

On Fillman's Question for Separable Continuum Fibonacci Operators: Half-Lines for Two Couplings and the High-Energy Fricke–Vogt Invariant

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October 18, 2026

Abstract

J. Fillman asked (Oberwolfach Report 46/2017) whether, for the continuum Fibonacci potential V_ω built from two bump functions $f_0, f_1 \in L^2[0, 1)$ on unit cells, the spectrum of the separable operator $-\Delta + \lambda_1 V_\omega(x) + \lambda_2 V_\omega(y)$ on $L^2(\mathbb{R}^2)$, which is the sum $\sigma(L_{\lambda_1 V_\omega}) + \sigma(L_{\lambda_2 V_\omega})$ of two one-dimensional spectra, contains a half-line for all $\lambda_1, \lambda_2 > 0$. We give a partial answer. For locally constant bumps $f_j = c_j \chi_{[0, 1)}$ the answer is yes for all real λ_1, λ_2 (Theorem A). For $\lambda_1 = \lambda_2$ this is a theorem of Damanik, Fillman and Gorodetski (J. Funct. Anal. 2021); our proof uses their Proposition 4.2 and their construction of curve pieces as their proof does, together with a two-set version of their abstract Bethe–Sommerfeld lemma; the windows that carry thick Cantor parts of the one-dimensional spectra do not depend on the coupling. For bumps of bounded variation whose difference $h = f_0 - f_1$ is absolutely continuous with h' of bounded variation we prove that the Fricke–Vogt invariant at energy t^2 is $\sin^2 t |h(1)e^{2it} - h(0)|^2 / (16t^4) + O(t^{-5})$, with the same error bound for the t -derivative (Theorem C). If moreover $h(0) + h(1) \neq 0$, this verifies the two hypotheses on the invariant in Proposition 4.2 of Damanik–Fillman–Gorodetski, one of which they name as the main obstacle for general bumps, and the half-line follows for all real couplings (Theorem B). Theorem B is conditional: it assumes the assertion of their Remark 4.25, not proved there, that their construction of the curve pieces and its itinerary property extend to general bumps. Bumps with $h(0) + h(1) = 0$, which include all continuous potentials V_ω , and all other bumps outside the hypotheses of Theorem B, for instance bumps whose difference is not absolutely continuous, remain open. This is an unrefereed note.

1 Introduction

1.1 The question

For a real potential $V \in L^2_{\text{loc, unif}}(\mathbb{R})$ let $L_V = -\frac{d^2}{dx^2} + V$ on $L^2(\mathbb{R})$. In his abstract in the Oberwolfach Report 46/2017, Fillman [Fil17] considers the potentials

$$V_\omega(x) = \sum_{n \in \mathbb{Z}} ((1 - \omega_n)f_0(x - n) + \omega_n f_1(x - n)), \quad f_0, f_1 \in L^2[0, 1), \quad (1)$$

where ω is the Fibonacci sequence

$$\omega_n = \chi_{[1-\alpha, 1)}(n\alpha \bmod 1) \quad (n \in \mathbb{Z}), \quad \alpha = \frac{\sqrt{5}-1}{2} \quad (2)$$

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(the source prints the interval as $[1-\alpha]$; we read it as $[1-\alpha, 1)$, the coding of [DFG21, Remark 1.3(c)]). For two potentials, the separable operator $L_{V_1, V_2}^{(2)} = -\Delta + V_1(x) + V_2(y)$ on $L^2(\mathbb{R}^2)$ has the spectrum $\sigma(L_{V_1, V_2}^{(2)}) = \sigma(L_{V_1}) + \sigma(L_{V_2})$, a Minkowski sum [Fil17, p. 2798], [DFG21, (1.2)]. Fillman asks [Fil17, p. 2798]: if V_ω is given by (1)–(2), does $\sigma(L_{\lambda_1 V_\omega, \lambda_2 V_\omega}^{(2)})$ contain a half-line for every $\lambda_1, \lambda_2 > 0$? This is record OWR-15951-003 of the UnsolvedMath dataset [ula26].

The bumps f_0, f_1 are arbitrary in the source, so a complete positive answer must cover every pair (f_0, f_1) and every pair of couplings. All spectra here are bounded below; a half-line means a set $[E_1, \infty)$.

1.2 Results

We use the notation of Damanik, Fillman and Gorodetski [DFG21]: for real $g_a, g_b \in L^2[0, 1)$, $\Sigma(g_a, g_b)$ is the common spectrum of the operators $-\frac{d^2}{dx^2} + V_\omega$, where V_ω carries g_{ω_n} on $[n, n+1)$ and ω runs through the Fibonacci subshift over $\{a, b\}$ [DFG21, (3.7)]; a is the more frequent letter. In (2) the symbol 1 plays the role of a , so $\sigma(L_{\lambda V_\omega}) = \Sigma(\lambda f_1, \lambda f_0)$ (Section 2). Following [DFG21] we write $\Sigma_\mu = \Sigma(0, \mu \chi_{[0,1)})$. The Fricke–Vogt invariant $I(E)$ of a pair (g_a, g_b) is defined in Section 2.

Theorem A. *Let $f_0 = c_0 \chi_{[0,1)}$ and $f_1 = c_1 \chi_{[0,1)}$ with $c_0, c_1 \in \mathbb{R}$. For all $\lambda_1, \lambda_2 \in \mathbb{R}$ the spectrum $\sigma(L_{\lambda_1 V_\omega, \lambda_2 V_\omega}^{(2)})$ contains a half-line; equivalently, $\Sigma_{\mu_1} + \Sigma_{\mu_2}$ contains a half-line for all $\mu_1, \mu_2 \in \mathbb{R}$. The proof uses [DFG21, Proposition 4.2] and the construction of the curve pieces in [DFG21, p. 13] as the proof of [DFG21, Theorem 3.4] does, for two families of windows, with two consequences of that construction stated in Section 4: the hulls of the Cantor sets are the windows, and the cut parameters satisfy $\alpha, \beta < \pi/4$.*

Theorem C. *Let $g_a, g_b: [0, 1] \rightarrow \mathbb{R}$ be of bounded variation, and assume that $h := g_b - g_a$ is absolutely continuous and that h' is of bounded variation. Then for $t \geq 1$*

$$I(t^2) = L(t) + R(t), \quad L(t) = \frac{\sin^2 t |h(1)e^{2it} - h(0)|^2}{16t^4}, \quad |R(t)| + |R'(t)| \leq C t^{-5},$$

where C depends only on upper bounds for $\|g_a\|_\infty, \|g_b\|_\infty, \text{Var } g_a, \text{Var } g_b, \|h'\|_\infty$ and $\text{Var } h'$.

Theorem B (conditional). *Let $f_0, f_1: [0, 1] \rightarrow \mathbb{R}$ be of bounded variation, such that $h := f_0 - f_1$ is absolutely continuous, h' is of bounded variation, and*

$$f_0(0+) + f_0(1-) \neq f_1(0+) + f_1(1-). \quad (3)$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}$, and assume statement (P) of Section 4 for the curves of initial conditions of the pairs $(\lambda_j f_1, \lambda_j f_0)$ with $\lambda_j \neq 0$. Then $\sigma(L_{\lambda_1 V_\omega, \lambda_2 V_\omega}^{(2)})$ contains a half-line.

Statement (P) says that the construction of the curve pieces of [DFG21, p. 13], with property (4) of [DFG21, Proposition 4.2], applies to the given curve; for general bumps it is asserted in [DFG21, Remark 4.25] but not proved there (Section 4). Under (3), Corollary 5.4 turns Theorem C into hypotheses (2) and (3) of [DFG21, Proposition 4.2], the hypotheses on the invariant; DFG name (3) as the main difficulty for general bumps. Hypothesis (1) holds for every pair of bumps (Lemma 5.3). For $c_0 \neq c_1$, Theorem A is a special case of Theorem B, but for constant bumps the inputs of Section 4 are those of DFG’s own proof. Both theorems use DFG’s statement that the cut parameters α, β are small for a large iterate of the trace map.

