_{N \geq 1}$ be a computable sequence of real numbers and $N_0 \geq 1$. Then $F_g(N_0) = \{x \in [0, 1] : |E_N(x)| \leq g(N) \text{ for all } N \geq N_0\}$ is a Π_1^0 class.*

Proof. We have $F_g(N_0) = [0, 1] \setminus \bigcup_{N \geq N_0} U_N$ with $U_N = \{x \in [0, 1] : |E_N(x)| > g(N)\}$, so it suffices that U_N is a Σ_1^0 class, uniformly in N . Since $|E'_N| \leq 2\pi \sum_{n \leq N} 2^n < L_N = 2\pi \cdot 2^{N+1}$, we have $||E_N(s)| - |E_N(t)|| \leq L_N |s - t|$, and $|E_N(c)|$ is computable from N and a rational c . Enumerate, for all rationals c and $m, k \in \mathbb{N}$, the interval $J = (c - 2^{-m}, c + 2^{-m})$ whenever the computed approximations q of $|E_N(c)|$ and r of $g(N)$, both within 2^{-k} , satisfy $q - r > L_N 2^{-m} + 2^{1-k}$. For $t \in J$ this gives $|E_N(t)| \geq q - 2^{-k} - L_N 2^{-m} > r + 2^{-k} \geq g(N)$. Conversely, if $x \in U_N$ and $\varepsilon = |E_N(x)| - g(N)$,

take m, k with $L_N 2^{-m} \leq \varepsilon/8$ and $2^{-k} \leq \varepsilon/8$, and a rational c with $|x - c| < 2^{-m}$; then $q - r \geq \varepsilon - L_N 2^{-m} - 2^{1-k} \geq 5\varepsilon/8 > 3\varepsilon/8 \geq L_N 2^{-m} + 2^{1-k}$, so x lies in an enumerated interval. Hence U_N is $[0, 1]$ intersected with the union of the enumerated intervals. $\square$

Lemma 4.3. *If $\limsup_N g(N)/\sqrt{N \log \log N} < 1$, then $\lambda(F_g(N_0)) = 0$ for every N_0 .*

Proof. Every $x \in F_g(N_0)$ satisfies $\limsup_N |E_N(x)|/\sqrt{N \log \log N} \leq \limsup_N g(N)/\sqrt{N \log \log N} < 1$, and by Theorem 4.1 the set of such x is null. $\square$

First proof of Theorem 1.1, and of Theorem 1.2 with the constant $1/(2\pi)$. Let $x \in [0, 1)$ be Kurtz random and g a computable sequence with $\limsup_N g(N)/\sqrt{N \log \log N} < 1/(2\pi)$. Then $h = 2\pi g$ is computable and $\limsup_N h(N)/\sqrt{N \log \log N} < 1$. If $ND_N(x) \leq g(N)$ for all $N \geq N_0$, then $|E_N(x)| \leq 2\pi ND_N(x) \leq h(N)$ for these N by Lemma 2.3, so x lies in $F_h(N_0)$, which is a null Π_1^0 class by Lemmas 4.2 and 4.3; this is impossible. Hence $ND_N(x) > g(N)$ for infinitely many N . For Theorem 1.1 take $g(N) = C' \log N$ with a rational $C' \geq C$. Taking $g(N) = c\sqrt{N \log \log N}$ for $N \geq 3$ and $g(1) = g(2) = 0$ with rational $c < 1/(2\pi)$ gives $\limsup_N ND_N(x)/\sqrt{N \log \log N} \geq c$ for each such c . $\square$

5 Second proof: counting binary digits

For a word $w = w_1 \cdots w_{N+1} \in \{0, 1\}^{N+1}$ let $I_w = [k2^{-N-1}, (k+1)2^{-N-1}]$ with $k = \sum_j w_j 2^{N+1-j}$, the closed dyadic interval of the reals whose binary expansion can begin with w , and let $z(w) = \#\{j : 2 \leq j \leq N+1, w_j = 0\}$.

