

Lattice Hulls of Euclidean Balls Need Not Be Normal: Counterexamples in Every Dimension at Least Six

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Abstract

Michalek asked in 2017 (Oberwolfach Report 44/2017, Conjecture 1) whether the lattice hull of a Euclidean ball $B \subset \mathbb{R}^n$, the convex hull of the lattice points in B , is always a normal polytope, and whether there is a very ample, non-normal simple polytope, or even simplex. We answer the first question in the negative in every dimension $n \geq 6$. The convex hull P of the points $z \in \mathbb{Z}^6$ with $|z|^2 - z_3 - z_5 \leq 158906$, the lattice hull of the ball with centre $(0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$ and squared radius $\frac{317813}{2}$, is a spanning lattice polytope that is neither normal nor very ample. For $\mathbf{n} = (-36, -46, 50, -57, -41, 1)$, the Cauchy–Schwarz inequality gives $\mathbf{n} \cdot z \geq -41504$ on the lattice points of P ; exactly six of them lie on the hyperplane $\mathbf{n} \cdot z = -41504$ (a computer-assisted finite enumeration, done by three independent methods); they span an empty simplex facet whose vertices generate a sublattice of index 2; and half their sum is a lattice point of $3P$ that is not a sum of three lattice points of P . Padding with zeros gives examples in every dimension $n \geq 6$; further certificates are given, one in dimension 7. For ellipsoids, Gubeladze (2021) gave non-normal examples in every dimension $d \geq 6$; for $n \leq 2$ the answer is positive by Pick’s theorem, and for $n = 3$ by a theorem of Bruns, Gubeladze and Michalek (2016). No answer for round balls was found in the literature accessible to us. Dimensions 4 and 5, and balls centred at lattice points, remain open; the face mechanism used here cannot work in dimensions at most 5. The second question was answered by an example of Cox, Haase, Hibi and Higashitani (2014), the very ample, non-normal simplex $3 \operatorname{conv}(0, e_1, \dots, e_6, (3, 3, 3, 3, 3, 4))$, as stated explicitly by Adeyemo, Bunnett and Levicán-Santibáñez (2025), who thank W. Bruns for informing them of it; we claim nothing new for it. This is an unrefereed note.

1 Introduction

1.1 The question

For a set $X \subseteq \mathbb{R}^d$ we write $L(X) = X \cap \mathbb{Z}^d$. A lattice polytope $P \subset \mathbb{R}^d$ has the *integer decomposition property* (IDP) if for every integer $k \geq 1$ every point of $L(kP)$ is a sum of k points of $L(P)$. In his contribution to the Oberwolfach mini-workshop “Lattice Polytopes: Methods, Advances, Applications” (2017), Michalek considers *spanning* lattice polytopes, whose lattice points generate the lattice affinely; he calls such a polytope *normal* if it has IDP, and *very ample* if the same condition is required only for all sufficiently large k [Mic17, p. 2684]. His Conjecture 1 [Mic17, p. 2684] consists of two questions.

- (1) “Let B be a ball in $\mathbb{R}^n$. Is $\operatorname{conv}(B \cap \mathbb{Z}^n)$ a normal polytope?” (With a reference to [BGM16, Remark 6.6].)

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$F_0 = (136, 177, -192, 218, 157, -3),$	$F_1 = (137, 175, -190, 220, 158, -4),$
$F_2 = (137, 176, -191, 219, 157, -6),$	$F_3 = (138, 176, -190, 220, 156, -4),$
$F_4 = (139, 175, -192, 217, 158, -3),$	$F_5 = (139, 177, -189, 218, 158, -4).$
$F_0 + \cdots + F_5 = (826, 1056, -1144, 1312, 944, -24) = 2w$	

Table 1: The lattice points of P on the hyperplane $\mathbf{n} \cdot z = -41504$, $\mathbf{n} = (-36, -46, 50, -57, -41, 1)$. For every i , $\mathbf{n} \cdot F_i = -41504$ and $f(F_i) = 158906$, so all six points lie on the boundary sphere of B .

(2) Is there a very ample, non-normal simple polytope, or even a very ample, non-normal simplex?

He adds that a positive answer to (2) would be a step towards a counterexample to Oda’s conjecture that smooth polytopes are normal. Question (1) had been posed by Michałek before, in his abstract “Toric structures in nontoric varieties” in the report of the Oberwolfach workshop “Toric Geometry” (Oberwolfach Report 19/2016), as the first part of a conjecture presented during his lecture: “Is the convex hull of all lattice points in any ball a normal polytope?” [Mic16, p. 1000, Conjecture 5(1)], with a reference to [BGM16, Question 7.2(b)], which asks the same for ellipsoids; the abstract adds that this question was still unanswered [Mic16, p. 1001]. Bruns had raised question (2) before [Bru13, Section 8]: he knew no simple very ample polytope that is not normal, and had not found a very ample non-normal simplex. The two questions form record OWR-15587-006 of the UnsolvedMath collection [ula26], where the problem is listed as open; the 2016 version of question (1) is part of record OWR-14601-005 of the same collection.

We call $\text{conv}(L(B))$ the *lattice hull* of B . Balls are closed Euclidean balls $B(c, R) = \{y \in \mathbb{R}^n : |y - c| \leq R\}$ with arbitrary centre and radius, the literal reading of (1); this agrees with “any ball” in [Mic16, p. 1000], with the balls $B(z, r)$ of [BGM16, Section 6], and with [BGM16, Theorem 6.5] and [Gub21], which treat ellipsoids with arbitrary centres. Open balls give the same lattice hulls (Remark 2.3). Balls centred at lattice points, a natural special case, are not reached here (Section 9).

1.2 Results

Theorem A. *Let $f(z) = z_1^2 + z_2^2 + \cdots + z_6^2 - z_3 - z_5$ and*

$$P = \text{conv}\{z \in \mathbb{Z}^6 : f(z) \leq 158906\},$$

the lattice hull of the closed ball $B \subset \mathbb{R}^6$ with centre $c = (0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$ and squared radius $R^2 = \frac{317813}{2}$ ($R \approx 398.6$). Then P is a full-dimensional spanning lattice polytope which is neither normal, in any of the conventions of Section 2, nor very ample. More precisely, let $\mathbf{n} = (-36, -46, 50, -57, -41, 1)$.

- (a) *Every lattice point z of P satisfies $\mathbf{n} \cdot z \geq -41504$.*
- (b) *Exactly six lattice points of P lie on the hyperplane $\mathbf{n} \cdot z = -41504$, the points $F_0, \dots, F_5$ of Table 1. They are affinely independent; the simplex $\text{conv}(F_0, \dots, F_5)$ is a facet of P whose only lattice points are its vertices, and its vertices affinely generate a sublattice of index 2 of the integer points of that hyperplane.*
- (c) *The point $w = \frac{1}{2}(F_0 + \cdots + F_5) = (413, 528, -572, 656, 472, -12)$ is a lattice point of $3P$ but not a sum of three lattice points of P . More generally, for every j and every integer $t \geq 0$ the point $w + tF_j$ lies in $(3+t)P$ and is not a sum of $3+t$ lattice points of P .*

Part (a) follows from the Cauchy–Schwarz inequality. Part (b), the only computer-assisted step, is the enumeration of the integer points of a five-dimensional ball of squared radius $\frac{270429}{43372} \approx 6.24$, made by three independent methods (Section 4); the rest follows from a general lemma (Lemma 3.2).

Corollary B. *For every $n \geq 6$ the polytope*

$$P^{(n)} = \text{conv}\{z \in \mathbb{Z}^n : z_1^2 + \dots + z_n^2 - z_3 - z_5 \leq 158906\},$$

the lattice hull of the closed ball in $\mathbb{R}^n$ with centre $(0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \dots, 0)$ and squared radius $\frac{317813}{2}$, is a full-dimensional spanning lattice polytope which is neither normal nor very ample. In particular, the answer to question (1) is negative in every dimension $n \geq 6$.

In Section 6, a non-unimodular simplex with an empty circumscribed sphere in a section $\mathbf{n}^\perp \cap \mathbb{Z}^d$ becomes a facet of the lattice hull of a large ball touching the hyperplane; three further certificates follow (Theorem 6.4), one in dimension 7. The mechanism needs a face of dimension at least five:

Proposition C. *Let $E \subset \mathbb{R}^d$ be an ellipsoid and $P = \text{conv}(L(E))$. If a face F of P is a simplex of dimension at most 4 whose only lattice points are its vertices, then F is unimodular, that is, its vertices affinely generate $\text{aff}(F) \cap \mathbb{Z}^d$.*

Consequently, in dimensions $n \leq 5$ no spanning lattice hull of a ball has a face of the kind used in Theorem A, and a counterexample in dimension 4 or 5, if there is one, needs another source of non-normality (Section 7).