For two couplings we use Lemma A (Section 3), a two-set version of [DFG21, Lemma 2.3], and Lemma E (Section 4): the windows that carry thick Cantor sets come from the free curve, not from

the coupling, so they are aligned for any two couplings. We also supply three details that the proof of [DFG21, Theorem 3.4] leaves implicit: the hull of each Cantor set is its window; a second family of windows is needed and is handled by a reflection; and the cut parameters satisfy $\alpha, \beta < \pi/4$. *Open* (Section 7): pairs with $f_0(0+) + f_0(1-) = f_1(0+) + f_1(1-)$, among them all pairs with continuous V_ω ; pairs whose difference is not absolutely continuous, such as $f_1 = 0$, $f_0 = \chi_{[0,1/2)}$, or has a derivative that is not of bounded variation; and bumps not of bounded variation.

1.3 Relation to earlier work

For $\lambda_1 = \lambda_2$ and constant bumps the half-line is [DFG21, Theorem 3.4]; the survey [DF20, Theorem 8.4] states it for d -fold sums. Question 1 of [DFG21] asks whether the results of [DFG21] hold for general bumps (and general cell lengths) with equal couplings; Theorem B with $\lambda_1 = \lambda_2$ answers its half-line part, conditionally on (P), for the bumps of Theorem B and unit cells. The characterization of the spectrum by bounded trace-map orbits is [DFG14, Proposition 6.3]. For general bumps, Fillman and Mei show $I(E) = O(E^{-1/2})$ in the proof of [FM18, Theorem 1.3]; Theorem C gives the leading term for the bumps considered here. Fillman and Tidwell [FT23] prove half-lines in sums of the form $g[F] + g[F]$ for one thick set F ; Lemma A treats two sets for $g(x) = x^2$ and uses their Lemma 3.2. The gap lemma of Newhouse [New70, New79] enters through [DFG21, Lemma 2.1], which is [DG11, Lemma 6.2]. The strongly coupled regime with one bump equal to zero is studied in [DEF⁺26].

2 Preliminaries

Letters. The density of the n with $n\alpha \bmod 1 \in [1 - \alpha, 1)$ is $\alpha > \frac{1}{2}$, so in (2) the symbol 1 is the more frequent one. With $1 \mapsto a$, $0 \mapsto b$, the sequence (2) is the element of the Fibonacci subshift Ω used in [DFG21, Remark 1.3(c)] (see also the coding in [DFG21, Section 6]), and λV_ω carries λf_1 on the a -cells and λf_0 on the b -cells. By [DFG21, (3.7)] and the Minkowski-sum formula,

$$\sigma(L_{\lambda_1 V_\omega, \lambda_2 V_\omega}^{(2)}) = \Sigma(\lambda_1 f_1, \lambda_1 f_0) + \Sigma(\lambda_2 f_1, \lambda_2 f_0). \quad (4)$$

Exchanging f_0 and f_1 replaces h by $-h$ and leaves (3) unchanged, so the letter convention does not affect Theorem B.

Bounded variation. A function on $[0, 1]$ of bounded variation is taken in its representative that is right-continuous on $[0, 1)$ and left-continuous at 1; $\text{Var } \psi$ is its total variation, and $\psi(0+) = \psi(0)$, $\psi(1-) = \psi(1)$. An absolutely continuous h is identified with its continuous representative, so for $h = f_0 - f_1$ in Theorem B we have $h(0) = f_0(0+) - f_1(0+)$, $h(1) = f_0(1-) - f_1(1-)$, and (3) reads $h(0) + h(1) \neq 0$.

Lemma 2.1 (aperiodicity). *Let $g_a, g_b \in L^2[0, 1]$ with $g_a \neq g_b$ (on a set of positive measure), and let V carry g_{ω_n} on $[n, n+1)$ for some $\omega \in \Omega$. Then V is not periodic: there is no $p > 0$ with $V(\cdot + p) = V$ almost everywhere. In particular [DFG21, Assumption 3.1] holds.*

Proof. The set $\mathcal{P}$ of periods $p \in \mathbb{R}$ with $V(\cdot + p) = V$ a.e. is a subgroup of $\mathbb{R}$, and it is closed because $s \mapsto V(\cdot + s)|_J$ is continuous into $L^2(J)$ for every bounded interval J . Suppose $\mathcal{P} \neq \{0\}$. If $\mathcal{P} = \mathbb{R}$, then all difference quotients of V vanish, V is a.e. constant, and $g_a = g_b$. Otherwise $\mathcal{P} = c\mathbb{Z}$ with $c > 0$. Let $\Phi(s) = V(s + \cdot)|_{[0,1)} \in L^2[0, 1]$; Φ is continuous, c -periodic, and $\Phi(n) = g_{\omega_n} \in \{g_a, g_b\}$ for $n \in \mathbb{Z}$. If c is irrational, $\{n + jc : n, j \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, so $\Phi(\mathbb{R})$ lies in the closed set $\{g_a, g_b\}$; as $\mathbb{R}$ is connected, Φ is constant, so V is a.e. constant, a contradiction. If $c = p/q$ is rational, the integer $p \geq 1$ is a period, so $g_{\omega_{n+p}} = g_{\omega_n}$ and hence $\omega_{n+p} = \omega_n$ for all n ; but no element of the Fibonacci subshift is periodic. $\square$

Transfer matrices and the trace map ([DFG21, Section 3], [DFG14, Section 6]). For real $g \in L^1[0, 1]$ and $E \in \mathbb{C}$ let $M_g(E)$ be the matrix that maps $(u(0), u'(0))$ to $(u(1), u'(1))$ for the solutions of $-u'' + gu = Eu$; it is entire in E and has determinant 1. For a pair (g_a, g_b) put

$$x_{-1} = \frac{1}{2} \operatorname{tr} M_{g_b}, \quad x_0 = \frac{1}{2} \operatorname{tr} M_{g_a}, \quad x_1 = \frac{1}{2} \operatorname{tr}(M_{g_b} M_{g_a}),$$

$\gamma(E) = (x_1, x_0, x_{-1})(E)$ (the curve of initial conditions), and $\Gamma(t) = \gamma(t^2)$ for $t \geq 0$. The trace map $T(x, y, z) = (2xy - z, x, y)$ preserves $G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$; the Fricke–Vogt invariant of the pair is $I(E) = G(\gamma(E))$. For $g_a = g_b = 0$ one obtains the free curve $\Pi(t) = (\cos 2t, \cos t, \cos t)$. Let Ω^+ be the set of $p \in \mathbb{R}^3$ with $G(p) \geq 0$ and bounded forward orbit $(T^k p)_{k \geq 0}$ [DFG21, p. 13].

Proposition 2.2 ([DFG14, Proposition 6.3]). *If $g_a \neq g_b$, then $\Sigma(g_a, g_b)$ is the set of $E \in \mathbb{R}$ for which $(T^k \gamma(E))_{k \geq 0}$ is bounded. In particular, if $t \geq 0$ and $\Gamma(t) \in \Omega^+$, then $t^2 \in \Sigma(g_a, g_b)$.*

In [DFG14, Section 6] the statement is made under aperiodicity and irreducibility. Aperiodicity holds by Lemma 2.1. Irreducibility in the sense of [DFG14, Section 4] holds with $\ell = 1$, since two concatenations of unit cells that agree on $[0, 1]$ begin with the same piece and $g_a \neq g_b$; for two letters the property for which irreducibility is assumed also follows from aperiodicity [DFG21, Remark 3.3]. For a closed set $S \subset \mathbb{R}$ write $S^{1/2} = \{t \geq 0 : t^2 \in S\}$, and for $F \subset [0, \infty)$ write $F^{(2)} = \{s^2 : s \in F\}$.