Lemma 5.1. *Let $(g(N))_{N \geq 1}$ be a computable sequence of rational numbers and $N_0 \geq 1$. For $N \geq 1$ let P_N be the union of the intervals I_w over the words $w \in \{0, 1\}^{N+1}$ with $|z(w) - N/2| \leq g(N)$, and let $P_g(N_0) = \bigcap_{N \geq N_0} P_N$.*

- (a) $P_g(N_0)$ is a Π_1^0 class.
- (b) If $x \in [0, 1)$ is not a dyadic rational and $ND_N(x) \leq g(N)$ for all $N \geq N_0$, then $x \in P_g(N_0)$.
- (c) If $x \in P_g(N_0)$ is not a dyadic rational, then $|Z_N(x) - N/2| \leq g(N)$ for all $N \geq N_0$.
- (d) $\lambda(P_N) = 2^{-N} \#\{v \in \{0, 1\}^N : |\#\{j : v_j = 0\} - N/2| \leq g(N)\}$.

Proof. (a) The condition on w compares two rational numbers, so P_N is a finite union of closed intervals with rational endpoints, computable from N . (b), (c) A real $x \in [0, 1)$ that is not a dyadic rational lies in exactly one interval I_w with $w \in \{0, 1\}^{N+1}$, namely the one with $w = b_1 \cdots b_{N+1}$, and $z(w) = Z_N(x)$. Under the hypothesis of (b), Lemma 2.2 gives $|Z_N(x) - N/2| \leq ND_N(x) \leq g(N)$ for $N \geq N_0$, so $x \in P_N$. In (c), $x \in P_N$ means that this interval satisfies the condition. (d) The intervals I_w have length 2^{-N-1} and overlap only in endpoints, and the condition does not involve w_1 . $\square$

Lemma 5.2. *Let g and N_0 be as in Lemma 5.1.*

- (a) If $(2g(N) + 1)/\sqrt{N+1} \rightarrow 0$, for instance if $0 \leq g(N) \leq K \log N + 1$ for a constant K , then $\lambda(P_g(N_0)) = 0$.
- (b) If $\limsup_N g(N)/\sqrt{N \log \log N} < 1/\sqrt{2}$, then $\lambda(P_g(N_0)) = 0$.

Proof. (a) If $g(N) < 0$ for some $N \geq N_0$, then P_N and hence $P_g(N_0)$ are empty; so let $g(N) \geq 0$ for all $N \geq N_0$. By Lemma 5.1(d), $\lambda(P_N)$ is the probability that a sum W_N of N independent fair ± 1 variables satisfies $|W_N| \leq 2g(N)$ (count $+1$ for each zero). The values of W_N have the parity of N , so at most $2g(N) + 1$ of them lie in $[-2g(N), 2g(N)]$, and each has probability at most $\beta_N = \binom{N}{\lfloor N/2 \rfloor} 2^{-N}$. For $N = 2m$ we have $\beta_N = \prod_{k=1}^m \frac{2k-1}{2k}$, and since $\left(\frac{2k-1}{2k}\right)^2 \leq \frac{2k-1}{2k+1}$, we get $\beta_N^2 \leq \prod_{k=1}^m \frac{2k-1}{2k+1} = \frac{1}{N+1}$. For $N = 2m+1$ we have $\beta_N = \beta_{N+1} \leq (N+2)^{-1/2}$, because $\binom{2m+2}{m+1} = 2\binom{2m+1}{m}$. Hence $\lambda(P_g(N_0)) \leq \lambda(P_N) \leq (2g(N) + 1)/\sqrt{N+1}$ for every $N \geq N_0$, and the right-hand side tends to 0.

(b) Let $S_n(x) = \sum_{j=1}^n b_j$. Khinchin's law of the iterated logarithm [Khi24], in the form stated in [Pan14, Theorem 3.4], says that

$$\limsup_{n \rightarrow \infty} \frac{|S_n(x) - n/2|}{\sqrt{(n/2) \log \log n}} = 1 \quad (2)$$

for almost every $x \in [0, 1]$. Let $x \in P_g(N_0)$ be not a dyadic rational. By Lemma 5.1(c), $|Z_N(x) - N/2| \leq g(N)$ for $N \geq N_0$. Since $S_{N+1}(x) = b_1 + N - Z_N(x)$,

$$\left| |S_{N+1}(x) - \frac{N+1}{2}| - |Z_N(x) - \frac{N}{2}| \right| \leq \frac{1}{2}. \quad (3)$$

Hence the upper limit in (2) is at most $\sqrt{2} \limsup_N g(N)/\sqrt{N \log \log N} < 1$. So $P_g(N_0)$ is contained in the union of a null set and the countable set of dyadic rationals. $\square$