The second question has a positive answer, which was in print before the question was posed:

Proposition D (Cox, Haase, Hibi, Higashitani [CHHH14, Example 2.3, Theorem 1.1]). *Let $m \geq 4$, $d = 2m - 1$ and $P_m = \text{conv}(0, e_1, \dots, e_{d-1}, (m-1, \dots, m-1, m)) \subset \mathbb{R}^d$. The lattice simplex $Q_m = 3P_m$ is spanning and very ample, and it is not normal: the point $(3, \dots, 3) \in L(2Q_m)$ is not a sum of two lattice points of Q_m . For $m = 4$ this is the simplex $Q_4 = \text{conv}(0, 3e_1, \dots, 3e_6, (9, \dots, 9, 12)) \subset \mathbb{R}^7$.*

A simplex is a simple polytope, so Proposition D answers both parts of question (2). Cox, Haase, Hibi and Higashitani show that $3P_m$ is not IDP; very ampleness follows from their Theorem 1.1 and a remark on p. 2 of their paper (Section 8). They do not formulate this as an answer to the question. Adeyemo, Bunnett and Levicán-Santibáñez state explicitly, citing [CHHH14, Example 2.3], that there is a 7-dimensional very ample non-normal simplex, and they thank W. Bruns for informing them about such an example [ABLS25, Section 2.3 and Acknowledgements]. The source of the question itself cites [CHHH14] (as its reference [5]) in another context [Mic17, p. 2684]. We give a short proof in Section 8 for the reader's convenience and claim nothing new.

1.3 What was known, what is new, and what remains open

For $n \leq 2$ the answer to question (1) is positive: lattice segments are normal, and lattice polygons are normal by Pick's theorem, as recalled in [BGM16, Section 3.1]. For $n = 3$ it is positive by [BGM16, Theorem 6.5(c)]: the lattice hull of every ellipsoid in $\mathbb{R}^3$ is normal; Gubeladze showed that it is even covered by unimodular simplices [Gub21, Theorem (b)]. In every dimension, every lattice point of $2P$ is a sum of two lattice points of P when P is the lattice hull of an ellipsoid [BGM16, Theorem 6.5(b)]; the failures in Theorem A occur in degree 3 and higher. Bruns, Gubeladze and Michałek tested several dozen lattice hulls of axis-parallel ellipsoids in dimensions 4 and 5 and found them all normal [BGM16, Remark 6.6(a)].

For ellipsoids, Gubeladze answered [BGM16, Question 7.2(b)]: for every $d \geq 6$ there are ellipsoids $E \subset \mathbb{R}^d$ whose lattice points affinely generate $\mathbb{Z}^d$ and whose lattice hulls are not normal, and for $d = 5$ there are such ellipsoids if the generation condition is dropped [Gub21, Theorem (c) and Remark 7]. His construction starts from the lattice $\Lambda(d) = \mathbb{Z}^d + \mathbb{Z}(\frac{1}{2}, \dots, \frac{1}{2})$ and the ball circumscribing $[0, 1]^d$, and produces an ellipsoidal set of points of $\Lambda(d)$ whose convex hull has a non-unimodular empty simplex as a facet [Gub21, Lemmas 4–6]; a linear map taking $\Lambda(d)$ to $\mathbb{Z}^d$ gives ellipsoids for the standard

lattice, which are not round balls ($\Lambda(d)$ has minimal norm 1 and determinant $\frac{1}{2}$, so it is not similar to $\mathbb{Z}^d$). Gubeladze’s survey [Gub23] does not treat lattice hulls of balls or ellipsoids.

The mechanism of Theorem A is the one of Gubeladze’s theorem: a non-unimodular empty simplex is a face of the lattice hull of a quadric, and a lattice point in a dilate of this face with non-integral barycentric coordinates cannot be decomposed (Lemma 3.2). What is new here, to our knowledge and relative to the literature we found, is that round balls for the standard lattice suffice: hyperplane sections $\mathbf{n}^\perp \cap \mathbb{Z}^6$ contain non-unimodular simplices with empty circumscribed spheres, and a ball touching the hyperplane from one side turns such a simplex into a facet (Lemma 6.1). Empty-sphere 5-simplices of multiplicity 2 in 5-dimensional lattices go back to a construction of Voronoi, as recalled in [Gub21, Remark 7]; what is used here is that such simplices occur in hyperplane sections of $\mathbb{Z}^6$. So Theorem A and Corollary B give counterexamples for round balls in every dimension $n \geq 6$ (and, as balls are ellipsoids, examples for Gubeladze’s theorem). Proposition C is a short consequence of [BGM16, Theorem 6.5(b)]; we do not know whether it has been stated before. No answer to question (1) for round balls was found in the literature accessible to us; a search that finds nothing is not a proof of novelty (“Scope and priority”, Section 10).

Open (Section 9): dimensions 4 and 5; balls centred at lattice points, $\text{conv}\{z \in \mathbb{Z}^n : |z|^2 \leq N\}$; the smallest suitable $\mathbf{n}$.

2 Definitions and conventions

Throughout, $P \subset \mathbb{R}^d$ is a lattice polytope, that is, a polytope with vertices in $\mathbb{Z}^d$; then $P = \text{conv}(\text{L}(P))$. For a closed ball B the lattice hull $P(B) = \text{conv}(\text{L}(B))$ is contained in B , since B is convex, and hence $\text{L}(P(B)) = \text{L}(B)$. We write $C(P) = \mathbb{R}_{\geq 0}(P \times \{1\}) \subset \mathbb{R}^{d+1}$ for the cone over P and $M(P)$ for the monoid generated by the points $(x, 1)$, $x \in \text{L}(P)$. The lattice points of $C(P)$ with last coordinate $k \geq 1$ are the points (x, k) with $x \in \text{L}(kP)$, and the elements of $M(P)$ with last coordinate k are the points $(x_1 + \cdots + x_k, k)$ with $x_i \in \text{L}(P)$.

Definition 2.1. Let $P \subset \mathbb{R}^d$ be a lattice polytope.

- (i) P has the *integer decomposition property* (IDP) if for every $k \geq 1$ every point of $\text{L}(kP)$ is a sum of k points of $\text{L}(P)$; equivalently, $M(P) = C(P) \cap \mathbb{Z}^{d+1}$.
- (ii) P is *spanning* if the points $(x, 1)$, $x \in \text{L}(P)$, generate $\mathbb{Z}^{d+1}$ as a group; equivalently, the lattice points of P affinely generate $\mathbb{Z}^d$.
- (iii) P is *normal in the sense of monoids* if $M(P) = C(P) \cap \text{gp}(M(P))$, where $\text{gp}(M(P))$ is the group generated by $M(P)$.
- (iv) For spanning P , Michałek calls P *normal* if it has IDP [Mic17, p. 2684].
- (v) P is *very ample* if there is k_0 such that for every $k \geq k_0$ every point of $\text{L}(kP)$ is a sum of k points of $\text{L}(P)$ [Mic17, p. 2684], [CHHH14, p. 2].
- (vi) P is *very ample at its vertices* if for every vertex v of P every lattice point of the cone $\mathbb{R}_{\geq 0}(P - v)$ is a sum of elements of $\text{L}(P) - v$, that is, $\text{Hilb}(\mathbb{R}_{\geq 0}(P - v)) \subset P - v$. This is the definition of very ampleness in [BGM16, Definition 5.2] and [Gub21, Section 1.2].

Property (i) is called “normal” in [BGM16, Definition 3.1] and [Gub21], “integrally closed” in [Gub23], and IDP in [CHHH14]; (iii) is “normal” in the book of Bruns and Gubeladze, as recalled in [BGM16, Section 3.1], [CHHH14, Remark 0.1] and [Gub21, Remark 7]. For lattice polytopes (v) and (vi) are equivalent (see the references in [CHHH14, p. 2] and after [BGM16, Definition 5.2]); we do not

use this, as our examples violate both. By the dictionary recalled in [Mic17, p. 2684], the projective toric variety given by the lattice points of a lattice polytope is normal exactly when the polytope is very ample; so our examples also give non-normal projective toric varieties.

