

Smooth 34-Dimensional IDP Polytopes with Nonunimodal h^* -Polynomials

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Abstract

Ferroni constructed smooth 40-dimensional lattice polytopes with the integer decomposition property whose h^* -polynomials are not unimodal. We refine the parameters in his Cayley-box construction to obtain infinitely many pairwise lattice-inequivalent examples in dimension 34. For every integer $t \geq 1$, the matrix $[100000t \mathbf{1}_{17} \mid 1310000t I_{17}] + J_{17 \times 18}$ defines such a polytope. Its h^* -polynomial has degree 17 and a strict trough $h_{15}^* > h_{16}^* < h_{17}^*$. We prove the inequalities uniformly in t by an exact positive-coefficient certificate and supply a standalone verifier using integer and rational arithmetic. The corresponding Ehrhart rings are standard graded normal Cohen–Macaulay domains of dimension 35. This is a dimension refinement of Ferroni’s counterexamples, not a first disproof of unimodality; it does not resolve unimodality for general standard graded Gorenstein domains.

1 Statement and relation to prior work

For a full-dimensional lattice polytope $P \subset \mathbb{R}^d$, write

$$\sum_{k \geq 0} |kP \cap \mathbb{Z}^d| z^k = \frac{h_P^*(z)}{(1-z)^{d+1}}, \quad h_P^*(z) = \sum_j h_j^*(P) z^j.$$

A coefficient sequence is unimodal if it first weakly increases and then weakly decreases. The integer decomposition property (IDP) means that every lattice point of kP is a sum of k lattice points of P , for every integer $k \geq 1$. A lattice polytope is smooth if it is simple and its primitive edge directions at each vertex form a lattice basis.

Ferroni [1] disproved unimodality for IDP polytopes and gave smooth examples in dimension 40. We use his Cayley sums of coordinate boxes and his positive-matrix modification. The contribution here is an explicit choice in dimension 34, together with a uniform certificate for an infinite family in that fixed dimension. We do not claim that 34 is minimal.

Put

$$A = 100000, \quad B = 1310000, \quad C = A + B = 1410000.$$

Let $\Delta_{17} = \text{conv}(0, e_1, \dots, e_{17})$, with barycentric coordinates $\lambda_0(y) = 1 - \sum_i y_i$ and $\lambda_i(y) = y_i$. For $t \in \mathbb{Z}_{\geq 1}$, define

$$P_t = \left\{ (x, y) \in \mathbb{R}^{17} \times \Delta_{17} : 0 \leq x_i \leq At\lambda_0(y) + Bt\lambda_i(y) + 1 \quad (1 \leq i \leq 17) \right\}. \quad (1)$$

Theorem 1. *For every integer $t \geq 1$, P_t is a smooth, 34-dimensional IDP lattice polytope. Its h^* -polynomial has degree 17 and satisfies*

$$h_{15}^*(P_t) > h_{16}^*(P_t) < h_{17}^*(P_t).$$

The polytopes P_t , $t = 1, 2, \dots$, are pairwise lattice-inequivalent.

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The source discussion [2, pp. 2695–2696], represented by OWR-15587-015 in *UnsolvedMath* [3], includes several different questions about unimodality and log-concavity. The Cohen–Macaulay unimodality variant is already refuted by Ferroni. Our examples refine that negative result. They are not Gorenstein, so they do not answer the general Gorenstein-domain question.

2 Geometry and integer decomposition

The geometry below is a specialization of [1, Propositions 2.1–2.3]; we include direct proofs of the properties used here. Let $v_0 = 0$, $v_j = e_j$, and write

$$M_t = [At \mathbf{1}_{17} \mid Bt I_{17}] + J_{17 \times 18}, \quad D_j = \prod_{i=1}^{17} [0, (M_t)_{ij}].$$

Here J is the all-ones matrix. The fiber of $\text{conv}(\bigcup_{j=0}^{17} D_j \times \{v_j\})$ over y is $\sum_j \lambda_j(y) D_j$. Since these are coordinate boxes, the fiber is exactly the box in (1). Thus this Cayley sum is P_t .

Lemma 2. *The polytope P_t has dimension 34, is a lattice polytope, is smooth, and has IDP.*

Proof. All entries of M_t are positive integers. The Cayley description shows that the vertices lie in the lattice: they are the corners of the boxes D_j in the layers $y = v_j$. The fibers have dimension 17, and the base has dimension 17, giving dimension 34.