Second proof of Theorems 1.1 and 1.2. Let $x \in [0, 1)$ be Kurtz random; by Lemma 3.1(b), x is not a dyadic rational. Let g be a computable sequence of real numbers with $\limsup_N g(N)/\sqrt{N \log \log N} < 1/\sqrt{2}$. Computing a rational r_N with $|r_N - g(N)| \leq \frac{1}{2}$, put $g'(N) = r_N + \frac{1}{2}$; then g' is a computable sequence of rationals with $g \leq g' \leq g + 1$, and $\limsup_N g'(N)/\sqrt{N \log \log N} < 1/\sqrt{2}$. If $ND_N(x) \leq g'(N)$ for all $N \geq N_0$, then $x \in P_{g'}(N_0)$ by Lemma 5.1(b), and $P_{g'}(N_0)$ is a null Π_1^0 class by Lemma 5.1(a) and Lemma 5.2(b); this is impossible. Hence $ND_N(x) > g'(N) \geq g(N)$ for infinitely many N , which is the first statement of Theorem 1.2; the examples are clear, and the lower bound for the upper limit follows with $g(N) = c\sqrt{N \log \log N}$ ($N \geq 3$) for rational $c < 1/\sqrt{2}$. For Theorem 1.1 take $g(N) = C' \log N$ with rational $C' \geq C$; then $0 \leq g'(N) \leq C' \log N + 1$, so Lemma 5.2(a) can replace Lemma 5.2(b), and this proof of Theorem 1.1 uses no law of the iterated logarithm. $\square$

Corollary 5.3. *If x is Martin-Löf random, then $\limsup_N ND_N(x)/\sqrt{N \log \log N} \geq 1/\sqrt{2}$.*

Proof. We may assume $x \in [0, 1)$. The sequence (b_j) of binary digits of a Martin-Löf random real is a Martin-Löf random sequence [DH10, Nie09]. By Vovk's law of the iterated logarithm [Vov87], in the form stated in [Pan14, Theorem 3.5], the upper limit in (2) is at least 1 for this x . By (3), $\limsup_N |Z_N(x) - N/2|/\sqrt{N \log \log N} \geq 1/\sqrt{2}$, and Lemma 2.2 completes the proof. $\square$

So for Martin-Löf random x the negative answer to (Q) follows at once from [Vov87] and Lemma 2.2; Theorems 1.1 and 1.2 extend it to Kurtz random reals.

6 Remarks and open questions

Remark 6.1 (Other lacunary sequences). The first proof works for every computable sequence (n_k) of positive integers with $n_{k+1}/n_k \geq q > 1$, for instance $n_k = b^k$ with an integer $b \geq 2$ (in Lemma 4.2 the Lipschitz constant becomes $2\pi \sum_{k \leq N} n_k$): for Kurtz random x and computable g with $\limsup_N g(N)/\sqrt{N \log \log N} < 1/(2\pi)$, the discrepancy of $(n_k x)_{k \geq 1}$ satisfies $ND_N > g(N)$ for infinitely many N .

Remark 6.2 (The quantifier). The eventual bound in (Q) is used once, to make $F_g(N_0)$ and $P_g(N_0)$ effectively closed. If $ND_N(x) \leq C \log N$ is required only for infinitely many N , our arguments say nothing (the digit count cannot help: for almost every x , $Z_N(x) = N/2$ for infinitely many N); this variant is not treated.

Remark 6.3 (No upper bounds). Theorems 1.1 and 1.2 give lower bounds for infinitely many N , the kind of statement that null effectively closed sets control. Kurtz randomness gives no upper bound: there are Kurtz random reals whose binary digits do not obey the law of large numbers [Avi13, Section 2] (see also [Pan14, Lemma 3.1]), and for these x , $D_N(x)$ does not tend to 0, by Lemma 2.2. Whether every Martin-Löf random x satisfies $\limsup_N ND_N(x)/\sqrt{N \log \log N} < \infty$, as believed in [BBG⁺23, p. 2], is not addressed here.