Lemma 2.2. *If P is spanning, the properties (i), (iii) and (iv) of Definition 2.1 are equivalent.*

Proof. (iv) is (i) by definition. If P is spanning, then $\text{gp}(M(P)) = \mathbb{Z}^{d+1}$, and (iii) reads $M(P) = C(P) \cap \mathbb{Z}^{d+1}$, which is (i). $\square$

Remark 2.3 (Open balls). For an open ball $B^\circ = \{y : |y - c| < R\}$ with $L(B^\circ) \neq \emptyset$, $L(B^\circ) = L(B(c, R'))$ with $R' = \max\{|z - c| : z \in L(B^\circ)\}$; conversely, $L(B(c, R)) = L(\{y : |y - c| < R''\})$ for every R'' with $R < R'' \leq \min\{|z - c| : z \in \mathbb{Z}^d, |z - c| > R\}$. So open and closed balls have the same lattice hulls.

3 A face with an indecomposable point

Lemma 3.1. *Let $X \subset \mathbb{R}^d$ be finite, let ϕ be an affine function on $\mathbb{R}^d$, and let $h \in \mathbb{R}$ with $\phi(x) \geq h$ for all $x \in X$. Then $\text{conv}(X) \cap \{\phi = h\} = \text{conv}(X \cap \{\phi = h\})$, and this set is a face of $\text{conv}(X)$.*

Proof. If $p = \sum_{x \in X} \lambda_x x$ with $\lambda_x \geq 0$, $\sum \lambda_x = 1$, and $\phi(p) = h$, then $\sum \lambda_x (\phi(x) - h) = 0$ with all terms non-negative, so $\lambda_x = 0$ whenever $\phi(x) > h$. This proves the inclusion “ $\subseteq$ ”; the other one is clear. Since $\phi \geq h$ on $\text{conv}(X)$, the hyperplane $\{\phi = h\}$ supports $\text{conv}(X)$, and the intersection is a face. $\square$

Lemma 3.2. *Let $P \subset \mathbb{R}^d$ be a lattice polytope, $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$ and $h \in \mathbb{Z}$ with $\mathbf{n} \cdot z \geq h$ for all $z \in L(P)$. Suppose that the lattice points of P on the hyperplane $\{\mathbf{n} \cdot y = h\}$ are affinely independent points $v_0, \dots, v_k$, and that a point $x \in \mathbb{Z}^d$ can be written as $x = \sum_{i=0}^k \lambda_i v_i$ with $\lambda_i \geq 0$, $\sum_i \lambda_i = q \in \mathbb{Z}$, and $\lambda_i \notin \mathbb{Z}$ for some i . Then the following hold.*

- (a) $F = \text{conv}(v_0, \dots, v_k) = P \cap \{\mathbf{n} \cdot y = h\}$ is a face of P with $L(F) = \{v_0, \dots, v_k\}$, and every v_j is a vertex of P .
- (b) For every j and every integer $t \geq 0$, the point $x + tv_j$ lies in $L((q+t)P)$ and is not a sum of $q+t$ points of $L(P)$.
- (c) P does not have IDP, and it is neither very ample nor very ample at its vertices: for every j , the point $x - qv_j$ is a lattice point of $\mathbb{R}_{\geq 0}(P - v_j)$ which is not a sum of elements of $L(P) - v_j$.
- (d) If P is spanning, then P is not normal in any of the senses (i), (iii) and (iv) of Definition 2.1.

Proof. (a) Apply Lemma 3.1 to $X = L(P)$ and $\phi(y) = \mathbf{n} \cdot y$. The lattice points of the face F are the lattice points of P on the hyperplane, and the vertices of the simplex F are vertices of the face F of P , hence of P .

(b) Put $\lambda'_i = \lambda_i + t\delta_{ij}$. Then $\lambda'_i \geq 0$ and $\sum_i \lambda'_i = q + t$, so $y = x + tv_j = \sum_i \lambda'_i v_i$ lies in $(q+t)F \subseteq (q+t)P$, and $y \in \mathbb{Z}^d$. As $\sum_i \lambda_i$ is an integer and some λ_i is not, at least two of the λ_i are not integers; so $\lambda'_i = \lambda_i \notin \mathbb{Z}$ for some $i \neq j$. Suppose that $y = p_1 + \dots + p_{q+t}$ with $p_l \in L(P)$. Then $\mathbf{n} \cdot p_l \geq h$ for all l , and $\sum_l \mathbf{n} \cdot p_l = \mathbf{n} \cdot y = \sum_i \lambda'_i (\mathbf{n} \cdot v_i) = (q+t)h$; so $\mathbf{n} \cdot p_l = h$ for all l , that is, every p_l is one of the v_i . Hence $y = \sum_i a_i v_i$ with $a_i \in \mathbb{Z}_{\geq 0}$ and $\sum_i a_i = q + t$. Since the v_i are affinely independent, the coefficients of a representation $y = \sum_i \mu_i v_i$ with $\sum_i \mu_i = q + t$ are unique; so $a_i = \lambda'_i$ for all i , which contradicts $\lambda'_i \notin \mathbb{Z}$.

(c) The case $t = 0$ of (b) shows that P does not have IDP, and as t is arbitrary, no k_0 as in Definition 2.1(v) exists. The point $u = x - qv_j = \sum_i \lambda_i (v_i - v_j)$ is a lattice point of $\mathbb{R}_{\geq 0}(P - v_j)$. Suppose that $u = \sum_{l=1}^k (p_l - v_j)$ with $p_l \in L(P)$ and $k \geq 0$. Then $x + (k - q)v_j = p_1 + \dots + p_k$. If

$k \geq q$, this contradicts (b) with $t = k - q$; if $k < q$, then $x = p_1 + \cdots + p_k + (q - k)v_j$ is a sum of q points of $L(P)$, which contradicts (b) with $t = 0$. As v_j is a vertex of P by (a), P is not very ample at its vertices.

(d) follows from (c) and Lemma 2.2. $\square$

4 Proof of Theorem A

Let B, c, R, f, P and $\mathbf{n}$ be as in Theorem A, and put $h = -41504$ and $H_h = \{y \in \mathbb{R}^6 : \mathbf{n} \cdot y = h\}$.

Integer form and spanning. For $y \in \mathbb{R}^6$ we have $|y - c|^2 = |y|^2 - y_3 - y_5 + \frac{1}{2}$, since $(y_3 - \frac{1}{2})^2 = y_3^2 - y_3 + \frac{1}{4}$ and similarly for y_5 . Hence a point $z \in \mathbb{Z}^6$ lies in B if and only if $f(z) + \frac{1}{2} \leq \frac{317813}{2}$, that is, $f(z) \leq 158906$; so P is the lattice hull of B and $L(P) = L(B)$. The values $f(0) = 0$ and $f(e_1), \dots, f(e_6) = 1, 1, 0, 1, 0, 1$ show that $0, e_1, \dots, e_6 \in L(P)$. These points affinely generate $\mathbb{Z}^6$, so P is spanning and full-dimensional.

Proof of (a). For $y \in B$, the Cauchy–Schwarz inequality gives

$$\mathbf{n} \cdot y = \mathbf{n} \cdot c + \mathbf{n} \cdot (y - c) \geq \mathbf{n} \cdot c - |\mathbf{n}| R = \frac{9}{2} - |\mathbf{n}| R, \quad |\mathbf{n}|^2 = 10843.$$

Since $4|\mathbf{n}|^2 R^2 = 2 \cdot 10843 \cdot 317813 = 6\,892\,092\,718 < 6\,892\,154\,361 = 83019^2$, we have $|\mathbf{n}| R < \frac{83019}{2}$ and hence $\mathbf{n} \cdot y > \frac{9-83019}{2} = -41505$ for every $y \in B$. (Numerically, $\min_{y \in B} \mathbf{n} \cdot y \approx -41504.81$.) As $\mathbf{n} \cdot z$ is an integer for $z \in \mathbb{Z}^6$, every $z \in L(P)$ satisfies $\mathbf{n} \cdot z \geq h$.

