

Seed Aggregation for Competing First-Passage Percolation

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25 October 2026

Abstract

We give an exact reduction from competing first-passage percolation with initially infected hubs to the same process with many seeds on a regular configuration model. Merging vertices already occupied by the same type preserves the entire infection process outside those vertices. Combining this observation with the regular-graph theorem of Antunović, Dekel, Mossel and Peres yields a sharp seed-count criterion within an almost regular family. If the slow-to-fast rate ratio is $r \in (0, 1)$ and there are $q_N = o(N)$ slow hubs, a construction with convergent finite second degree moment in which the slow type captures asymptotically all vertices exists exactly when $q_N/N^{1-2r} \rightarrow \infty$. In particular, one diverging-degree hub suffices precisely when $r > 1/2$; at $r = 1/2$, arbitrarily slowly diverging hub counts suffice. The high-probability conclusions also hold for the corresponding uniform simple graphs. The growth estimates are imported from prior work. The result concerns deterministic degree sequences with planted hubs and does not settle the question for a prescribed fixed i.i.d. degree law.

1 Question, scope and prior input

Deijfen’s contribution to Oberwolfach Report 57/2017 [4] asks whether degree-based starting positions can allow a slower infection to capture a positive fraction of a finite-variance configuration model. The contribution describes i.i.d. degrees. The related paper of Ahlberg, Deijfen and Janson [2] treats deterministic degree sequences under empirical degree convergence and convergence of second moments, and raises the same question in its final section.

We record a deduction for an almost regular deterministic family. The key probabilistic result is already available in the work of Antunović, Dekel, Mossel and Peres [1]. The contribution here is an exact aggregation of initially infected vertices, a check of the second-moment constraint, and the resulting seed-count tradeoff. We do not prove a new growth-limit theorem. The constructed degrees are triangular arrays, not samples from a fixed i.i.d. law. Neither general finite-variance degree sequences nor a critical-window distribution are classified here.

For a random variable sequence, $X_N = O_{\mathbb{P}}(a_N)$ means that X_N/a_N is tight, and $X_N = o_{\mathbb{P}}(a_N)$ means convergence to zero in probability after division by a_N .

2 Model and exact aggregation

Fix $d \geq 3$ and two rates $0 < \lambda_s < \lambda_f$, and put

$$r = \frac{\lambda_s}{\lambda_f} \in (0, 1).$$

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Let N tend to infinity through integers with dN even. Take integers $q = q_N \geq 1$ and block sizes $t_1, \dots, t_q \geq 1$, and set

$$k = \sum_{i=1}^q t_i, \quad n = N - k + q.$$

We will assume $k \rightarrow \infty$ and $k = o(N)$. On a d -regular configuration model with N labelled vertices, mark k vertices as slow seeds and one disjoint vertex as a fast seed. Partition the slow seeds into blocks of sizes $t_1, \dots, t_q$, and merge each block into one vertex. All edges, including loops and parallel edges, are retained. The quotient has n vertices and degree sequence

$$dt_1, \dots, dt_q, \underbrace{d, \dots, d}_{N-k \text{ vertices}}. \quad (1)$$

The q merged vertices are the slow seeds; the fast seed has degree d .

An occupied vertex transmits its type across each incident edge at the rate of that type. A vacant vertex keeps the first type reaching it permanently. Parallel edges provide separate transmission channels and loops do not change colours. Let C_s, C_f be the final numbers of slow and fast vertices in the quotient.

Lemma 1 (Exact aggregation). *The quotient is the configuration model with degree sequence (1). Its infection process can be coupled with the process on the original regular configuration model so that all vertices outside the merged blocks have identical infection times and colours. If S_s, S_f are the original final counts, then*

$$C_f = S_f, \quad C_s = S_s - k + q. \quad (2)$$

Proof. A configuration is a uniform perfect matching of labelled half-edges. Merging blocks changes the ownership of half-edges but not this matching or its distribution. This proves the graph assertion.

Every vertex in a merged block is slow from time zero and never changes colour. When colours outside the blocks agree in the two processes, each outside vacant vertex has exactly the same numbers of incident edges to slow and fast vertices. Its two transition rates therefore agree. One may use the same type-specific Poisson clocks on corresponding edges to couple the processes. Edges whose endpoints were already slow cannot produce a new infection, including those that become loops. The coupled processes thus agree at every outside vertex. Only the count of time-zero slow vertices changes, from k to q , proving (2). $\square$

The lemma does not permit merging vacant vertices, or seeds of different types. Fixed labels for the seed sets cause no restriction: exchangeability of the regular configuration model gives the same count distribution as uniform disjoint seed sets of the prescribed sizes.