Remark 6.4 (Normal numbers). Levin [Lev99] constructed x with $D_N((2^j x)_{j \geq 0}) = O((\log N)^2/N)$ [ABSS17, Section 1], and such x can be computed [Bec21, slide 19], so it is not Kurtz random. Schmidt [Sch72] showed that every sequence has $ND_N \geq c \log N$ for infinitely many N , with an absolute $c > 0$ [ABSS17, Section 1]; Theorem 1.1 strengthens this for $(2^n x)$ with Kurtz random x . Korobov’s question on the optimal order of $D_N((b^j x)_{j \geq 0})$ [Kor56, ABSS17] is not affected, except that a real attaining the order $(\log N)/N$ is not Kurtz random.

Open questions. (1) The variant of Remark 6.2. (2) Does every Martin-Löf random x satisfy $\limsup_N ND_N(x)/\sqrt{N \log \log N} = \sqrt{84/9}$, the almost-everywhere value [Fuk08], or at least $\geq \sqrt{84/9}$, or $< \infty$ [BBG⁺23]? (3) Korobov’s question.

7 Computations and reproducibility

All theorems and lemmas are proved in the text, and no proof depends on a computation. The program of this note, `check_note.py` in the folder `reproducibility/` of the source archive (Python 3, standard library only, about 30 seconds), tests the finite ingredients: (i) the closed formulas $D_N = \frac{1}{N} + \max_i(\frac{i}{N} - z_i) - \min_i(\frac{i}{N} - z_i)$ and $D_N^* = \max_i \max(\frac{i}{N} - z_i, z_i - \frac{i-1}{N})$ for sorted points $z_1 \leq \dots \leq z_N$, against the definitions in exact rational arithmetic, on 360 point sets with ties, zeros and points near 1 (the formula without $\frac{1}{N}$ is rejected on all 60 test sets); (ii) Lemma 2.3 and its cosine form $|\sum_n \cos 2\pi y_n| \leq 4ND_N^*$ on 3000 point sets, and, for the points $\{2^n x\}$ on 200 pairs (x, N) with $N \leq 4096$, Lemma 2.2 and $D_N^* \leq D_N \leq 2D_N^*$ exactly and Lemma 2.3 with E_N in floating point; (iii) as an illustration only, for 24 pseudo-random x and $N = 2^{10}, \dots, 2^{20}$, the mean of $ND_N/\log N$ grows from 5.0 to 94.1, while $ND_N/\sqrt{N \log \log N}$ has sample means between 0.72 and 0.86, minima at least 0.38 and maxima at most 1.58 (pseudo-random reals are computable, hence not Kurtz random, and finitely many N say nothing about a limit); (iv) $\binom{2m}{m} 4^{-m} \leq (2m+1)^{-1/2}$ for $m \leq 3000$, and the bound of Lemma 5.2(a) in eight cases. All checks passed. The package also contains the floating-point program with which the orders of magnitude were first examined, and the programs of the three verification runs with their outputs (those of the third run in the folder `reproducibility/independent_run_2/`).

Verification. The results were first obtained with the first proof. Two independent verification runs, both AI-assisted, followed; each read the source, re-derived the proofs, wrote its own programs, re-ran the original program with the same output, and found independently the digit-count proof of Section 5 and its reduction to Vovk’s theorem for Martin-Löf random x . The first run also tested the enumeration of Lemma 4.2, and it found that a clause of the first version of Theorem 1.2, that $ND_N(x)$ is not $O(\sqrt{N \log \log N}/7)$, was false, as it contradicts the almost-everywhere law of Philipp and Fukuyama; Theorem 1.2 now states what the proof gave, $ND_N(x) > \sqrt{N \log \log N}/7$ for infinitely many N . The second run read the theorem of Erdős and Gál in the original, tested its programs

with brute-force discrepancies and with deliberately wrong inputs, which were rejected, checked the bibliography, searched the literature and found [Bec21, BBG⁺23]. Neither run found an error in the proof of Theorem 1.1. All corrections they required were made: the clause above; the digit-count proof with the constant $1/\sqrt{2}$, and Corollary 5.3; the description of the numerical results; separate citations of [BSS17] and [ABSS17]; the attribution of the almost-everywhere results; the year of [BG22]; a neutral statement on Korobov’s question; the citations of [Bec21, BBG⁺23]. This note was then written, and `check_note.py` tests the statements as printed.