Proof of (b): the enumeration. Let $p = c + \frac{h - \mathbf{n} \cdot c}{|\mathbf{n}|^2} \mathbf{n}$ be the orthogonal projection of c to H_h . For $y \in H_h$ we have $|y - c|^2 = |y - p|^2 + \delta^2$ with $\delta^2 = \frac{(h - \mathbf{n} \cdot c)^2}{|\mathbf{n}|^2} = \frac{83017^2}{43372}$. Hence every $z \in L(B) \cap H_h$ satisfies $|z - p|^2 \leq \rho^2 := R^2 - \delta^2 = \frac{270429}{43372} \approx 6.2351$, and in particular $|z_k - p_k| \leq \rho < 2.5$ for $k = 1, \dots, 6$. As $p \approx (137.81, 176.09, -190.91, 218.20, 157.45, -3.83)$, the coordinates $z_1, \dots, z_6$ lie in the intervals

$$[136, 140], [174, 178], [-193, -189], [216, 220], [155, 159], [-6, -2].$$

(In integers: $(2|\mathbf{n}|^2 z_k - C_k)^2 \leq D$ with $C_k = 2|\mathbf{n}|^2 p_k = 10843 \cdot 2c_k - 83017 n_k$ and $D = 10843 \cdot 270429$.) The lattice points of B on H_h were enumerated in three independent ways:

- (I) the $5^5 = 3125$ choices of $(z_1, \dots, z_5)$ in the first five intervals, with $z_6 = h - (n_1 z_1 + \cdots + n_5 z_5)$ (recall $n_6 = 1$), tested for $f(z) \leq 158906$;
- (II) all $5^6 = 15625$ points of the box, tested for $\mathbf{n} \cdot z = h$ and $f(z) \leq 158906$;
- (III) a Fincke–Pohst enumeration of the points $z = h e_6 + \sum_{k=1}^5 a_k (e_k - n_k e_6)$, $a \in \mathbb{Z}^5$ (these are exactly the integer points of H_h), with $|z - c|^2 \leq R^2$, using an exact $LDL^\top$ decomposition of the Gram matrix of the five vectors $e_k - n_k e_6$.

Each method finds exactly the six points of Table 1. All three are implemented in the program `check_note.py` of this note, (I) and (II) in integer arithmetic and (III) in exact rational arithmetic. The same set was found by the programs with which the result was first obtained (among them a full box of $9^6 = 531441$ points) and by the three verification runs with programs of their own (among them full boxes of $8^6 = 262144$ points, Fincke–Pohst enumerations, one of them in an LLL-reduced basis, and an enumeration with z_1 solved from the equation of H_h); see Section 10.

The facet and its index. The 5×5 minors of the matrix with rows $F_i - F_0$ ($i = 1, \dots, 5$), obtained by deleting the column $k = 1, \dots, 6$, are

$$72, -92, -100, -114, 82, 2, \quad \text{that is,} \quad -2(-1)^{k-1}n_k.$$

In particular $F_0, \dots, F_5$ are affinely independent. Let $\Lambda = \mathbf{n}^\perp \cap \mathbb{Z}^6$ and let Λ_F be the lattice generated by the $F_i - F_0$. Since $\mathbb{Z}^6/\Lambda \cong \mathbb{Z}$ is torsion-free, a basis of Λ extends to a basis of $\mathbb{Z}^6$, so the maximal minors of a basis matrix of Λ have greatest common divisor 1; by the Cauchy–Binet formula the maximal minors of the matrix with rows $F_i - F_0$ are $\det U$ times those of the basis matrix, where U expresses the rows in the basis and $|\det U| = [\Lambda : \Lambda_F]$. Hence $[\Lambda : \Lambda_F] = \gcd(72, 92, 100, 114, 82, 2) = 2$. By (a) and Lemma 3.1, $P \cap H_h = \text{conv}(F_0, \dots, F_5)$ is a face of P ; it has dimension 5, and P has dimension 6, so it is a facet.

Proof of (c) and of the theorem. The sum $F_0 + \dots + F_5 = (826, 1056, -1144, 1312, 944, -24)$ is even, so $w = \frac{1}{2} \sum_i F_i$ is a lattice point, with all barycentric coordinates equal to $\frac{1}{2}$ (sum 3). We apply Lemma 3.2 with $v_i = F_i$, $x = w$, $\lambda_i = \frac{1}{2}$ and $q = 3$. Its part (b) is part (c) of the theorem, and its parts (c) and (d), together with the spanning property, show that P is neither normal in any of the senses of Definition 2.1 nor very ample, in either sense. This proves Theorem A. $\square$

Remark 4.1. The data are tight: the bound in (a) holds with a margin of about 0.19, all vertices of the facet lie on the boundary sphere, and with 158907 in place of 158906 the hyperplane $\mathbf{n} \cdot z = -41504$ contains 14 lattice points of the ball. For the ball of the same radius centred at the origin the bound (a) fails.

5 All dimensions at least six

Proof of Corollary B. Let $n \geq 6$ and write points of $\mathbb{R}^n$ as (y, u) with $y \in \mathbb{R}^6$ and $u \in \mathbb{R}^{n-6}$. The ball B_n of the corollary has centre $(c, 0)$ and radius R , so $(y, u) \in B_n$ if and only if $|y - c|^2 + |u|^2 \leq R^2$; for integer points this is the inequality defining $P^{(n)}$, by the identity $|y - c|^2 = |y|^2 - y_3 - y_5 + \frac{1}{2}$ of Section 4. Put $\mathbf{n}' = (\mathbf{n}, 0) \in \mathbb{Z}^n$. The points $0, e_1, \dots, e_n$ satisfy the inequality (the values of the left-hand side are 0 and 1, 1, 0, 1, 0, 1, 1, $\dots$, 1), so $P^{(n)}$ is spanning and full-dimensional. If $(z, u) \in L(B_n)$, then $z \in L(B)$, and Theorem A(a) gives $\mathbf{n}' \cdot (z, u) = \mathbf{n} \cdot z \geq -41504$. If moreover $\mathbf{n} \cdot z = -41504$, then z is one of the points F_i by Theorem A(b), and since $|F_i - c|^2 = R^2$ (Table 1), we get $|u|^2 \leq R^2 - |z - c|^2 = 0$, that is, $u = 0$. So the lattice points of $P^{(n)}$ on the hyperplane $\mathbf{n}' \cdot y = -41504$ are the affinely independent points $(F_i, 0)$, and $(w, 0) = \sum_i \frac{1}{2}(F_i, 0)$. Lemma 3.2 with $q = 3$ gives the claim. $\square$

For $n \geq 7$ this 5-dimensional face is not a facet. The argument uses that the vertices of the face lie on the boundary sphere, so it also applies to Theorem 6.4; `check_note.py` repeats it for $n = 7, 8$.

6 How the examples arise, and further certificates

6.1 A ball touching a hyperplane

Lemma 6.1. Let $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$, $H = \mathbf{n}^\perp$, $c_0 \in H$, and let $r, s > 0$ with $2s + 1/|\mathbf{n}|^2 > r^2$. Put $c = c_0 + s\mathbf{n}$, $R^2 = r^2 + s^2|\mathbf{n}|^2$ and $B = B(c, R)$. Then

- (a) $\mathbf{n} \cdot z \geq 0$ for every $z \in L(B)$;
- (b) $L(B) \cap H = \{z \in H \cap \mathbb{Z}^d : |z - c_0| \leq r\}$.

Proof. For $y \in \mathbb{R}^d$ let $m = \mathbf{n} \cdot y$ and $y_H = y - \frac{m}{|\mathbf{n}|^2} \mathbf{n} \in H$. Then $y - c = (y_H - c_0) + (\frac{m}{|\mathbf{n}|^2} - s) \mathbf{n}$ with $y_H - c_0 \in H$ orthogonal to $\mathbf{n}$, so

$$|y - c|^2 = |y_H - c_0|^2 + \frac{m^2}{|\mathbf{n}|^2} - 2ms + s^2 |\mathbf{n}|^2. \quad (1)$$

If $z \in \mathbb{Z}^d$ and $m = \mathbf{n} \cdot z \leq -1$, then $m^2/|\mathbf{n}|^2 \geq 1/|\mathbf{n}|^2$ and $-2ms \geq 2s$, so $|z - c|^2 \geq 1/|\mathbf{n}|^2 + 2s + s^2 |\mathbf{n}|^2 > r^2 + s^2 |\mathbf{n}|^2 = R^2$. This proves (a). If $m = 0$, then $z = z_H$, and (1) gives $|z - c|^2 = |z - c_0|^2 + s^2 |\mathbf{n}|^2$, which is at most R^2 if and only if $|z - c_0| \leq r$. This proves (b). $\square$

Proposition 6.2. *Let $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$, $H = \mathbf{n}^\perp$, $c_0 \in H$ and $r > 0$, and suppose that the points of $H \cap \mathbb{Z}^d$ at distance at most r from c_0 are affinely independent points $v_0, \dots, v_k$. Let $x \in \mathbb{Z}^d$ with $x = \sum_i \lambda_i v_i$, $\lambda_i \geq 0$, $\sum_i \lambda_i = q \in \mathbb{Z}$ and $\lambda_i \notin \mathbb{Z}$ for some i . Let $s > 0$ with $2s + 1/|\mathbf{n}|^2 > r^2$, and let P be the lattice hull of the ball B of Lemma 6.1. Then P does not have IDP and is neither very ample nor very ample at its vertices; if P is spanning, it is not normal in any of the senses of Definition 2.1.*

Proof. By Lemma 6.1, $\mathbf{n} \cdot z \geq 0$ on $L(P) = L(B)$, and the lattice points of P on H are $v_0, \dots, v_k$. Apply Lemma 3.2 with $h = 0$. $\square$

P is spanning, for instance, if $z_0, z_0 + e_1, \dots, z_0 + e_d \in B$ for some $z_0 \in \mathbb{Z}^d$, which holds when $R \geq \frac{\sqrt{d}}{2} + 1$ and z_0 is a lattice point nearest to the centre. The natural choice is $k = d - 1$ and $c_0 = c_S$, $r = r_S$, the circumcentre and circumradius of $S = \text{conv}(v_0, \dots, v_{d-1}) \subset H$; the hypothesis then says that S is an *empty-sphere simplex* of the lattice $H \cap \mathbb{Z}^d$ (no lattice point inside its circumscribed sphere in H , and only its vertices on it). If S is not unimodular, a non-zero lattice point $(x, q) = \sum_i r_i(v_i, 1)$ with $0 \leq r_i < 1$ provides x . In all our examples the vertices of S generate a sublattice of index 2, and $q = 3$.

