

The Maximal Finite CP-Plus-Rank Equals the Maximal CP-Rank

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Abstract

The cp-plus-rank of a symmetric matrix is the smallest number of columns in an entrywise strictly positive Gram factor. We prove that, in every order n , its largest finite value equals the maximal cp-rank p_n . The key observation is that a strictly positive factor remains strictly positive after a sufficiently small contraction along the Perron eigendirection. Combining this negative deformation with the positive Perron lift of Bomze, Dickinson and Still realizes every finite cp-plus-rank as an ordinary cp-rank in the same order and with the same ordinary matrix rank. The argument includes singular matrices and extends the corresponding interior two-sided Perron-ray identity. It determines the relation between the two maxima, not a numerical formula for p_n or the largest pointwise gap.

1 The maximum relation

All matrices and factors are real. An $n \times n$ symmetric matrix A is *completely positive* if $A = BB^T$ for some entrywise nonnegative matrix B . Its cp-rank $\text{cp}(A)$ is the least number of columns of such a factor. The cp-plus-rank $\text{cp}^+(A)$ uses entrywise strictly positive factors instead. A minimum over an empty set is ∞ ; both ranks of the zero matrix are set to zero. Write

$$p_n = \max_{A \in \mathcal{CP}_n} \text{cp}(A), \quad p_n^+ = \sup\{\text{cp}^+(A) : \text{cp}^+(A) < \infty\}.$$

Conic Carathéodory in the space of symmetric matrices gives $\text{cp}(A) \leq n(n+1)/2$, so p_n is a finite attained integer. Always $\text{cp}(A) \leq \text{cp}^+(A)$.

Theorem 1. *For every $n \geq 1$, the maximum finite cp-plus-rank is attained and*

$$\boxed{p_n^+ = p_n.}$$

Bomze, Dickinson and Still introduced cp-plus-rank and studied its relation to cp-rank [1]. Their accessible preprint [2, Theorem 5.1] gives $p_n \leq p_n^+ \leq p_n + 1$, and the passage preceding Lemma 5.2 asks whether equality of the maxima holds in all orders. Dickinson's Oberwolfach contribution repeats this bracket and asks for the maximum finite cp-plus-rank [3, p. 3083]. The essential positive Perron lift below is theirs. Our additional step is a negative deformation of the factor, which also applies on the singular part of the finite cp-plus-rank domain.

Theorem 1 identifies the maximum in terms of p_n ; it does not determine p_n numerically for general n . The separate question in [3] concerning the maximum finite difference $\text{cp}^+(A) - \text{cp}(A)$ is not answered here. No absolute priority claim is made.

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2 Moving in the Perron direction

The notation $x > 0$ or $B > 0$ means entrywise strict positivity. For a unit vector x , put $P = xx^T$. If a symmetric matrix A satisfies $Ax = \lambda x$, then

$$P^2 = P, \quad PA = AP = \lambda P, \quad (I + tP)A(I + tP) = A + \lambda(2t + t^2)P. \quad (1)$$

Lemma 2 (Positive Perron lift, [1, 2]). *Suppose $M \in \mathcal{CP}_n \setminus \{0\}$ has a unit eigenvector $x > 0$ with eigenvalue $\lambda > 0$. Then, for every $\mu > 0$,*

$$\text{cp}^+(M + \mu xx^T) \leq \text{cp}(M).$$

Proof. Choose a minimal factor $M = DD^T$ with $D \geq 0$. Every column of D is nonzero, so $D^T x > 0$. Thus $E = (I + \nu xx^T)D$ is strictly positive for $\nu > 0$. Taking $\nu = \sqrt{1 + \mu/\lambda} - 1$ in (1) gives $EE^T = M + \mu xx^T$ without changing the column count. This is the argument of [2, Theorem 4.8]. $\square$

Lemma 3 (Negative deformation). *Let $A \neq 0$ have $q = \text{cp}^+(A) < \infty$. Let $x > 0$ be its unit Perron eigenvector and $\lambda > 0$ its Perron eigenvalue. For all sufficiently small $\varepsilon > 0$,*

$$\text{cp}(A - \varepsilon xx^T) = \text{cp}^+(A - \varepsilon xx^T) = q, \quad \text{rank}(A - \varepsilon xx^T) = \text{rank}(A). \quad (2)$$

Proof. Take $A = BB^T$ with $B > 0$ and q columns. In particular A is entrywise positive, so the specified positive Perron pair exists, whether or not A is singular. Set $P = xx^T$ and $H = PB = x(B^T x)^T > 0$. The finite minimum

$$\eta = \min_{i,j} \frac{B_{ij}}{H_{ij}}$$

is strictly positive. For $0 < s < \min\{1, \eta\}$ define

$$B_s = (I - sP)B = B - sH > 0, \quad \varepsilon = \lambda(2s - s^2) > 0.$$

Equation (1) shows that

$$A_s := B_s B_s^T = A - \varepsilon P, \quad A_s x = \lambda(1 - s)^2 x.$$

Apply Lemma 2 to A_s with its positive eigenvector x and eigenvalue $\lambda(1 - s)^2$. It gives

$$q = \text{cp}^+(A_s + \varepsilon P) \leq \text{cp}(A_s) \leq \text{cp}^+(A_s) \leq q.$$

All inequalities are equalities. The matrix $I - sP$ is invertible, so $\text{rank}(B_s) = \text{rank}(B)$ and hence the Gram matrices have the same rank. Finally, $s \mapsto \lambda(2s - s^2)$ is strictly increasing on $(0, 1)$ and maps a sufficiently small positive interval onto a positive interval. This proves the assertion for every sufficiently small ε . $\square$

In particular, every nonzero finite cp-plus-rank is realized as a cp-rank of a matrix of the same order and ordinary rank. The proof does not invoke upper semicontinuity of cp-plus-rank in an ambient neighborhood of a boundary point.