A third independent verification run, also AI-assisted, then examined the text of this note and the package as if nothing had been checked before. It read the source again (pp. 16–18 and 34–36), the theorem of [EG55] on p. 65 of the original, the statements of [Pan14] that are used, [ABSS17, Section 1], [BSS17, pp. 1–2], [Avi13, Section 2], [Bec21, slides 17–21] and [BBG⁺23, pp. 1–3], and re-derived all proofs line by line. Its own programs, written independently of `check_note.py`, compute D_N and D_N^* exactly by a decomposition into cells of constant count and test the formulas of this section, Lemmas 2.1 to 2.3 (Lemma 2.2 exactly for $N \leq 16384$) and the remark on the index shift in Section 1; they carry out the construction of Lemma 3.1(a) exactly for the middle-third Cantor set; they test the enumeration of Lemma 4.2 with the worst admissible approximations (soundness on 127,100 intervals, completeness on 24,662 points, and a control with a wrong Lipschitz constant, which fails as it must); and they test Lemma 5.1(d), Lemma 5.2(a) for $N \leq 2000$ and the bound $\binom{2m}{m}4^{-m} \leq (2m+1)^{-1/2}$ for $m \leq 20,000$ in exact arithmetic. The run also re-ran all programs of the package from the source archive, with the recorded outputs reproduced apart from running times, checked the bibliographic data with Crossref and zbMATH, and repeated the literature search. It found no mathematical error. The corrections it required were made: the abstract states the constant of each proof and describes Korobov’s question as the source does; the natural logarithm is specified; the proof of Lemma 5.2(a) treats negative values of g ; the citation of [Avi13, Section 2] is restricted to what it states; and two descriptions of the computations in this section were made precise.

Scope and priority. The note answers Question 1 of problem (1) of [BSJ⁺18], read as (Q), in the negative, with the quantitative form Theorem 1.2. The question is Becher’s and Slaman’s. The tools are classical: Koksma’s inequality [KN74], the laws of the iterated logarithm of Erdős and Gál [EG55], Khinchin [Khi24] and Vovk [Vov87] (see also [Pan14]), and null Π_1^0 classes and Kurtz randomness [DH10, Nie09]; the laws [GG64, Phi75, Fuk08] are quoted for context only. For Martin-Löf random x the negative answer follows at once from [Vov87] and Lemma 2.2; the argument is short and may be folklore. No printed statement of this negative answer was found in the sources searched: arXiv (discrepancy with Martin-Löf, Kurtz or Schnorr randomness, algorithmic randomness or random reals; normal numbers and Korobov’s question; the law of the iterated logarithm for lacunary sequences; the proposers), Crossref, OpenAlex (no work cites [BSJ⁺18]), zbMATH, Becher’s list of publications, the sources [Bec21, BBG⁺23], which present the question, or a stronger form, as open, and the open problems of the Oberwolfach workshop on computability theory of 2021; Semantic Scholar refused automated queries. The searches were made on 15 October 2026 by the first two verification runs, repeated in part when this note was written, and repeated by the third run (arXiv, Crossref, OpenAlex, zbMATH and one web search). A search that finds nothing is not a proof of novelty, and we claim no priority. Not treated are the variant of (Q) with infinitely many N , Korobov’s question, upper bounds for Martin-Löf random reals, and the other problems of [BSJ⁺18], in particular problem (2), Jockusch’s question on Hindman’s theorem for sums of exactly n elements.

What was read: [BSJ⁺18] in the publisher’s open PDF, pp. 16–18 and 34–36; the theorem of [EG55] (p. 65, open copy) and its zbMATH review Zbl 0068.05403; [ABSS17, Section 1] and [BSS17, pp. 1–2] (arXiv versions), from which we quote [GG64, Phi75, Fuk08, Lev99, Sch72] and Korobov’s question; [Pan14] (Theorems 2.2, 3.4, 3.5, Lemma 3.1); [Avi13, Section 2] (arXiv version); [Bec21, slides 17–21];

[BBG⁺23, pp. 1–3]. Not read: [GG64, Phi75, Fuk08, Lev99, Sch72, Kor56, Khi24, Vov87, KN74, DH10, Nie09]; the statements used from [Khi24, Vov87] are those of [Pan14] (for [Vov87] also of Zbl 0645.60006), and of [BG22] only the abstract was read. The proofs of the cited theorems were not checked; the bibliographic data were checked with Crossref.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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