6.2 Moving the centre

Lemma 6.3. *Let $\mathbf{n} \in \mathbb{Z}^d$ be primitive, $H = \mathbf{n}^\perp$, and let $v_0 = 0$ and $v_1, \dots, v_{d-1} \in H \cap \mathbb{Z}^d$ be affinely independent, with circumcentre $c_S \in H$ and circumradius r_S . Suppose that the points of $H \cap \mathbb{Z}^d$ at distance at most r_S from c_S are exactly $v_0, \dots, v_{d-1}$, and let x and q be as in Proposition 6.2. Let $Q \in \mathbb{R}^d$.*

- (a) *If there is $y \in \mathbb{Z}^d$ with $v_i \cdot (Q - y) = \frac{1}{2}|v_i|^2$ for $i = 1, \dots, d - 1$, then $Q - y = c_S + \tau \mathbf{n}$ for some real τ , and y can be chosen so that $\tau > 0$ and $2\tau + 1/|\mathbf{n}|^2 > r_S^2$. With $R^2 = r_S^2 + \tau^2 |\mathbf{n}|^2$, the lattice hull of $B(Q, R)$ is the translate by y of the lattice hull of the ball of Lemma 6.1 with $c_0 = c_S$, $r = r_S$ and $s = \tau$; so it is not very ample, and not normal if it is spanning.*
- (b) *Suppose moreover that $v_0, \dots, v_{d-1}$ generate a sublattice of index 2 of $H \cap \mathbb{Z}^d$ and that $x = \frac{1}{2}(v_0 + \dots + v_{d-1})$. Then a point y as in (a) exists if and only if the numbers $\omega_i = v_i \cdot Q - \frac{1}{2}|v_i|^2$ ($1 \leq i \leq d - 1$) are integers with an even sum. For $Q \in \mathbb{Z}^d$ this condition does not depend on Q .*

Proof. (a) Since $v_0 = 0$ lies on the circumscribed sphere, $|v_i - c_S|^2 = |c_S|^2$, that is, $v_i \cdot c_S = \frac{1}{2}|v_i|^2$ for all i . Hence $v_i \cdot (Q - y - c_S) = 0$ for $i = 1, \dots, d - 1$, and as these v_i span H , the vector $Q - y - c_S$ is a multiple $\tau \mathbf{n}$ of $\mathbf{n}$. Replacing y by $y - j \mathbf{n}$ with $j \in \mathbb{Z}$ replaces τ by $\tau + j$, so τ can be taken as large as needed. Then $B(Q, R) = y + B(c_S + \tau \mathbf{n}, R)$, and translation by the lattice vector y maps the lattice points of one ball onto those of the other. All properties in Definition 2.1 are invariant under such translations, and Proposition 6.2 applies.

(b) The image I of the map $\mathbb{Z}^d \rightarrow \mathbb{Z}^{d-1}$, $y \mapsto (v_i \cdot y)_{i=1}^{d-1}$, is a sublattice whose index is the greatest common divisor of the maximal minors of the matrix with rows $v_1, \dots, v_{d-1}$ (Smith normal form). As

in Section 4, these minors are ± 2 times the maximal minors of a basis matrix of $H \cap \mathbb{Z}^d$, which have greatest common divisor 1; so $[\mathbb{Z}^{d-1} : I] = 2$. Since $\sum_{i \geq 1} v_i \cdot y = 2x \cdot y$ is even for $y \in \mathbb{Z}^d$, I is contained in the sublattice $\{\omega : \sum_i \omega_i \text{ even}\}$ of index 2, hence equal to it. A point y as in (a) exists if and only if $(\omega_i)_i \in I$. For $Q \in \mathbb{Z}^d$ we have $\omega_i \equiv -\frac{1}{2}|v_i|^2 \pmod{1}$ and $\sum_i \omega_i = 2x \cdot Q - \sum_i \frac{1}{2}|v_i|^2 \equiv -\sum_i \frac{1}{2}|v_i|^2 \pmod{2}$, so the condition is the same as for $Q = 0$. $\square$

6.3 Further certificates

We use the following data; c_S and r_S are the circumcentre and the circumradius of the v_i in $\mathbf{n}^\perp$.

Data (a), $d = 6$: $\mathbf{n} = (387, 25, -279, -355, 219, 1)$, $|\mathbf{n}|^2 = 402222$, $v_0 = 0$,

$$\begin{aligned} v_1 &= (-3, 2, -3, -2, -2, 2), \quad v_2 = (-2, 5, 1, -2, 1, -1), \quad v_3 = (-3, 1, -2, -1, 1, 4), \\ v_4 &= (-4, 0, -3, -2, 0, 1), \quad v_5 = (-4, 2, 1, -5, 0, 2), \quad x = \frac{1}{2} \sum_i v_i = (-8, 5, -3, -6, 0, 4), \\ c_S &= \left(-\frac{3637}{1559}, \frac{828533}{402222}, -\frac{16650}{67037}, -\frac{392254}{201111}, \frac{54251}{134074}, \frac{205615}{201111} \right), \quad r_S^2 = \frac{5936999}{402222} \approx 14.76. \end{aligned}$$

Data (c), $d = 7$: $\mathbf{n} = (-240, 288, -250, -21, -68, -192, 1)$, $|\mathbf{n}|^2 = 244974$, $v_0 = 0$,

$$\begin{aligned} v_1 &= (1, 1, 1, -3, -2, 0, 3), \quad v_2 = (0, 2, 2, 1, -2, 1, 1), \quad v_3 = (1, 1, 2, 0, -1, -2, 0), \\ v_4 &= (1, 3, 2, 0, -1, 1, 0), \quad v_5 = (1, 2, 1, 1, 1, 0, 3), \quad v_6 = (-1, 2, 3, 0, 1, 0, 2), \\ x &= \frac{1}{2}(v_0 + v_2 + v_3 + v_4 + v_5 + v_6) = (1, 5, 5, 1, -1, 0, 3), \\ c_S &= \left(\frac{41465}{81658}, \frac{56779}{40829}, \frac{174517}{122487}, -\frac{37739}{40829}, -\frac{69139}{122487}, -\frac{7657}{81658}, \frac{215369}{122487} \right), \quad r_S^2 = \frac{2081093}{244974} \approx 8.50. \end{aligned}$$

Theorem 6.4. *The lattice hulls P of the following closed balls are full-dimensional and spanning, and they are neither normal, in any of the senses of Definition 2.1, nor very ample.*

- (a) *The ball in $\mathbb{R}^6$ with centre $c_S + 8\mathbf{n}$ and squared radius $r_S^2 + 64|\mathbf{n}|^2 = \frac{10354088323175}{402222}$, for the data (a). The lattice points of P on $\mathbf{n}^\perp$ are $v_0, \dots, v_5$, they span a facet of index 2, and $x \in L(3P)$ is not a sum of three lattice points of P .*
- (b) *The ball in $\mathbb{R}^6$ with centre $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$ and squared radius $\frac{51196593}{2}$, whose lattice hull is $P = \text{conv}\{z \in \mathbb{Z}^6 : |z|^2 - z_2 - z_5 \leq 25598296\}$. For the vector $\mathbf{n}$ of the data (a), every lattice point of P satisfies $\mathbf{n} \cdot z \geq -3208646$, and the lattice points on $\mathbf{n} \cdot z = -3208646$ are the six points $y + v_i$ with $y = (-3085, -201, 2226, 2834, -1747, -9)$; they span a facet of index 2, and $x + 3y = (-9263, -598, 6675, 8496, -5241, -23)$ is not a sum of three lattice points of P .*
- (c) *The ball in $\mathbb{R}^7$ with centre $c_S + 8\mathbf{n}$ and squared radius $r_S^2 + 64|\mathbf{n}|^2 = \frac{3840786764357}{244974}$, for the data (c). The lattice points of P on $\mathbf{n}^\perp$ are $v_0, \dots, v_6$, they span a facet of index 2, and $x \in L(3P)$ is not a sum of three lattice points of P .*