3 A two-set Bethe–Sommerfeld lemma

A Cantor set $C \subset \mathbb{R}$ is a compact, perfect, nowhere dense set; its hull is $\operatorname{hull} C = [\min C, \max C]$, and its gaps are the bounded components of $\operatorname{hull} C \setminus C$. We use Newhouse thickness as in [DFG21, Section 2]: for an ordering $\mathcal{U} = (U_n)_{n \geq 1}$ of the gaps and an endpoint u of U_n , the bridge $B(\mathcal{U}, u)$ is the component of $\operatorname{hull} C \setminus (U_1 \cup \dots \cup U_n)$ that contains u , and $\tau(C) = \sup_{\mathcal{U}} \inf_{n, u} |B(\mathcal{U}, u)|/|U_n|$. We use the following consequence of the gap lemma of Newhouse [New70, New79].

Lemma 3.1 ([DFG21, Lemma 2.1], proved as [DG11, Lemma 6.2]). *Let C, K be Cantor sets with $\tau(C)\tau(K) > 1$, such that no gap of C is longer than $\operatorname{diam} K$ and no gap of K is longer than $\operatorname{diam} C$. Then $C + K = [\min C + \min K, \max C + \max K]$.*

Lemma 3.2. *Let C be a Cantor set with $\operatorname{hull} C = [x, y]$.*

- (a) *If $x > 0$, then $C^{(2)}$ is a Cantor set with $\operatorname{hull} [x^2, y^2]$ and $\tau(C^{(2)}) \geq \frac{x}{y} \tau(C)$.*
- (b) ([FT23, Lemma 3.2]) *If $\tau(C) > \beta \geq 0$, every gap of C has length at most $(y - x)/(1 + 2\beta)$.*

Proof. (a) $s \mapsto s^2$ is an increasing homeomorphism of $[x, y]$ onto $[x^2, y^2]$; it maps the gaps of C bijectively and in order onto those of $C^{(2)}$, and bridges onto bridges. For an interval $P \subset [x, y]$ we have $2x|P| \leq |P^{(2)}| \leq 2y|P|$, so each ratio $|B|/|U|$ changes at most by the factor x/y . (b) If a gap U had $|U| > (y - x)/(1 + 2\beta)$, one of the two components of $[x, y] \setminus U$ would have length less than $\beta(y - x)/(1 + 2\beta) < \beta|U|$; in every ordering the bridge at the endpoint of U on that side lies in this component, so $\tau(C) \leq \beta$. $\square$

Lemma A (two-set Bethe–Sommerfeld lemma). *Let $F_1, F_2 \subset [0, \infty)$ be closed. Suppose that for $i = 1, 2$ and all integers $k \geq k_0$ there are Cantor sets $K_k^i \subset F_i$ with $\operatorname{hull} K_k^i = [x_k^i, y_k^i]$, and constants $0 < a < A, B, D$, such that for $i = 1, 2$ and $k \geq k_0$:*

- (a) $y_k^i < x_{k+1}^i$, $A \leq y_k^i - x_k^i \leq B$ and $x_{k+1}^i - y_k^i \leq a$;
- (b) $|x_k^1 - x_k^2| \leq D$;

(c) $\tau(K_k^i) \rightarrow \infty$ as $k \rightarrow \infty$.

Then $F_1^{(2)} + F_2^{(2)}$ contains a half-line.

Proof. Write $X_k^i = (x_k^i)^2$, $Y_k^i = (y_k^i)^2$ and $\tilde{K}_k^i = (K_k^i)^{(2)}$. By (a), $x_{k+1}^i - x_k^i \geq A$, so $x_k^i \rightarrow \infty$; we may assume $x_k^i \geq 1$. Fix $N \geq 1$ with $2NA > a$. Since $x_{l+1}^2 - x_l^2 \leq B + a$, (b) gives for $|l - k| \leq N$, $l \geq k_0$:

$$x_k^1, y_k^1, x_l^2, y_l^2 \in [x_k^1 - \Xi, x_k^1 + \Xi], \quad \Xi := D + N(B + a) + B. \quad (5)$$

By Lemma 3.2(a) and (a), $\tau(\tilde{K}_k^i) \geq (1 - B/y_k^i)\tau(K_k^i) \rightarrow \infty$.

Step 1. For all large k and all l with $|l - k| \leq N$: $\tilde{K}_k^1 + \tilde{K}_l^2 = [X_k^1 + X_l^2, Y_k^1 + Y_l^2]$. Indeed, $\text{diam } \tilde{K}_k^1 = (y_k^1 - x_k^1)(y_k^1 + x_k^1) \geq 2Ax_k^1$, and by Lemma 3.2(b) with $\beta = \tau/2$ every gap of $\tilde{K}_l^2$ has length at most $\text{diam } \tilde{K}_l^2 / (1 + \tau(\tilde{K}_l^2)) \leq 2B(x_k^1 + \Xi) / (1 + \tau(\tilde{K}_l^2))$, which is less than $2Ax_k^1$ for large k , uniformly in the $2N + 1$ values of l . In the same way the gaps of $\tilde{K}_k^1$ are shorter than $\text{diam } \tilde{K}_l^2 \geq 2A(x_k^1 - \Xi)$, and $\tau(\tilde{K}_k^1)\tau(\tilde{K}_l^2) > 1$. Apply Lemma 3.1.

Step 2. Let $U_k = \bigcup_{l=k-N}^{k+N} [X_k^1 + X_l^2, Y_k^1 + Y_l^2]$, a subset of $F_1^{(2)} + F_2^{(2)}$ by Step 1. Both endpoints increase with l , and the intervals for l and $l + 1$ overlap if $X_{l+1}^2 - Y_l^2 \leq Y_k^1 - X_k^1$. By (a) and (5) the left side is $(x_{l+1}^2 - y_l^2)(x_{l+1}^2 + y_l^2) \leq 2a(x_k^1 + \Xi)$, and the right side is at least $2Ax_k^1$. As $a < A$, for large k we get $U_k = [p_k, q_k]$ with $p_k = X_k^1 + X_{k-N}^2$ and $q_k = Y_k^1 + Y_{k+N}^2$.

Step 3. $p_{k+1} \leq q_k$ for large k , i.e. $X_{k+1}^1 - Y_k^1 \leq Y_{k+N}^2 - X_{k+1-N}^2$. The left side is at most $a(2y_k^1 + a) \leq a(2x_k^1 + 2B + a)$. The interval $[x_{k+1-N}^2, y_{k+N}^2]$ contains the $2N$ disjoint hulls of $K_{k+1-N}^2, \dots, K_{k+N}^2$, so it has length at least $2NA$, and by (5) $y_{k+N}^2 + x_{k+1-N}^2 \geq 2(x_k^1 - \Xi)$; the right side is at least $4NA(x_k^1 - \Xi)$. Since $4NA > 2a$, the inequality holds for large k .

Step 4. Let k_1 be such that Steps 1–3 hold for $k \geq k_1$. The sequence p_k increases to ∞ . Given $E \geq p_{k_1}$, let $k \geq k_1$ be the largest index with $p_k \leq E$; then $E < p_{k+1} \leq q_k$, so $E \in U_k$. Hence $[p_{k_1}, \infty) \subset F_1^{(2)} + F_2^{(2)}$. $\square$

With $F_1 = F_2$, $K_k^1 = K_k^2$ and $N = 1$ this is the argument of [DFG21, Lemma 2.3], which uses the pairs (n, n) and $(n, n + 1)$.