At a vertex over v_j , 17 base inequalities and one bound on each of the 17 fiber coordinates are tight. Their normals are independent. The two bounds on a fiber coordinate never coincide, so the vertex is simple. The edges in its layer have primitive directions $(\pm e_i, 0)$. For each $\ell \neq j$, keep the selected lower or upper fiber bound tight and move the base from v_j to v_ℓ . The resulting edge vector has integral fiber coordinates and base component $v_\ell - v_j$. This base component is primitive. The 17 vectors $v_\ell - v_j$ form a basis of $\mathbb{Z}^{17}$, so these vectors together with the fiber directions form a block triangular unimodular matrix. This proves smoothness.

For IDP, take an integer point $(x, y) \in kP_t$. Its base coordinates determine nonnegative integers

$$b_i = y_i \ (i \geq 1), \quad b_0 = k - \sum_{i=1}^{17} y_i, \quad \sum_{j=0}^{17} b_j = k.$$

Choose a list of k base vertices containing b_j copies of v_j . In coordinate i , the corresponding integer capacities are the k values $(M_t)_{ij}$, with repetitions. Their sum is the upper bound on x_i in kP_t . Every integer between zero and the sum of nonnegative integer capacities can be allocated among those capacities: successively use the smaller of the remaining integer and the next capacity. Do this independently for each i . The resulting k integer fiber points, paired with their chosen base vertices, belong to P_t and sum to (x, y) . $\square$

3 Three interior counts

Let $I_k(t) = |\text{int}(kP_t) \cap \mathbb{Z}^{34}|$. Strict base inequalities require all 18 barycentric integers b_j to be positive. The strict fiber bounds then give

$$I_k(t) = \sum_{\substack{b_0 + \dots + b_{17} = k \\ b_j \geq 1}} \prod_{i=1}^{17} (Atb_0 + Btb_i + k - 1). \quad (2)$$

In particular $I_k(t) = 0$ for $1 \leq k < 18$, and $I_{18}(t) > 0$. The codegree is therefore 18, and Ehrhart reciprocity gives $\deg h_{P_t}^* = 35 - 18 = 17$.

For the three relevant values of k , (2) becomes

$$I_{18}(t) = (Ct + 17)^{17}, \quad (3)$$

$$I_{19}(t) = ((C + A)t + 18)^{17} + 17((C + B)t + 18)(Ct + 18)^{16}, \quad (4)$$

$$\begin{aligned} I_{20}(t) = & ((C + 2A)t + 19)^{17} \\ & + 17((C + A + B)t + 19)((C + A)t + 19)^{16} \\ & + 17((C + 2B)t + 19)(Ct + 19)^{16} \\ & + 136((C + B)t + 19)^2(Ct + 19)^{15}. \end{aligned} \quad (5)$$

To obtain these formulas, start with all $b_j = 1$. For $k = 19$ the extra unit goes either to b_0 or to one of the other 17 coordinates. For $k = 20$, the two extra units go to b_0 ; to b_0 and one other coordinate; to one other coordinate twice; or to two distinct other coordinates. These four cases account for the multiplicities 1, 17, 17, $\binom{17}{2} = 136$.

Ehrhart reciprocity in generating-function form reads

$$\sum_{k \geq 1} I_k(t) z^k = \frac{z^{35} h_{P_t}^*(z^{-1})}{(1 - z)^{35}}.$$

Comparing the coefficients in degrees 18, 19 and 20 yields

$$\begin{aligned} h_{17}^*(P_t) &= I_{18}(t), \\ h_{16}^*(P_t) &= I_{19}(t) - 35I_{18}(t), \\ h_{15}^*(P_t) &= I_{20}(t) - 35I_{19}(t) + 595I_{18}(t). \end{aligned}$$

Consequently it suffices to prove

$$\begin{aligned} L(t) &:= I_{20}(t) - 36I_{19}(t) + 630I_{18}(t) > 0, \\ R(t) &:= 36I_{18}(t) - I_{19}(t) > 0. \end{aligned} \quad (6)$$