Proof. (a) and (c) follow from Proposition 6.2 with $c_0 = c_S$, $r = r_S$ and $s = 8$, given the following finite facts: $c_S \in \mathbf{n}^\perp$ is at squared distance r_S^2 from every v_i (linear algebra); the points of $\mathbf{n}^\perp \cap \mathbb{Z}^d$ at distance at most r_S from c_S are exactly the v_i (an enumeration as in Section 4); the v_i are affinely independent and generate a sublattice of index 2 of $\mathbf{n}^\perp \cap \mathbb{Z}^d$ (maximal minors), and x has the coefficients stated; $2s = 16 > r_S^2$; and $z_0, z_0 + e_1, \dots, z_0 + e_d$ lie in the ball for $z_0 = (3094, 202, -2232, -2842, 1752, 9)$ in (a) and $z_0 = (-1919, 2305, -1999, -169, -545, -1536, 10)$ in (c). The face spanned by the v_i has dimension $d - 1$, so it is a facet. For (b) we argue as in Section 4: the centre $c' = (0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$ has $\mathbf{n} \cdot c' = 122$, and $(\mathbf{n} \cdot c' + 3208646 + 1)^2 = 3208769^2 = 10\,296\,198\,495\,361 > 10\,296\,198\,014\,823 = |\mathbf{n}|^2 R^2$, so $\mathbf{n} \cdot z \geq -3208646$ on $L(P)$; the enumeration of the integer points of the ball on the hyperplane $\mathbf{n} \cdot z = -3208646$ gives the six points $y + v_i$; and $0, e_1, \dots, e_6$ lie in the ball. Lemma 3.2 with $q = 3$ concludes. $\square$

These finite facts are checked by `check_note.py` (25088, 25088 and 38880 candidate points), by the programs with which the certificates were first verified (161051, 161051 and 1771561 candidate points, one coordinate being solved from the equation of the hyperplane) and by all three verification runs. In (c), x lies in three times the 5-face opposite v_1 . The certificates (a) and (c) were found by an exploratory search whose program is not part of the package; the search of Section 10 found the section of Theorem A.

The centres of Theorem A and Theorem 6.4(b). For the data (a) and $Q = 0$, the numbers of Lemma 6.3(b) are $\omega = (-17, -18, -16, -15, -25)$, with the odd sum -91 ; so Lemma 6.3 gives no ball centred at a lattice point for this simplex. For $Q = (0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$ they are $\omega = (-17, -15, -15, -15, -24)$, with an even sum, and Lemma 6.3(a) gives the point y of Theorem 6.4(b), with $\tau = \frac{1604384}{201111}$; this is how the ball in (b) was found. For Theorem A, take the simplex with vertices $F_i - F_1$ in the section $\mathbf{n}^\perp \cap \mathbb{Z}^6$, $\mathbf{n} = (-36, -46, 50, -57, -41, 1)$; its squared circumradius is $r_S^2 = \frac{270429}{43372}$, and its witness is $w - 3F_1$. For $Q = 0$ some of the numbers ω_i are not integers, while $Q = (0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$ gives $\omega = (-9, -5, -4, -10, -6)$ (for the vertices $F_i - F_1$, $i = 0, 2, 3, 4, 5$), $y = F_1$, $\tau = \frac{83017}{21686}$ and $R^2 = r_S^2 + \tau^2|\mathbf{n}|^2 = \frac{317813}{2}$: the ball of Theorem A, which was proved directly in Section 4.

7 Why dimension six

For a lattice simplex $F = \text{conv}(v_0, \dots, v_k) \subset \mathbb{R}^d$ (with affinely independent vertices) let

$$\text{Box}(F) = \left\{ b = \sum_i r_i(v_i, 1) \in \mathbb{Z}^{d+1} : 0 \leq r_i < 1 \right\}, \quad \deg b = \sum_i r_i \in \mathbb{Z}_{\geq 0}.$$

The coefficients r_i of a box point are unique, as the $(v_i, 1)$ are linearly independent. F is unimodular if and only if $\text{Box}(F) = \{0\}$: with Λ' the set of lattice points in the span of the $(v_i, 1)$ and $\Lambda'' = \bigoplus_i \mathbb{Z}(v_i, 1)$, every $y = \sum_i \mu_i(v_i, 1) \in \Lambda'$ satisfies $y - \sum_i \lfloor \mu_i \rfloor (v_i, 1) \in \text{Box}(F)$, so $\text{Box}(F) = \{0\}$ if and only if $\Lambda' = \Lambda''$; and $\Lambda' = \Lambda''$ holds if and only if the v_i affinely generate $\text{aff}(F) \cap \mathbb{Z}^d$ (for $(y, m) \in \Lambda'$ the point $v_0 + (y - mv_0)$ lies in $\text{aff}(F) \cap \mathbb{Z}^d$).

Proof of Proposition C. Let $F = \text{conv}(v_0, \dots, v_k)$ be a face of $P = \text{conv}(L(E))$ with $L(F) = \{v_0, \dots, v_k\}$, and let $b = \sum_i r_i(v_i, 1) = (y, g)$ be a non-zero point of $\text{Box}(F)$ of degree $g \geq 1$.

If $g = 1$, then $y = \sum_i r_i v_i \in L(F)$, say $y = v_a$; by uniqueness of the coefficients $r_a = 1$, which is impossible.

If $g = 2$, then $y \in L(2F) \subseteq L(2P)$, and by [BGM16, Theorem 6.5(b)] there are $w_1, w_2 \in L(P)$ with $y = w_1 + w_2$. Let ϕ be a linear function and h a number with $\phi \leq h$ on P and $F = P \cap \{\phi = h\}$. From $\phi(w_1) + \phi(w_2) = \phi(y) = 2h$ we get $\phi(w_1) = \phi(w_2) = h$, so w_1 and w_2 are vertices of F , say $w_1 = v_a$ and $w_2 = v_c$. Then $(y, 2) = (v_a, 1) + (v_c, 1)$, and uniqueness of the coefficients gives $r_a \geq 1$, which is impossible.

Hence $g \geq 3$. Let $S = \{i : r_i > 0\}$. The point $b^* = \sum_{i \in S} (1 - r_i)(v_i, 1) = \sum_{i \in S} (v_i, 1) - b$ is a lattice point with coefficients in $(0, 1)$, so it is a non-zero point of $\text{Box}(F)$, of degree $|S| - g$ (this reflection is the argument of [BGM16, Lemma 2.3]); by the same argument $|S| - g \geq 3$. Therefore $k + 1 \geq |S| \geq g + 3 \geq 6$, that is, $\dim F = k \geq 5$. So if $\dim F \leq 4$, then $\text{Box}(F) = \{0\}$ and F is unimodular. $\square$

Consequences for $n \leq 5$. The hypotheses of Lemma 3.2 require a face F that is a simplex with $L(F) = \text{vert}(F)$ and a non-zero box point of F . If P is the spanning lattice hull of an ellipsoid in $\mathbb{R}^n$, $n \leq 5$, every proper face has dimension at most 4, so Proposition C excludes this. A counterexample

in dimension 4 or 5, if there is one, must come from a face that is not a simplex, a face with further lattice points, or a failure not detected on a single face. This explains the dimension 6 in Corollary B; it is also the threshold of Gubeladze's theorem for ellipsoids whose lattice points affinely generate $\mathbb{Z}^d$ [Gub21, Theorem (c)]. (In the facet of Theorem A the only non-zero box point is $(w, 3)$, of the smallest degree allowed by the proof.)

8 The second question: a very ample non-normal simplex

We recall the example of Cox, Haase, Hibi and Higashitani and prove Proposition D for the reader's convenience; nothing in this section is new. In [CHHH14, Example 2.3], stated for $m \geq 4$ and used with $m = 4$ in their Example 2.5, they determine the box points of P_m , show that the Hilbert basis of the cone over P_m has largest degree 2, and show that $(3, \dots, 3) \in L(2 \cdot 3P_m)$ is not a sum of two lattice points of $3P_m$. By their Theorem 1.1 ($\mu_{\text{va}} \leq \mu_{\text{Hilb}}$) $2P_m$ is very ample, and by their remark on p. 2 so is $3P_m$. This is stated explicitly in [ABLS25, Section 2.3] for $m = 4$.