4 The input from Damanik–Fillman–Gorodetski

Windows. For small $I \geq 0$, DFG use a Markov partition $R_I^1, \dots, R_I^6$ of $T|_{S_I}$, $S_I = \{G = I\}$, formed by pieces of stable and unstable manifolds of eight periodic points (of periods 2 and 6) near the singular points of S_0 , and put $R^i = \bigcup_{I \geq 0} R_I^i$ [DFG21, pp. 12–13]. For an integer m the curve $T^{6m}(\Pi([2\pi n, (2n+1)\pi]))$ runs from $(1, 1, 1)$ to $(1, -1, -1)$, and each component of its intersection with an element of the partition connects one stable boundary with another. Removing the first and the last component and pulling back by T^{6m} leaves $\tilde{\Pi} = \Pi([2\pi n + \alpha, (2n+1)\pi - \beta])$ with $\alpha = \alpha(m) > 0$, $\beta = \beta(m) > 0$, which are small when m is large [DFG21, p. 13]. Since $\Pi(2\pi(2n+1) - s) = \Pi(s)$, the same curve is $\Pi([(2n+1)\pi + \beta, 2\pi(n+1) - \alpha])$; DFG mention this second parametrization, with the upper end printed as $2\pi n - \alpha$.

(*C-small*) We fix m so large that $0 < \alpha, \beta < \pi/4$. This uses only DFG’s statement that the removed pieces are small for large m . (The reason: $\Pi([0, \pi])$ is the image of the diagonal segment from $(0, 0)$ to $(\frac{1}{2}, \frac{1}{2})$ under the semiconjugacy of the torus automorphism $(\theta, \varphi) \mapsto (\theta + \varphi, \theta)$ [DFG21, pp. 8, 12], whose $6m$ -th iterate expands the segment by a factor comparable to $(\frac{1+\sqrt{5}}{2})^{6m}$, while the removed components have bounded length.)

Proposition 4.1 ([DFG21, Proposition 4.2]). *Let Γ_n be curves in $\mathbb{R}^3$ parametrized by $J_n = [2\pi n + \alpha_n, (2n+1)\pi - \beta_n]$ with $\alpha_n \rightarrow \alpha$, $\beta_n \rightarrow \beta$, such that (1) Γ_n converges to $\tilde{\Pi}$ in the C^2 topology; (2)*

$G(\Gamma_n(t)) > 0$ on J_n ; (3) $|\frac{d}{dt}G(\Gamma_n(t))| \leq C G(\Gamma_n(t))$ with C independent of n ; (4) for all $k \geq 0$ the images $T^k(\Gamma_n)$ meet exactly the same sets R^i , in the same order, as $T^k(\tilde{\Pi})$. Then $K_n = \{t \in J_n : \Gamma_n(t) \in \Omega^+\}$ is a Cantor set for all large n , and $\tau(K_n) \rightarrow \infty$.

DFG prove this for curves satisfying (1)–(4) (Sections 4.1–4.4 of [DFG21]). For the curve Γ of a pair (g_a, g_b) they choose the pieces Γ_n so that the T^{6m} -images connect points of the boundaries of the same R^i as $T^{6m}(\tilde{\Pi})$ [DFG21, p. 13], and in their model they obtain (4) from this construction [DFG21, p. 23]; (4) is understood in this sense below. We formulate the input for a curve Γ as follows.

Statement (P) for Γ . There are n_0 and numbers $\alpha_n, \alpha'_n \rightarrow \alpha, \beta_n, \beta'_n \rightarrow \beta$ ($n \geq n_0$) such that, with

$$J_n^1 = [2\pi n + \alpha_n, (2n+1)\pi - \beta_n], \quad J_n^2 = [(2n+1)\pi + \beta'_n, 2\pi(n+1) - \alpha'_n],$$

the curves $\Gamma|_{J_n^1}$ and $\Gamma'_n(s) = \Gamma(2\pi(2n+1) - s)$, $s \in [2\pi n + \alpha'_n, (2n+1)\pi - \beta'_n]$, have property (4) of Proposition 4.1, and T^{6m} maps the endpoints of these curves into the stable boundary of the Markov partition of the level surface through them.

For DFG's model $(0, \mu\chi_{[0,1)})$, (P) is the construction that the proof of [DFG21, Theorem 3.4] uses; for the second family we use the same construction in the second parametrization, transported by the reflection $s \mapsto 2\pi(2n+1) - s$. For general bumps, DFG write that “a modified version of property (1) and property (4) can be established” in the general case by the same argument [DFG21, Remark 4.25]; this is not proved in [DFG21], and Theorem B assumes it in the form (P). For unit cells no modification is needed for property (1): by Lemma 5.3, every pair of bumps has the free curve Π itself as its high-energy limit, whereas for cells of lengths ℓ_a, ℓ_b the limit would be $(\cos(\ell_a + \ell_b)t, \cos \ell_a t, \cos \ell_b t)$. The choice of the pieces involves Γ only through its T^{6m} -image near $T^{6m}(\tilde{\Pi})$, which is controlled by the C^1 distance between Γ and Π on the windows.

Lemma 4.2 (hulls). *Assume (P) for Γ and $G(\Gamma(t)) > 0$ at the endpoints of J_n^1 and J_n^2 . Then Γ maps these endpoints into Ω^+ .*

Proof. Let p be such a point and $I = G(p) > 0$. By (P), $T^{6m}p$ lies on the stable boundary of the Markov partition of $T|_{S_I}$, which consists of pieces of stable manifolds of the eight periodic points [DFG21, p. 13]. So the forward orbit of p converges to a periodic orbit and is bounded. $\square$

Proposition 4.3. *Let $g_a \neq g_b$ and let Γ be the curve of (g_a, g_b) . Put $d = \min(\alpha, \beta)/4$ and assume*

- (H1) $\Gamma^{(r)}(t) - \Pi^{(r)}(t) \rightarrow 0$ as $t \rightarrow \infty$, for $r = 0, 1, 2$;
- (H2) there is t_0 such that $G(\Gamma(t)) > 0$ for all $t \geq t_0$ with $\text{dist}(t, \pi\mathbb{Z}) \geq d$;
- (H3) there is C such that $|\frac{d}{dt}G(\Gamma(t))| \leq C G(\Gamma(t))$ for the same t ;
- (P) statement (P) for Γ .

Then there are n_1 and Cantor sets $K_n^j \subset \Sigma(g_a, g_b)^{1/2}$ ($j = 1, 2$, $n \geq n_1$) with $\text{hull } K_n^j = J_n^j$ and $\tau(K_n^j) \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. For large n all of $\alpha_n, \beta_n, \alpha'_n, \beta'_n$ exceed $2d$, so $J_n^1, J_n^2 \subset \{t \geq t_0 : \text{dist}(t, \pi\mathbb{Z}) \geq d\}$. The curves $\Gamma|_{J_n^1}$ satisfy (1) of Proposition 4.1 by (H1), (2) and (3) by (H2) and (H3), and (4) by (P). So $K_n^1 = \{t \in J_n^1 : \Gamma(t) \in \Omega^+\}$ is a Cantor set for large n with $\tau(K_n^1) \rightarrow \infty$; its hull is J_n^1 by Lemma 4.2, and $K_n^1 \subset \Sigma^{1/2}$ by Proposition 2.2. The map $s \mapsto 2\pi(2n+1) - s$ is an isometry onto J_n^2 that preserves $\text{dist}(\cdot, \pi\mathbb{Z})$, and $\Pi(2\pi(2n+1) - s) = \Pi(s)$. Hence the curves Γ'_n satisfy (1) by (H1), (2) and (3) by (H2) and (H3) (the derivative only changes sign), and (4) by (P). Proposition 4.1 gives Cantor sets K'_n with $\tau(K'_n) \rightarrow \infty$, and $K_n^2 := 2\pi(2n+1) - K'_n = \{t \in J_n^2 : \Gamma(t) \in \Omega^+\}$ has the same thickness, hull J_n^2 by Lemma 4.2, and lies in $\Sigma^{1/2}$ by Proposition 2.2. $\square$

Both families are needed: [DFG21, Lemma 2.3] asks for hulls I_n with $|I_n| \geq A > a \geq \text{dist}(I_n, I_{n+1})$, while with the first family alone consecutive hulls have length about $\pi - \alpha - \beta$ and distance about $\pi + \alpha + \beta$.