4 A uniform exact certificate

Set $u = t - 1$. Write

$$L(1 + u) = \sum_{j=0}^{17} \ell_j u^j, \quad R(1 + u) = \sum_{j=0}^{17} r_j u^j, \quad D_j = \binom{17}{j} C^j (C + 17)^{17-j}.$$

Thus $D_j > 0$ is the coefficient of u^j in $I_{18}(1 + u)$. The following table gives exact integer floors, rather than floating-point approximations.

j	$\lfloor 10^6 \ell_j / D_j \rfloor$	$\lfloor 10^6 r_j / D_j \rfloor$	j	$\lfloor 10^6 \ell_j / D_j \rfloor$	$\lfloor 10^6 r_j / D_j \rfloor$
0	1951	80	9	1979	179
1	1954	91	10	1983	189
2	1957	102	11	1986	200
3	1960	113	12	1989	211
4	1964	124	13	1992	222
5	1967	135	14	1995	233
6	1970	146	15	1998	244
7	1973	157	16	2002	255
8	1976	168	17	2005	266

Here is a precise finite recipe for reproducing every entry. For integers s, d and $m \geq 0$, form the coefficient vector

$$\mathcal{B}(s, d, m)_j = \binom{m}{j} s^j (s + d)^{m-j} \quad (0 \leq j \leq m).$$

This is the vector of $(s(1 + u) + d)^m$. Substitute these vectors into (3)–(5), using ordinary convolution for products. Form the two integer linear combinations in (6), multiply each coefficient by 10^6 , and use integer division by D_j . This gives the table.

All 36 entries are positive, so every ℓ_j and r_j is strictly positive. It follows that $L(t) > 0$ and $R(t) > 0$ for every real $t \geq 1$, and in particular for every integer $t \geq 1$. This proves the strict trough in Theorem 1. For example, the first row provides the exact bounds

$$L(1) \geq \frac{1951}{10^6} I_{18}(1) > 0, \quad R(1) \geq \frac{80}{10^6} I_{18}(1) > 0.$$

The supplied standard-library Python verifier expands these polynomials with integer arithmetic and checks the table. Independently, it enumerates the positive compositions in (2) at $t = 1, \dots, 18$ and reconstructs each degree-at-most-17 polynomial by exact rational Newton interpolation. The resulting coefficient arrays agree. For $t = 1, 2$, it also computes the entire h^* -vector from forward lattice-point counts

$$E_t(k) = \sum_{\substack{b_0 + \dots + b_{17} = k \\ b_j \geq 0}} \prod_{i=1}^{17} (1 + Atb_0 + Btb_i + k),$$

and checks that the three terminal coefficients match the interior calculation. The quantifier over all t follows from coefficient positivity, not from testing finitely many polytopes.

To finish Theorem 1, observe that $P_t \subset P_{t+1}$. For every base point in the interior of Δ_{17} , each fiber length strictly increases with t . Integrating the positive difference in fiber volumes shows $\text{vol}_{34}(P_t) < \text{vol}_{34}(P_{t+1})$. Lattice equivalence preserves volume, which proves pairwise inequivalence.

5 The algebraic consequence and its scope

Corollary 3. *Over every field, there are standard graded normal Cohen–Macaulay domains of Krull dimension 35 whose Hilbert numerators have a strict trough in degrees 15, 16, 17. The domains furnished here are not Gorenstein.*

Proof. Use the Ehrhart semigroup ring of P_t . Its normality follows from its saturated cone semigroup, and Hochster’s theorem makes it Cohen–Macaulay; these standard Ehrhart-ring facts are recalled in [2, p. 2696]. IDP says that its grading is generated in degree one. Its Krull dimension is $34 + 1$, and its Hilbert numerator is $h_{P_t}^*$. The latter is not symmetric, because

$$h_0^*(P_t) = 1, \quad h_{17}^*(P_t) = (Ct + 17)^{17} > 1.$$

A Gorenstein Ehrhart ring has symmetric h^* -polynomial, so these rings are not Gorenstein. $\square$

The 2017 discussion and the frozen dataset record should therefore not be read as a single conjecture settled by this note. Ferroni’s work already supplies the disproof in the Cohen–Macaulay setting; our result reduces the dimension of his displayed smooth construction from 40 to 34. The unrestricted standard graded Gorenstein-domain unimodality question is outside this result. Neither a minimal-dimension theorem nor an exhaustive priority claim is made.

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References

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