Proof of Proposition D. Write $P = P_m$, $Q = 3P$, $w_m = (m-1, \dots, m-1, m)$, and let $\mathcal{V}$ be the vertex set $\{0, e_1, \dots, e_{d-1}, w_m\}$ of P . The vectors $(v, 1)$, $v \in \mathcal{V}$, form a basis of $\mathbb{R}^{d+1}$. Let C be the cone over $P \times \{1\}$ and $\mathcal{M}$ the monoid generated by the $(v, 1)$. Every lattice point y of C has unique coordinates $y = \sum_v \lambda_v(v, 1)$, $\lambda_v \geq 0$, and $y = \sum_v \{\lambda_v\}(v, 1) + \sum_v \lfloor \lambda_v \rfloor (v, 1)$ is the unique decomposition of y as a box point plus an element of $\mathcal{M}$.

Box points. Let $b = \sum_v r_v(v, 1) \in \mathbb{Z}^{d+1}$ with $0 \leq r_v < 1$. The d -th coordinate of b is mr_{w_m} , so $r_{w_m} = j/m$ with $0 \leq j \leq m-1$. For $i < d$ the i -th coordinate is $r_{e_i} + (m-1)j/m \in \mathbb{Z}$, so $r_{e_i} = j/m$. The last coordinate is $r_0 + (d-1)j/m + j/m = r_0 + 2j - j/m \in \mathbb{Z}$, so $r_0 = j/m$. As $\sum_{v \in \mathcal{V}} v = (m, \dots, m)$, the box points are $\beta_j = \frac{j}{m} \sum_v (v, 1) = (j, \dots, j, 2j) = j\beta_1$, $0 \leq j \leq m-1$, of degree $2j$.

Spanning. Since $0 \in P$, we have $P \subseteq 2P \subseteq 3P = Q$; so $0, e_1, \dots, e_{d-1} \in Q$, and $\beta_1 \in C$ means $(1, \dots, 1) \in 2P \subseteq Q$. These $d+1$ lattice points affinely generate $\mathbb{Z}^d$.

Not normal. Since $3 \leq m-1$, $\beta_3 = ((3, \dots, 3), 6)$ is a box point, so $(3, \dots, 3) \in L(6P) = L(2Q)$. Suppose that $(3, \dots, 3) = p + p'$ with $p, p' \in L(Q) = L(3P)$, and write $(p, 3) = \beta_a + \mu$ and $(p', 3) = \beta_{a'} + \mu'$ with $\mu, \mu' \in \mathcal{M}$. Comparing degrees, $2a \leq 3$ and $2a' \leq 3$, so $a, a' \leq 1$. Then $\beta_3 = \beta_{a+a'} + (\mu + \mu')$, where $\beta_{a+a'}$ is a box point because $a + a' \leq 2 \leq m-1$; by uniqueness of the decomposition $\beta_3 = \beta_{a+a'}$, that is, $3 = a + a' \leq 2$, a contradiction. So Q does not have IDP, and as it is spanning, it is not normal in any of the senses of Definition 2.1 (Lemma 2.2).

Very ample. Let $\ell \geq m-1$ and $\alpha \in L(\ell Q) = L(3\ell P)$. Then $(\alpha, 3\ell) = \beta_j + \mu$ with $0 \leq j \leq m-1 \leq \ell$ and μ a sum of $3\ell - 2j \geq j$ points $(v, 1)$, $v \in \mathcal{V}$. Pair each of the j copies of β_1 in $\beta_j = j\beta_1$ with one of these points: $\beta_1 + (v, 1) = ((1, \dots, 1) + v, 3)$, and $(1, \dots, 1) + v \in 2P + P = 3P = Q$. Group the remaining $3\ell - 3j$ points $(v, 1)$ into $\ell - j$ triples; each triple sums to $(v + v' + v'', 3)$ with $v + v' + v'' \in 3P = Q$. This writes α as a sum of $j + (\ell - j) = \ell$ lattice points of Q . So Q is very ample, with $k_0 = m-1$ in Definition 2.1(v). $\square$

By computer, for $m = 4$: $Q_4, \dots, 4Q_4$ have 128, 2083, 15784 and 77987 lattice points; $(3, \dots, 3)$ is the only point of $2Q_4$ that is not a sum of two lattice points of Q_4 ; all lattice points of $3Q_4, 4Q_4$ (and $5Q_4$, in one run) decompose; and the four box points of each of the eight corner cones are sums of elements of $L(Q_4) - v$, so Q_4 is very ample at its vertices too. For $m = 5$ two runs found the analogous picture: $(3, \dots, 3)$ is the only point of $2Q_5$ that is not a sum of two lattice points of Q_5 , ten lattice points of $3Q_5$ are not sums of three (threshold $m-1 = 4$), and all lattice points of $4Q_5$ decompose.

9 Open questions

1. Dimensions 4 and 5. Is the lattice hull of every ball in $\mathbb{R}^4$ and in $\mathbb{R}^5$ normal? By Proposition C a counterexample cannot come from an empty simplex face. Lattice hulls of several dozen axis-parallel ellipsoids in these dimensions were found to be normal [BGM16, Remark 6.6(a)], and Gubeladze does not know an ellipsoid in $\mathbb{R}^5$ whose lattice points affinely generate $\mathbb{Z}^5$ and whose lattice hull is not normal [Gub21, Remark 7, last paragraph].

2. Balls centred at lattice points. Are the polytopes $\text{conv}\{z \in \mathbb{Z}^n : |z|^2 \leq N\}$ normal? For the two simplices behind Theorem A and Theorem 6.4(a, b), Lemma 6.3(b) shows that Lemma 6.3 gives no ball centred at a lattice point. In all 17 sections found by the search of Section 10, the centres admitted by Lemma 6.3(a) contain no lattice point (those with an integral last coordinate form one class modulo $\mathbb{Z}^6$, of denominator 2). Two exploratory searches, over other centres c_0 in Proposition 6.2 near the circumcentre (seven simplices in four sections, stopped early) and over all 10843 classes of slices of balls centred at the origin for the section of Theorem A, found no such ball either. These are observations, not proofs; the third verification run confirmed the statement on the 17 sections, and no verification run repeated the two searches.

3. Small normal vectors. The smallest $|\mathbf{n}|^2$ found, 10843 (Theorem A), is only the smallest found by a non-exhaustive random search: the runs with entries bounded by 20, 35 and 60 (980 sections) found nothing, and the vector of Theorem A (entries at most 57 in absolute value) was sampled in a run with bound 80. An exhaustive search could determine the smallest example of this kind.

10 Computations and reproducibility

Apart from the enumerations in Theorem A(b) and in the proof of Theorem 6.4, and routine exact arithmetic on the given data (determinants, circumcentres), every theorem, proposition, corollary and lemma of this note is proved without computation; the paragraph after the proof of Proposition D, Section 9 and Table 2 report computations that no proof uses. The reproducibility package (folder `reproducibility/` of the source archive) contains the following.

The program of this note, `check_note.py` (standard library, exact arithmetic, about two seconds). It checks, for Theorem A, the identity behind the integer form (on 28 points, which determine a quadratic polynomial in six variables), the inequality of (a), the enumerations (I)–(III) of Section 4, the values $\mathbf{n} \cdot F_i$ and $f(F_i)$, the minors and the index, the witness and its coefficients, and the spanning points, with the two controls of Remark 4.1; the padded balls for $n = 7, 8$; the three certificates of Theorem 6.4; and, for Proposition D with $m = 4$, the box points, the point $(3, \dots, 3)$ of $2Q_4$, the spanning points and the decomposition of all 15784 lattice points of $3Q_4$. All 79 checks pass.

The programs with which the results were first obtained (original/), with recorded outputs: `verify_ball.py` (hypotheses of Proposition 6.2; 59049, 161051 and 1771561 candidate points for the section of Theorem A and Theorem 6.4(a), (c)), `verify_direct.py` (Theorem A and Theorem 6.4(b) from the balls), `independent_check_thmA.py` (Theorem A, full box of 531441 points), `verify_padded.py`, `verify_centered.py` and `verify_simplex_chhh.py` (padding to $d = 7, 8$, Lemma 6.3, Proposition D for $m = 4$), and the search programs described below. The folders `verification_run_A/` and `verification_run_B/` contain the programs and outputs of the first two verification runs, and the folder `independent_run_2/` those of the third (see the paragraph “Verification”). The script `rerun_quick.py` runs again the 23 programs of the package outside `independent_run_2/` that have a recorded output and need neither a third-party PDF file nor numpy or scipy, writing into a new folder; on 14 October 2026 all 23 outputs were reproduced exactly,

up to recorded running times (about ten minutes in all), and the third verification run obtained the same result from the source archive.