Lemma 2 (Second moments). *If*

$$\sum_{i=1}^q t_i^2 = o(N), \quad (3)$$

then the degree D_n of a uniformly chosen quotient vertex satisfies

$$D_n \xrightarrow{\text{law}} d, \quad \mathbb{E}D_n^2 \longrightarrow d^2, \quad n/N \longrightarrow 1.$$

All degrees are at least d . Conversely, under $k = o(N)$, convergence of $\mathbb{E}D_n^2$ to d^2 is equivalent to (3).

Proof. Since $q \leq k \leq \sum_i t_i^2$, condition (3) gives $q, k = o(N)$ and $n/N \rightarrow 1$. At most q vertices have degree different from d . Directly from (1),

$$\mathbb{E}D_n^2 = d^2 \frac{N - k + \sum_i t_i^2}{N - k + q}. \quad (4)$$

This proves the forward implication. If $k = o(N)$, then $q = o(N)$ and $n \sim N$, so (4) also proves the converse. $\square$

In particular, the hypotheses (A1) and (A2) of [2] hold, with limiting degree identically d . The rare hubs do not hide a positive amount of second moment that is absent from the limiting distribution.

3 Transfer of the regular-graph growth theorem

The following is the precise input from [1, Theorem 2.6]. For one fast seed, $k \rightarrow \infty$ uniformly placed slow seeds and $k = o(N)$,

$$\frac{k}{N^{1-r}} \rightarrow \infty \implies S_f = O_{\mathbb{P}}((N/k)^{1/r}) = o_{\mathbb{P}}(N), \quad (5)$$

$$\frac{k}{N^{1-r}} \rightarrow 0 \implies S_s = O_{\mathbb{P}}(kN^r) = o_{\mathbb{P}}(N). \quad (6)$$

In the notation of that theorem, take $B_0 = 1$, $\beta = \lambda_f$, $R_0 = k$ and $\rho = \lambda_s$. Its stronger distributional conclusions imply these tightness bounds. Section 3 and Remark 4.2 of [1] establish the results first for the configuration-model process; we therefore use an unconditioned result here. The regular configuration model is connected with probability tending to one, as also used there.

Proposition 3 (Off-critical outcomes). *Suppose $k \rightarrow \infty$ and (3) holds. If $k/N^{1-r} \rightarrow \infty$, then*

$$C_f = O_{\mathbb{P}}((N/k)^{1/r}), \quad C_s/n \rightarrow 1 \quad \text{in probability.}$$

If $k/N^{1-r} \rightarrow 0$, then

$$C_s = O_{\mathbb{P}}(kN^r), \quad C_s/n \rightarrow 0 \quad \text{in probability.}$$

These conclusions hold both for the configuration model and for the uniform simple graph with the same degree sequence.

Proof. For the configuration model, apply Lemma 1 to (5)–(6). In the second case $C_s \leq S_s$. In the first case, connectivity and (2) give $C_s + C_f = n$ with probability tending to one. Lemma 2 gives $n \sim N$.

For the simple graph, the sum of degrees is dN , whereas the sum of their squares is $d^2(N - k + \sum_i t_i^2) = O(N)$. Janson's simplicity theorem [3, Theorem 1.1] therefore bounds the probability of a simple configuration away from zero. If an event E_N has probability tending to zero, then

$$\mathbb{P}(E_N \mid \text{simple}) \leq \frac{\mathbb{P}(E_N)}{\mathbb{P}(\text{simple})} \rightarrow 0.$$

The same bound transfers tightness. Conditioning a uniform matching on a simple graph produces the uniform simple graph with the specified degrees, proving the assertions. $\square$

The conditioning argument above is only used for probability convergence and tightness. It is not a claim that arbitrary distributional limits survive conditioning unchanged.

4 A sharp criterion for the number of hubs

Theorem 4. *Fix a sequence of positive integers $q_N = o(N)$. There exist block sizes $t_1, \dots, t_{q_N}$ such that $k_N = \sum_i t_i \rightarrow \infty$, (3) holds, and $C_s/n \rightarrow 1$ in probability, if and only if*

$$\frac{q_N}{N^{1-2r}} \rightarrow \infty. \quad (7)$$

When (7) holds, all blocks can have the same size $t_N \rightarrow \infty$. Thus every slow seed has diverging degree, while the fast seed has degree d . The equivalence holds for both graph models in Proposition 3.

Proof. Suppose (7) fails. Along a subsequence, $q_N \leq CN^{1-2r}$ for some finite constant C . For any admissible blocks, Cauchy–Schwarz and (3) give along this subsequence

$$k_N^2 \leq q_N \sum_i t_i^2 = o(N^{2-2r}).$$

Consequently $k_N/N^{1-r} \rightarrow 0$ on that subsequence. Proposition 3 forces $C_s/n \rightarrow 0$ there, excluding convergence to one along the full sequence.