Lemma E (two couplings). *Let two pairs (g_a^i, g_b^i) , $i = 1, 2$, satisfy the conclusion of Proposition 4.3, with the same m , α , β . Then $\Sigma(g_a^1, g_b^1) + \Sigma(g_a^2, g_b^2)$ contains a half-line.*

Proof. For the i -th pair put $K_{2n}^i = K_n^1$ and $K_{2n+1}^i = K_n^2$. As $n \rightarrow \infty$ the hull lengths tend to $L = \pi - \alpha - \beta$, and the distances between consecutive hulls tend to 2β (between J_n^1 and J_n^2) or 2α (between J_n^2 and J_{n+1}^1); the left ends of the k -th hulls of the two pairs differ by $\alpha_n^1 - \alpha_n^2$ or $\beta_n^1 - \beta_n^2$, which tends to 0. Let $g = 2\max(\alpha, \beta)$. Since $\alpha, \beta < \pi/4$, we have $g < \pi/2 < L$. With $\delta = (L - g)/3$, $A = L - \delta$, $a = g + \delta < A$, $B = \pi$ and $D = 1$, hypotheses (a)–(c) of Lemma A hold for $k \geq k_0$ with $F_i = \Sigma(g_a^i, g_b^i)^{1/2}$, which is closed. Lemma A gives a half-line in $F_1^{(2)} + F_2^{(2)}$, which is contained in $\Sigma(g_a^1, g_b^1) + \Sigma(g_a^2, g_b^2)$. $\square$

5 The invariant at high energy

Complex coordinates. Identify $\mathbb{R}^2$ with $\mathbb{C}$. Every real 2×2 matrix X acts as $z \mapsto \alpha z + \zeta \bar{z}$ for unique $\alpha = \alpha(X)$, $\zeta = \zeta(X) \in \mathbb{C}$; explicitly $\alpha = \frac{X_{11} + X_{22}}{2} - i \frac{X_{12} - X_{21}}{2}$ and $\zeta = \frac{X_{11} - X_{22}}{2} + i \frac{X_{12} + X_{21}}{2}$. We write $q(X) = -\text{Im } \alpha(X)$. Then $\frac{1}{2} \text{tr } X = \text{Re } \alpha$, $\det X = |\alpha|^2 - |\zeta|^2$, $\|X\| = |\alpha| + |\zeta|$ (operator norm), and

$$(\alpha_1, \zeta_1) \cdot (\alpha_2, \zeta_2) = (\alpha_1 \alpha_2 + \zeta_1 \bar{\zeta}_2, \alpha_1 \zeta_2 + \zeta_1 \bar{\alpha}_2). \quad (6)$$

The rotation $R(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} = e^{\varphi J}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, is $(e^{-i\varphi}, 0)$, and $n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is $(\frac{i}{2}, \frac{i}{2})$.

Lemma 5.1. *For $A, B \in SL(2, \mathbb{R})$ with $x = \frac{1}{2} \text{tr}(AB)$, $y = \frac{1}{2} \text{tr } A$, $z = \frac{1}{2} \text{tr } B$,*

$$G(x, y, z) = |q(A)\zeta(B) - q(B)\zeta(A)|^2 - (\text{Im } \bar{\zeta}(A)\zeta(B))^2.$$

Proof. Write $q_A, \zeta_A, \dots$. By (6), $x = \text{Re}(\alpha_A \alpha_B + \zeta_A \bar{\zeta}_B) = yz + b$ with $b = -q_A q_B + \text{Re}(\bar{\zeta}_A \zeta_B)$. Then $G = (yz + b)^2 + y^2 + z^2 - 2(yz + b)yz - 1 = b^2 - (1 - y^2)(1 - z^2)$, and $1 - y^2 = q_A^2 - |\zeta_A|^2$, $1 - z^2 = q_B^2 - |\zeta_B|^2$ since the determinants are 1. Expanding, $G = q_A^2 |\zeta_B|^2 + q_B^2 |\zeta_A|^2 - 2q_A q_B \text{Re}(\bar{\zeta}_A \zeta_B) + (\text{Re } \bar{\zeta}_A \zeta_B)^2 - |\zeta_A|^2 |\zeta_B|^2$, which is the claim because $|\zeta_A|^2 |\zeta_B|^2 = (\text{Re } \bar{\zeta}_A \zeta_B)^2 + (\text{Im } \bar{\zeta}_A \zeta_B)^2$. $\square$

The rotating frame. For real $g \in L^1[0, 1]$ and $k \in \mathbb{C} \setminus \{0\}$ let $\widetilde{M}_g(k) = D_k M_g(k^2) D_k^{-1}$ with $D_k = \text{diag}(1, 1/k)$. Since $D_k \begin{pmatrix} 0 & 1 \\ g - k^2 & 0 \end{pmatrix} D_k^{-1} = kJ + \frac{g}{k}n$, we have $\widetilde{M}_g(k) = Y(1)$, where $Y' = (kJ + \frac{g}{k}n)Y$, $Y(0) = \mathbf{1}$. Conjugation does not change traces, so by Lemma 5.1 the invariant of a pair is, for $k > 0$,

$$I(k^2) = |q_a \zeta_b - q_b \zeta_a|^2 - (\text{Im } \bar{\zeta}_a \zeta_b)^2, \quad q_a = q(\widetilde{M}_{g_a}(k)), \quad \zeta_a = \zeta(\widetilde{M}_{g_a}(k)), \text{ etc.} \quad (7)$$

Variation of constants gives the Dyson series: with $\Delta_m = \{1 > s_1 > \dots > s_m > 0\}$ and $T_0 = R(k)$,

$$\widetilde{M}_g(k) = \sum_{m \geq 0} T_m, \quad T_m = \frac{1}{k^m} \int_{\Delta_m} g(s_1) \cdots g(s_m) R(k(1 - s_1)) n R(k(s_1 - s_2)) \cdots n R(k s_m) ds. \quad (8)$$

The arguments of the rotations are k times non-negative numbers with sum 1, and $\|R(\varphi)\| = e^{|\text{Im } \varphi|}$, $\|n\| = 1$; hence $\|T_m\| \leq e^{|\text{Im } k|} (\|g\|_1 / |k|)^m / m!$. For real k we use the form $T_m = R(k) W_m$ with

$$W_m = \frac{1}{k^m} \int_{\Delta_m} g(s_1) \cdots g(s_m) N(s_1) \cdots N(s_m) ds, \quad N(\sigma) = R(-k\sigma) n R(k\sigma) = \left(\frac{i}{2}, \frac{i}{2} e^{2ik\sigma}\right).$$

For real k , $\|N(\sigma)\| = 1$ and $\|\partial_k N(\sigma)\| = \sigma \leq 1$, so $\|W_m\| \leq (\|g\|_1/k)^m/m!$ and $\|\partial_k W_m\| \leq (m/k + m)(\|g\|_1/k)^m/m!$. By (6),

$$W_1 = \left(\frac{iQ}{2k}, \frac{i}{2k}F_g(k)\right), \quad \zeta(W_2) = \frac{1}{4k^2} \int_0^1 \varphi(s)e^{2iks} ds, \quad (9)$$

where $Q = \int_0^1 g$, $F_g(k) = \int_0^1 g(s)e^{2iks} ds$, $\varphi = g(G_0 - H_0)$, $G_0(s) = \int_0^s g$, $H_0(s) = \int_s^1 g$: indeed $\zeta(N(s_1)N(s_2)) = \frac{1}{4}(e^{2iks_1} - e^{2iks_2})$.