The search. For random $a \in [-A, A]^5$ with non-zero entries of distinct absolute values (in the runs marked “odd”, all entries odd), the program `search_sections.py` reduces the basis $e_k - a_k e_6$ ($k = 1, \dots, 5$) of $\mathbf{n}^\perp \cap \mathbb{Z}^6$, $\mathbf{n} = (a, 1)$, by an exact LLL algorithm, lists the lattice points near 0 by a Fincke–Pohst enumeration, computes a Delaunay triangulation of this patch in floating point (scipy), and checks every simplex with vertex 0 and normalised volume at least 2 exactly: no lattice point may lie inside its circumscribed sphere, only its vertices on it, and half the sum of its vertices must be a lattice point. Floating point is used only to find candidates.

bound A	20	35	60	80	100	150	200 odd	400 odd	total
sections sampled	180	400	400	900	400	400	1080	560	4320
sections with simplices	0	0	0	2	3	4	4	4	17

Table 2: The randomized search, $\mathbf{n} = (a, 1)$, $a \in [-A, A]^5$. Counts from the logs, written every 20 sections (a stopped run may have tested up to 19 more); the run with $A = 20$ ended with an error of the triangulation library, which later versions of the program catch.

Table 2 summarises the runs. In each of the 17 sections the program recorded 12 simplices (two simplices up to translation, each with each of its six vertices placed at 0), 204 records in all; all have index 2 and a witness of degree 3, and the second and the third verification run confirmed all 204 records with their own exact programs. The smallest $|\mathbf{n}|^2$ among them is 10843, the section of Theorem A; this is only the smallest found by a non-exhaustive random search. The programs of Section 9 (item 2) are `analyze_hits.py`, `search_origin.py` and `search_origin2.py`.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs, without using those of the first version. The first run re-derived Lemmas 3.1, 3.2, 6.1 and 6.3, Proposition 6.2, Proposition C (with the proof of [BGM16, Theorem 6.5(a),(b)]), the arguments for Theorem A and Corollary B, and Proposition D for all $m \geq 4$; it checked Theorem A by a full box of 262144 points and by a Fincke–Pohst enumeration (31 checks), and Theorem A, Theorem 6.4 and the padding to $d = 7, 8$ by a general program starting from the balls (85 checks); for Proposition D with $m = 4$ it checked the box points, the lattice point counts, the unique indecomposable point of $2Q_4$ and all corner cones; and it read the sources. The second run validated its exact library against naive enumeration on 400 random balls and on eleven deliberately wrong inputs (all detected), enumerated the facet of Theorem A in three ways, checked Theorem 6.4, the padding to $d = 7, 8, 9$, the numbers used with Lemma 6.3, all 204 search records, and Proposition D for $m = 4, 5$ with an exact corner-cone certificate; it checked the proofs, re-ran ten programs of `original/` with byte-identical outputs, and repeated the literature search. Both runs confirmed the results. Their required corrections, all applied here, concerned the description of [CHHH14] (the range $m \geq 4$; very ampleness of $3P_m$ via Theorem 1.1 and the remark on p. 2) and of [ABLS25], the attribution of the cases $n \leq 3$, the weight of a database search that returned nothing, two statements on Delaunay simplices of lattices of small rank that had not been checked against a source and one statement without recorded evidence (all three removed), the form of Corollary B for the ball of Theorem A, the word “smallest” for the search, and bibliographic data. When this note was written, `check_note.py` and some short remarks not seen by the runs were added (the last sentence of Lemma 6.3(b), the spanning condition after Proposition 6.2, Remark 4.1, and the remarks on toric varieties and on $\Lambda(d)$), and the package was

re-run.

A third verification run, also AI-assisted, then examined the text of this note line by line, as if nothing had been checked before. It read the sources again, re-derived every proof, including those of [BGM16, Theorem 6.5(a),(b)] and of the inequality $\mu_{\text{va}} \leq \mu_{\text{Hilb}}$ of [CHHH14, Theorem 1.1], and wrote its own exact programs (folder `independent_run_2/`), without using those of the package. Its enumerator of the lattice points of a ball on a hyperplane (Fincke–Pohst in an LLL-reduced basis obtained from a unimodular column reduction of $\mathbf{n}$) agrees with naive enumeration on 300 random cases. It enumerated the facet of Theorem A in three further ways (in that basis, with z_1 solved from the equation of the hyperplane, and in a full box with exact bounds for each coordinate), and checked Theorem A, Remark 4.1, Corollary B for $n = 7, 8, 9$, Theorem 6.4 and the numbers used with Lemma 6.3 (92 checks); Proposition D for $m = 4$ (lattice point counts from the inequalities of the simplex, box points, decompositions up to $4Q_4$, all corner cones) and for $m = 5$ (up to $4Q_5$); and all 204 search records, Table 2 and the observation of Section 9 (item 2) on lattice-point centres. It re-ran the 23 programs of `rerun_quick.py` from the source archive (all outputs identical) and repeated the literature search. It found no mathematical error and no gap. Its required corrections, all applied here, concerned the description of the 2016 occurrence of question (1), the wording of the credit in [ABLS25], credit to Voronoi’s construction as recalled in [Gub21, Remark 7] and to [BGM16, Lemma 2.3], a citation and a wording concerning [Gub21], the first sentence of this section, the notation of Corollary B, and the record of this run.

Scope and priority. The note answers question (1) of [Mic17] in the negative, for balls with arbitrary centres, in every dimension $n \geq 6$ (Theorem A, Corollary B); the examples are spanning, so the answer does not depend on the convention for “normal”. Dimensions 4 and 5, and balls centred at lattice points, remain open. Question (2) was answered in 2014 by an example of Cox, Haase, Hibi and Higashitani [CHHH14], as stated explicitly by Adeyemo, Bunnett and Levicán-Santibáñez [ABLS25], who thank W. Bruns for informing them of it; we claim nothing for it. The questions are due to Michałek [Mic17, Mic16], following [BGM16, Remark 6.6 and Question 7.2(b)] and [Bru13, Section 8]. The face mechanism is the one of Gubeladze’s theorem for ellipsoids [Gub21], and Proposition C rests on [BGM16, Theorem 6.5(b)]. To our knowledge, and relative to the literature we found, the counterexamples for round balls are new. No answer for round balls was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and no priority is claimed.

What was read: [Mic17] (pp. 2684–2686) and [Mic16] (pp. 999–1001) in the publisher’s open files; [BGM16] in arXiv version 2, whose numbering we use (the journal version was not compared); [Gub21] in the journal version, whose numbering we use (in arXiv version 2, also read, Lemmas 4–6 and Remark 7 are Lemmas 3.2, 4.1, 4.2 and Remark 4.3); [CHHH14] in the journal version (introduction, Theorem 1.1 with its proof, Examples 2.2–2.5); [ABLS25], version 1 (introduction, acknowledgements, Section 2.3); [Bru13], Section 8; and [Gub23], arXiv version 2, searched for balls and ellipsoids. The third verification run read these sources again, and the whole report containing [Mic16], which has no problem session: the 2016 version of question (1) appears there only in Michałek’s abstract. The proofs of cited theorems were not checked, except those of [BGM16, Theorem 6.5(a),(b)] and of the inequality $\mu_{\text{va}} \leq \mu_{\text{Hilb}}$ of [CHHH14, Theorem 1.1], re-derived by the first and the third verification runs.

The searches were made on 14 October 2026, when the results were first obtained, by all three verification runs and when this note was written: arXiv API queries combining ball, sphere, ellipsoid, lattice points, empty sphere and Delaunay with normal polytope, integer decomposition property, integrally closed and very ample; Crossref for all DOIs; citing works in OpenAlex (nine for [BGM16], among them [Gub21]; none for [Gub21] and none for the report containing [Mic16]; three for [Gub23]; three for the whole report containing [Mic17]; and thirty-two for [CHHH14], none of them on balls);

zbMATH Open keyword queries; and seven web searches. Semantic Scholar returned no citing works even for [BGM16], so its empty answer for [Gub21] carries no information. Among the works found, the only one on lattice hulls of balls or ellipsoids after [BGM16] is [Gub21].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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