3 Attainment and the two-sided identity

We include the elementary closedness fact used to obtain the reverse inequality and the positive half of the ray identity; see also [2, Theorem 2.4].

Lemma 4. *For every nonnegative integer r , the set of symmetric matrices with cp-rank at most r is closed.*

Proof. The case $r = 0$ is immediate. Suppose $A_k \rightarrow A$ and $A_k = D_k D_k^T$ with $D_k \geq 0$ and r columns, padded with zeros when necessary. The squared norm of row i of D_k is $(A_k)_{ii}$, so all entries of D_k are bounded. A subsequence converges to a nonnegative $n \times r$ matrix D with $DD^T = A$. $\square$

Proof of Theorem 1. Lemma 3 shows that every finite cp-plus-rank is at most p_n , including trivially the value zero. For the opposite inequality, choose A_0 with $\text{cp}(A_0) = p_n$ and a minimal factor $D \geq 0$ with p_n nonzero columns. Let J be the all-ones $n \times n$ matrix. For $t > 0$, the matrix

$$C_t = (I + tJ)D$$

is strictly positive, since each column of D has positive sum. Also $C_t C_t^T \rightarrow A_0$ as $t \downarrow 0$. By Lemma 4, all sufficiently small $t > 0$ satisfy $\text{cp}(C_t C_t^T) \geq p_n$. On the other hand, C_t has p_n columns. Therefore

$$p_n \leq \text{cp}(C_t C_t^T) \leq \text{cp}^+(C_t C_t^T) \leq p_n.$$

The value p_n is attained by both ranks, completing the proof. $\square$

Corollary 5. *Suppose $A \neq 0$, $p = \text{cp}(A)$ and $q = \text{cp}^+(A) < \infty$. With x as in Lemma 3, for all sufficiently small $\varepsilon > 0$ one has*

$$\begin{aligned} \text{cp}(A + \varepsilon x x^T) &= \text{cp}^+(A + \varepsilon x x^T) = p, \\ \text{cp}(A - \varepsilon x x^T) &= \text{cp}^+(A - \varepsilon x x^T) = q. \end{aligned}$$

Both perturbed matrices have ordinary rank $\text{rank}(A)$.

Proof. The negative identity is Lemma 3. Closedness at A gives $p \leq \text{cp}(A + \varepsilon x x^T)$ for small ε , while Lemma 2 gives the reverse bound on cp-plus-rank. In the positive direction, only the nonzero eigenvalue associated with x is increased, so the ordinary rank is unchanged. $\square$

For interior matrices this is [2, Theorem 4.9]. The factor deformation extends it to all matrices with finite cp-plus-rank, including singular ones. A related criterion follows immediately.

Corollary 6. *Let A be an entrywise positive completely positive matrix, with unit Perron vector x and Perron eigenvalue λ . Then $\text{cp}^+(A) < \infty$ if and only if $A - \varepsilon x x^T$ is completely positive for some $0 < \varepsilon < \lambda$.*

Proof. The forward implication follows from the negative deformation. Conversely the remaining matrix has positive eigenvector x and eigenvalue $\lambda - \varepsilon > 0$; apply Lemma 2 to recover A . $\square$

Equality of the two extrema is not pointwise equality of the two rank functions. Nor can the finite cp-plus-rank hypothesis be replaced by entrywise positivity; examples distinguishing these conditions are given in [1, 2]. The result supplies neither a minimal-factor algorithm nor an ambient boundary semicontinuity theorem.

The accompanying standard-library Python checker verifies the deformation identities in exact rational arithmetic for 240 positive factors, including singular examples, as well as positive lifts and

invalid-hypothesis controls. These finite checks are supplementary; the proofs above do not depend on computational enumeration. This unrefereed preprint was developed with AI assistance in the derivations, literature search, code and exposition. Its verification consists of the stated self-audit and reproducible checks, not independent peer review or proof-assistant certification.

References

- [1] I. M. Bomze, P. J. C. Dickinson and G. Still, The structure of completely positive matrices according to their CP-rank and CP-plus-rank, *Linear Algebra and its Applications* **482** (2015), 191–206. [doi:10.1016/j.laa.2015.05.021](https://doi.org/10.1016/j.laa.2015.05.021).
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- [3] P. J. C. Dickinson, The structure of completely positive matrices according to their CP-rank and CP-plus-rank, in *Copositivity and Complete Positivity*, Oberwolfach Report 52/2017 (published 2018), 3082–3083. [doi:10.4171/owr/2017/52](https://doi.org/10.4171/owr/2017/52).