Oscillatory integrals. For ψ of bounded variation on $[0, 1]$ and real $w \neq 0$, integration by parts (Riemann–Stieltjes) gives

$$\left| \int_0^1 \psi(s)e^{iws} ds \right| \leq \frac{V(\psi)}{|w|}, \quad V(\psi) := |\psi(0)| + |\psi(1)| + \text{Var } \psi \leq 2\|\psi\|_\infty + \text{Var } \psi. \quad (10)$$

Proposition 5.2. *Let $g: [0, 1] \rightarrow \mathbb{R}$ be of bounded variation. There is C , depending only on bounds for $\|g\|_\infty$ and $\text{Var } g$, such that for $k \geq 1$, with $q = q(\widetilde{M}_g(k))$ and $\zeta = \zeta(\widetilde{M}_g(k))$:*

$$(C1) \quad |q - \sin k| + |\partial_k q - \cos k| \leq C/k;$$

$$(C2) \quad \zeta = \frac{ie^{-ik}}{2k}F_g(k) + \rho_g(k) \text{ with } |\rho_g| + |\rho'_g| \leq C/k^3;$$

$$(C3) \quad |F_g| \leq V(g)/(2k), \quad |F'_g| \leq V(sg)/k, \text{ and } |\zeta| + |\partial_k \zeta| \leq C/k^2.$$

Proof. By (6), $\alpha(R(k)X) = e^{-ik}\alpha(X)$ and $\zeta(R(k)X) = e^{-ik}\zeta(X)$. The bounds on W_m give $\|W_{\geq 2}\| + \|\partial_k W_{\geq 2}\| \leq C/k^2$ and $\|W_{\geq 3}\| + \|\partial_k W_{\geq 3}\| \leq C/k^3$, where $W_{\geq j} = \sum_{m \geq j} W_m$. (C1): $\alpha(W) = 1 + \frac{iQ}{2k} + O(k^{-2})$, so $q = -\text{Im}(e^{-ik}\alpha(W)) = \sin k - \frac{Q}{2k}\cos k + O(k^{-2})$, and the same holds for the k -derivatives, with $O(k^{-1})$ for the derivative of the middle term. (C2): $\rho_g = e^{-ik}(\zeta(W_2) + \zeta(W_{\geq 3}))$ and $|\zeta(X)| \leq \|X\|$. The function φ is of bounded variation, since $G_0 - H_0$ is Lipschitz with $\|G_0 - H_0\|_\infty \leq \|g\|_1$ and $\text{Var}(G_0 - H_0) \leq 2\|g\|_1$. By (10) with $w = 2k$, $|\zeta(W_2)| \leq V(\varphi)/(8k^3)$, and $\partial_k \zeta(W_2) = -\frac{1}{2k^3} \int \varphi e^{2iks} + \frac{1}{4k^2} \int 2is \varphi(s)e^{2iks} ds$ is $O(k^{-3})$ by (10) for φ and $s\varphi$. (C3): (10) with $w = 2k$ for g and for $F'_g(k) = \int_0^1 2is g(s)e^{2iks} ds$; then (C2). $\square$

Proof of Theorem C. We write k for t . Let $q_a = \sin k + \varepsilon_a$, $q_b = \sin k + \varepsilon_b$; by (C1), $|\varepsilon| + |\varepsilon'| \leq C/k$. Then $q_a \zeta_b - q_b \zeta_a = \sin k (\zeta_b - \zeta_a) + \varepsilon_a \zeta_b - \varepsilon_b \zeta_a$. By (C2) and linearity, $\zeta_b - \zeta_a = \frac{ie^{-ik}}{2k}F_h(k) + \rho_b - \rho_a$. Since h is absolutely continuous,

$$F_h(k) = \frac{h(1)e^{2ik} - h(0)}{2ik} + r(k), \quad r(k) = -\frac{1}{2ik} \int_0^1 h'(s)e^{2iks} ds,$$

and (10) for h' and sh' gives $|r| + |r'| \leq C/k^2$. As $\frac{i}{2k} \cdot \frac{1}{2ik} = \frac{1}{4k^2}$,

$$q_a \zeta_b - q_b \zeta_a = \Lambda + \rho_*, \quad \Lambda = \frac{\sin k e^{-ik}(h(1)e^{2ik} - h(0))}{4k^2}, \quad \rho_* = \sin k \left(\frac{ie^{-ik}}{2k}r + \rho_b - \rho_a \right) + \varepsilon_a \zeta_b - \varepsilon_b \zeta_a.$$

Each term of ρ_* is a product of a factor that is $O(k^{-1})$ together with its derivative ($\frac{\sin k e^{-ik}}{2k}$, ε_a , ε_b) and a factor that is $O(k^{-2})$ together with its derivative (r , ζ_a , ζ_b), or is $\sin k (\rho_b - \rho_a)$; so $|\rho_*| + |\rho'_*| \leq C/k^3$. Also $|\Lambda| + |\Lambda'| \leq C/k^2$, $|\Lambda|^2 = L(k)$, and by (7)

$$I(k^2) = L(k) + 2 \text{Re}(\bar{\Lambda}\rho_*) + |\rho_*|^2 - (\text{Im } \bar{\zeta}_a \zeta_b)^2,$$

where the last three terms and their derivatives are $O(k^{-5})$, $O(k^{-6})$ and $O(k^{-8})$ by (C3). The constants depend only on the quantities listed in Theorem C. $\square$

For $g_a = 0$, $g_b = \mu\chi_{[0,1]}$ the leading term is $L(t) = \mu^2 \sin^4 t / (4t^4)$, the leading term of the exact formula $I(t^2) = \mu^2 \sin^2 t \sin^2 \sqrt{t^2 - \mu} / (4t^2(t^2 - \mu))$ [DFG21, (3.11)] (stated there for the pair $(\mu\chi_{[0,1]}, 0)$; I is symmetric in the two pieces).

Lemma 5.3 (property (1) for all bumps). *Let $g_a, g_b \in L^1[0, 1]$ be real and Γ their curve. There is C such that $|\Gamma^{(r)}(t) - \Pi^{(r)}(t)| \leq r! C / (t - 1)$ for $t \geq 2$ and $r = 0, 1, 2$.*

Proof. For $|\operatorname{Im} k| \leq 1$ and $|k| \geq 1$, (8) gives $\|\widetilde{M}_g(k) - R(k)\| \leq e(e^{\|g\|_1/|k|} - 1) \leq C_1/|k|$ for $g = g_a, g_b$. Hence $|x_0(k^2) - \cos k|$, $|x_{-1}(k^2) - \cos k|$ and $|x_1(k^2) - \cos 2k|$ are at most $C/|k|$ there, using $\widetilde{M}_{g_b}\widetilde{M}_{g_a} - R(2k) = (\widetilde{M}_{g_b} - R(k))\widetilde{M}_{g_a} + R(k)(\widetilde{M}_{g_a} - R(k))$ and $|\operatorname{tr} X| \leq 2\|X\|$. The functions $x_j(k^2)$ and $\Pi(k)$ are entire in k , and for real $t \geq 2$ the disc of radius 1 about t lies in the region $\{|\operatorname{Im} k| \leq 1, |k| \geq t - 1 \geq 1\}$. Cauchy's estimates give the claim. $\square$

Corollary 5.4. *In the setting of Theorem C assume $h(0) + h(1) \neq 0$, and let $0 < d \leq \pi/2$. There are t_0 and C_d such that for all $t \geq t_0$ with $\operatorname{dist}(t, \pi\mathbb{Z}) \geq d$:*

$$I(t^2) \geq \frac{\sin^4 d (h(0) + h(1))^2}{32t^4} > 0 \quad \text{and} \quad \left| \frac{d}{dt} I(t^2) \right| \leq C_d I(t^2).$$