Conversely, assume (7), and abbreviate $q = q_N$. Set

$$L_N = \max \left\{ 1, \frac{N^{1-r}}{q} \right\}, \quad U_N = \sqrt{\frac{N}{q}}, \quad t_N = \lfloor \sqrt{L_N U_N} \rfloor.$$

Since $q = o(N)$, we have $U_N \rightarrow \infty$. Moreover,

$$\frac{L_N}{U_N} = \max \left\{ \sqrt{\frac{q}{N}}, \frac{N^{1/2-r}}{\sqrt{q}} \right\} \rightarrow 0.$$

It follows that $t_N \rightarrow \infty$, $t_N/U_N \rightarrow 0$ and $t_N/L_N \rightarrow \infty$; rounding does not alter these limits. Choose q blocks of size t_N , so $k_N = qt_N$. Then

$$\frac{qt_N^2}{N} = \left(\frac{t_N}{U_N} \right)^2 \rightarrow 0, \quad \frac{k_N}{N^{1-r}} = \frac{t_N}{N^{1-r}/q} \geq \frac{t_N}{L_N} \rightarrow \infty.$$

Also $k_N/N \leq qt_N^2/N \rightarrow 0$, so there is room for all seeds for sufficiently large N . Proposition 3 completes the construction. $\square$

Corollary 5 (One hub and the half-rate boundary). *Within this family, one slow hub of diverging degree and convergent second degree moment can capture asymptotically all vertices if and only if $r > 1/2$. For $r \leq 1/2$, every such one-hub sequence gives $C_s/n \rightarrow 0$. At $r = 1/2$, any choice $q_N \rightarrow \infty$ with $q_N = o(N)$, however slowly it diverges, permits a construction with $C_s/n \rightarrow 1$.*

Proof. For one hub, (3) says $k_N = o(\sqrt{N})$. If $r \leq 1/2$, this puts the process in (6). If $r > 1/2$, choose $k_N = \lfloor N^a \rfloor$ with $1-r < a < 1/2$ and apply Proposition 3. At $r = 1/2$, condition (7) is exactly $q_N \rightarrow \infty$. $\square$

Corollary 6 (An explicit construction for every rate ratio). *For every $r \in (0, 1)$, take*

$$q_N = \lceil N^{1-3r/4} \rceil, \quad t_N = \lfloor N^{r/4} \rfloor.$$

These blocks satisfy (3), all slow hubs have diverging degree, their fraction tends to zero, and

$$C_s/n \rightarrow 1, \quad C_f = O_{\mathbb{P}}(N^{1/2}).$$

Proof. Here $k_N = q_N t_N \sim N^{1-r/2}$ and $q_N t_N^2 \sim N^{1-r/4} = o(N)$. Thus $k_N/N^{1-r} \rightarrow \infty$ and $(N/k_N)^{1/r} \sim N^{1/2}$. Apply Proposition 3. $\square$

5 Interpretation and reproducibility

The high-degree starting set in these examples is determined by degrees: eventually every hub has degree larger than d , and every other vertex has degree d . The fast seed can be chosen uniformly among the latter vertices. The threshold concerns how the finite second-moment budget limits the total number of initially active half-edges. Aggregation turns that budget into Cauchy–Schwarz’s inequality.

This gives an existence result in the deterministic finite-variance framework associated with [2]. It does not establish the same behaviour for a specified i.i.d. degree distribution, arbitrary networks or graph-dependent adversarial seed choices. It also leaves the critical regime $k/N^{1-r} \rightarrow c \in (0, \infty)$ unanalyzed. The original workshop record is therefore not declared fully resolved.

The accompanying standard-library Python verifier exhausts small labelled matchings and compares the original and quotient infection generators, including loops and multiple edges. It also checks exact degree-moment identities and rational parameter inequalities. These finite diagnostics support the reduction and algebra; the asymptotic claims follow from the proofs above and the cited theorems. They are not proof-assistant formalization. AI tools assisted the derivation, source search, verification code and exposition. This is an unrefereed note and makes no absolute priority claim.

References

- [1] T. Antunović, Y. Dekel, E. Mossel and Y. Peres, Competing first passage percolation on random regular graphs, *Random Structures & Algorithms* **50** (2017), 534–583. [doi:10.1002/rsa.20699](https://doi.org/10.1002/rsa.20699). Theorem numbering used here is from [arXiv:1109.2575v2](https://arxiv.org/abs/1109.2575v2) (2014).
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