Proof. We have $|h(1)e^{2it} - h(0)|^2 = (h(1) - h(0))^2 + 4h(0)h(1)\sin^2 t$. If $h(0)h(1) \geq 0$ this is at least $\sin^2 t [(h(1) - h(0))^2 + 4h(0)h(1)] = \sin^2 t (h(0) + h(1))^2$; if $h(0)h(1) < 0$ it is at least $(h(1) - h(0))^2 + 4h(0)h(1) = (h(0) + h(1))^2$, which is at least $\sin^2 t (h(0) + h(1))^2$. So $L(t) \geq \sin^4 t (h(0) + h(1))^2 / (16t^4)$, and $|\sin t| \geq \sin d$ when $\operatorname{dist}(t, \pi\mathbb{Z}) \geq d$. Also $|L'(t)| \leq C/t^4$. By Theorem C, $I \geq L - C/t^5 \geq L/2$ and $|I'| \leq |L'| + C/t^5 \leq C'/t^4 \leq C_d I$ for large t . $\square$

6 Proofs of Theorems A and B

Proof of Theorem A. Let W_ω be the potential of the pair $(0, \chi_{[0,1]})$. Then $\lambda V_\omega = \lambda c_1 + \lambda(c_0 - c_1)W_\omega$, so $\Sigma(\lambda c_1 \chi_{[0,1]}, \lambda c_0 \chi_{[0,1]}) = \lambda c_1 + \Sigma_{\lambda(c_0 - c_1)}$, and by (4) the spectrum of $L_{\lambda_1 V_\omega, \lambda_2 V_\omega}^{(2)}$ is $(\lambda_1 + \lambda_2)c_1 + \Sigma_{\mu_1} + \Sigma_{\mu_2}$ with $\mu_j = \lambda_j(c_0 - c_1)$. It suffices to show that $\Sigma_{\mu_1} + \Sigma_{\mu_2}$ contains a half-line. If $\mu_1 = 0$, then $\Sigma_0 = [0, \infty)$ and $\Sigma_0 + \Sigma_{\mu_2} = [\min \Sigma_{\mu_2}, \infty)$; likewise if $\mu_2 = 0$. Let $\mu \neq 0$ and let Γ be the curve of $(0, \mu\chi_{[0,1]})$. (H1) holds by Lemma 5.3; DFG obtain it from the asymptotics of [PT87, Theorem 1.3] and Cauchy estimates [DFG21, Section 4.5]. For (H2) and (H3) we use the exact formula $I(t^2) = \mu^2 \sin^2 t \sin^2 \sqrt{t^2 - \mu} / (4t^2(t^2 - \mu))$ quoted in Section 5 and its logarithmic derivative

$$\frac{d}{dt} \log I(t^2) = 2 \cot t - \frac{2}{t} + \frac{2t \cot \sqrt{t^2 - \mu}}{\sqrt{t^2 - \mu}} - \frac{2t}{t^2 - \mu}$$

([DFG21, Section 4.5], where the last term is printed as $2t/\sqrt{t^2 - \lambda}$; boundedness holds for either form). Since $\sqrt{t^2 - \mu} = t - \frac{\mu}{2t} + O(t^{-3})$, we have $\operatorname{dist}(\sqrt{t^2 - \mu}, \pi\mathbb{Z}) \geq d/2$ when $\operatorname{dist}(t, \pi\mathbb{Z}) \geq d$ and t is large; so $I(t^2) > 0$ and both cotangents are bounded there. Statement (P) for Γ is the construction used in the proof of [DFG21, Theorem 3.4] (Section 4), and $(0, \mu\chi_{[0,1]})$ is aperiodic by Lemma 2.1. By Proposition 4.3, every $\mu \neq 0$ satisfies the conclusion of Proposition 4.3 with the same m, α, β , and Lemma E gives the claim. $\square$

Proof of Theorem B. In DFG's letters the pair is $(g_a, g_b) = (\lambda f_1, \lambda f_0)$, and $g_b - g_a = \lambda h$. Let $\lambda \neq 0$. Then λh is absolutely continuous with derivative of bounded variation, and $\lambda(h(0) + h(1)) \neq 0$; in particular h is continuous and not identically zero, so $g_a \neq g_b$. (H1) holds by Lemma 5.3, (H2) and (H3) by Corollary 5.4 with $d = \min(\alpha, \beta)/4$, and (P) is assumed. So Proposition 4.3 applies to $(\lambda_j f_1, \lambda_j f_0)$ whenever $\lambda_j \neq 0$. If $\lambda_1 \lambda_2 \neq 0$, Lemma E and (4) give the half-line. If $\lambda_1 = 0$, then $\Sigma(0, 0) = [0, \infty)$ and the sum is $[\min \Sigma(\lambda_2 f_1, \lambda_2 f_0), \infty)$; likewise if $\lambda_2 = 0$. $\square$

Examples of pairs (f_1, f_0) in Theorem B: $(0, 1 + x)$, $(\cos 2\pi x, x^2)$, every pair of C^2 functions with (3), and pairs with common jumps such as $(\chi_{[0,1/2)}, \chi_{[0,1/2)} + x)$.

7 What remains open, and numerical illustrations

The open part. The question stays open for pairs with $h(0) + h(1) = 0$, for pairs of bounded variation whose difference h is not absolutely continuous (for example $f_1 = 0$, $f_0 = \chi_{[0,1/2)}$, which satisfies (3)) or has a derivative that is not of bounded variation, and for bumps that are not of bounded variation. If V_ω is continuous, then, since the words aa , ab and ba occur in ω , $f_1(0+) = f_1(1-) = f_0(0+) = f_0(1-)$, so $h(0) = h(1) = 0$.

Why the method stops: if $h(0) + h(1) = 0$, the leading term of Theorem C is $L(t) = h(1)^2 \sin^2(2t)/(16t^4)$. If $h(1) \neq 0$ it vanishes at the points of $\frac{\pi}{2} + \pi\mathbb{Z}$, which lie inside every window because $\alpha, \beta < \pi/4$; if $h(0) = h(1) = 0$ it vanishes identically and Theorem C only gives $I(t^2) = O(t^{-5})$. In both cases Corollary 5.4 fails, and hypotheses (2)–(3) of Proposition 4.1 cannot be obtained from the leading term; for some such pairs, among them $(0, \sin \pi x)$, hypothesis (2) itself fails on the windows (see (ii) below). This is a limitation of the method, not evidence against a half-line. If h has an interior jump, F_h acquires further terms of order $1/t$ and the leading term of Theorem C is no longer correct; we do not treat this case.

Numerical illustrations (tested, not used in any proof). (i) Computations of I from its definition by traces in high-precision arithmetic (Taylor recursions with about 78 correct digits; for piecewise linear bumps, Airy and trigonometric matrices at 40 digits) give bounded values of $|I - L|t^5$ and $|I' - L'|t^5$ for ten pairs with $h(0) + h(1) \neq 0$, couplings between -2 and 40 and $25 \leq t \leq 286$, and bounded $|I - L|t^5$ for four pairs on both families of windows up to $t \approx 1011$, among them $(\chi_{[0,1/2)}, \chi_{[0,1/2)} + x)$. A separately written Taylor solver at 60 digits gives the same for six pairs up to $t \approx 410$, among them $(0, 1 - 3x)$, with $h(0)h(1) < 0$, and $(0, 1 + |x - \frac{1}{2}|)$; there $t^4 I(t^2)$ exceeds the bound of Corollary 5.4 by a factor of about 2. Wrong leading terms (factor $1/8$, or $+h(0)$ for $-h(0)$) give values growing like t , and so does the leading term of Theorem C for $f_1 = 0$, $f_0 = 1 + \chi_{[1/2,1)}$, where h has an interior jump. (ii) For $h(0) + h(1) = 0$ the bound $|I - L|t^5 = O(1)$ persists, but I has deep minima or zeros. For $(f_1, f_0) = (0, 1 - 2x)$ the first-order part of ζ_b is proportional to $e^{iu}(\sin u/t + \cos u/t^2)$, $u = t - t_*$, $t_* \in \frac{\pi}{2} + \pi\mathbb{Z}$, which vanishes near $u = -1/t$; there I is about $t^{-8}/16$, and on the window $[40\pi + 0.15, 41\pi - 0.15]$ the supremum of $|\frac{d}{dt} \log I|$ is about $3.2 \cdot 10^4$, against 13 to 27 for three pairs with $h(0) + h(1) \neq 0$; an equally spaced grid of step 0.02 misses this minimum, whose width is of order t^{-2} . If $f_1 = 0$ and f_0 is symmetric about $\frac{1}{2}$, the transfer matrix of f_0 has equal diagonal entries, so ζ_b is purely imaginary and $I(t^2) = \sin^2 t (\text{Im } \zeta_b)^2$. For $f_0 = \sin \pi x$ and $f_0 = x(1 - x)$ the computed $\text{Im } \zeta_b$ changes sign near t_* in every window examined ($33 \leq t \leq 256$), so I vanishes there: hypothesis (2) of Proposition 4.1 fails as well, and $\frac{d}{dt} \log I$ is unbounded. (iii) In double precision the trace definition of I suffers from cancellation: with the solver of the first computations its absolute error reaches about $4 \cdot 10^{-13}$ for $127 \leq t \leq 256$, more than I itself near the zeros of L when $h(0) + h(1) = 0$; formula (7), used by the programs of the first version, avoids this. (iv) For periodic approximants (Fibonacci words of lengths 144 to 377) the sums of the spectra for two different couplings, of either sign and also for pairs with $h(0) + h(1) = 0$, have no gaps from a low energy up to the top of the computed range in all seven cases tried, while each one-dimensional approximant has gaps up to the top. Such computations cannot distinguish a half-line from gaps at arbitrarily high energies.

8 Computations and reproducibility

No theorem, proposition or lemma of this note depends on a computation. The program `check_note.py` of the reproducibility package (standard library, a few seconds) checks the exact parts: the polynomial identity behind Lemma 5.1; the coordinates of J , n , $R(\varphi)$, (6), $N(\sigma)$ and $\zeta(N(s_1)N(s_2))$ in rational arithmetic at rational points of the unit circle, and (9) by quadrature; the algebra of the proofs of Theorem C and Corollary 5.4; the leading term against [DFG21, (3.11)] and the logarithmic derivative of (3.11); the constants of Lemma E; and, as an illustration, $(I - L)t^5$ for four pairs up to $t \approx 60$, with I computed both by traces and by (7). The package also holds the programs, with recorded outputs, of the first version (`original/`), of the two verification runs (`verification_run_A/`, `verification_run_B/`) and of the further run described below (`independent_run_2/`), and a `README.md` with command lines and running times.

Verification. The results were first obtained, with proofs, together with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each read the source and the closest papers and wrote its own programs. The first run re-derived Lemma A, Proposition 5.2, Theorem C, Lemma 5.3 and Corollary 5.4, Lemma E and the details of Section 4, compared the quotations with the sources, read the proof of [DFG21, Proposition 4.2] and found no use of the curves beyond its hypotheses (1)–(4), and computed I with exact Airy and trigonometric matrices and with an ODE solver. The second run computed I from its trace definition in high-precision arithmetic with wrong leading terms as controls, checked Lemma 5.1 exactly on 200 random pairs in $SL(2, \mathbb{Q})$, computed sums of periodic approximants, re-ran the programs of the first version (identical outputs), checked the bibliography and repeated the literature search. Neither run found an error in a proof. The corrections they required were made: the exact quotation of [DFG21, Remark 4.25] and why unit cells need no modification; aperiodicity (Lemma 2.1); the dependence on [DFG21], made visible in the statements and the abstract; pairs with non-absolutely continuous difference in the open part; a constant in Lemma 5.3; the description of the numerical values, with lower bounds in place of two coarse-grid values; and unequal cell lengths mentioned only with [DFG21, Question 1]. Neither run re-proved the hyperbolic estimates of [DFG21, Proposition 4.2], the construction of the pieces for curves other than DFG’s, or the smallness of α, β .

A further independent verification run, also AI-assisted, then examined the text of this note, with programs of its own (folder `independent_run_2/` of the package). It re-derived all proofs line by line, among them Lemma 2.1, Lemma A, Lemma E, Proposition 5.2, Theorem C, Lemma 5.3 and Corollary 5.4; compared every quotation with the source and with both versions of [DFG21]; checked Lemma 5.1, the coordinates of Section 5, (9) and [DFG21, (3.11)] symbolically; computed I from its trace definition at 60 digits with a Taylor solver of its own; re-ran 24 programs of the package from the archive, which reproduced their recorded outputs; and repeated the literature search. It found no error in a proof and no earlier answer to the question. It found that the invariant vanishes inside the windows for the symmetric bumps of Section 7 (ii), and it required eleven corrections, which were made: the comparison with the journal version of [DFG21]; the misprint in DFG’s logarithmic derivative; the description of the numerical values in Section 7; the complete list of open cases; the remark on irreducibility after Proposition 2.2; the descriptions of `check_note.py` and of the package; the record of this run; and a weaker priority statement for Proposition 5.2. This run did not re-prove the hyperbolic estimates of [DFG21, Proposition 4.2] either.

Scope and priority. The note answers Fillman’s question [Fil17] for constant bumps (Theorem A) and, conditionally on (P), for the bumps of Theorem B; the general question stays open. Theorem A for $\lambda_1 = \lambda_2$ is [DFG21, Theorem 3.4]; Proposition 4.1, the construction of the pieces, [DFG21, (3.11)]

with its logarithmic derivative and Lemma 3.1 are from [DFG21] (with [DG11, New70, New79]); Proposition 2.2 is [DFG14, Proposition 6.3]; Lemma 3.2(b) is [FT23, Lemma 3.2]. Proposition 5.2 is a quantitative form of standard high-energy asymptotics of the fundamental solutions, obtained from the first two terms of the Dyson series; we do not claim it as new. To our knowledge Lemma A, Lemma E with the extension to two couplings, Theorem C, Corollary 5.4 and the conditional Theorem B are not in print. We read [Fil17] in the open PDF; [DFG21] in the arXiv version v1 of 12 January 2020, whose page numbers we use, and in the journal version, which we compared with v1 in the parts used here (Lemmas 2.1 and 2.3, Section 3, the construction preceding Proposition 4.2, Proposition 4.2, Section 4.5, Remark 4.25 and Question 1): the texts agree, and so does the numbering; Section 6 of [DFG14], Section 3 and Theorem 1.6 of [FT23], the proof of [FM18, Theorem 1.3] and [DF20, Theorem 8.4] in their arXiv versions; and the abstract of [DEF⁺26]. The list of open problems in [DF25, pp. 1035–1037] was not accessible to us; [New70, New79, DG11, PT87] are used as quoted in [DFG21]. Searches on 18 October 2026, repeated in part by the last verification run: arXiv (authors of [DFG21, FM18, FT23] and related authors; Fibonacci with half-line, Bethe–Sommerfeld, separable, sums of Cantor sets, Fricke–Vogt), Crossref, zbMATH, the works citing [DFG21] in OpenAlex (4) and Semantic Scholar (5) and citing [DFG14] in Semantic Scholar (25), and five web searches. No printed answer to the question was found in the sources searched, which include [DFG21, DFG14, FM18, FT23, DF20, DEF⁺26] and these citing works; a search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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