

Strong Approximate Innerness for Diagonal Graph Actions without Aperiodicity

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Abstract

Let a finite abelian group act on the C^* -algebra of a finite strongly connected graph that is not a directed cycle, fixing the vertex projections and multiplying the edge generators by characters. We show that the action is strongly approximately inner, without assuming either outerness of the action or aperiodicity of the graph. The analytic input is the aperiodic outer-action theorem of Barlak and Szabó. We remove the graph period by passing to a return-path corner and lifting fixed-point implementers through a finite frame. We then identify inner edge scalings with vertex coboundaries and use character extension to remove the inner kernel. A final block-diagonal polar correction recovers implementers fixed by the original action. Consequently, every finite abelian diagonal quasi-free action on a simple purely infinite Cuntz–Krieger algebra is strongly approximately inner, answering the question in Barlak’s contribution to Oberwolfach Report 42/2017.

1 Statement and the known aperiodic theorem

For an action $\alpha : G \curvearrowright \mathcal{A}$ of a finite abelian group on a separable unital C^* -algebra, write $\mathcal{A}^\alpha = \{a : \alpha_g(a) = a \text{ for all } g \in G\}$. The action is *strongly approximately inner* if for each $g \in G$ there are unitaries $u_{g,n} \in \mathcal{A}^\alpha$ such that

$$\lim_{n \rightarrow \infty} \|u_{g,n} a u_{g,n}^* - \alpha_g(a)\| = 0 \quad (a \in \mathcal{A}).$$

This is Izumi’s notion [5, Definition 3.6]; it does not require the $u_{g,n}$ to form approximate representations. An action is *outer* if α_g is not inner for every $g \neq 1$.

Let $E = (E^0, E^1, s, r)$ be a finite directed graph. Our convention for its generators is

$$s_e^* s_f = \delta_{e,f} p_{r(e)}, \quad p_v = \sum_{s(e)=v} s_e s_e^*.$$

All graphs used in the main argument have no sinks. A path $\mu = e_1 \cdots e_k$ has $r(e_i) = s(e_{i+1})$ and $s_\mu = s_{e_1} \cdots s_{e_k}$. Vertices are paths of length zero. We use the usual spanning family $s_\mu s_\nu^*$ with $r(\mu) = r(\nu)$ [7].

Theorem 1.1. *Let E be finite and strongly connected, and suppose that E is not a directed cycle. Let G be a finite abelian group. For each edge e , let $\chi_e : G \rightarrow \mathbb{T}$ be a character. The action on $\mathcal{A} = C^*(E)$ given by*

$$\alpha_g(p_v) = p_v, \quad \alpha_g(s_e) = \chi_e(g) s_e$$

is strongly approximately inner.

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The essential analytic input is the following result, which we use rather than reprove.

Theorem 1.2 (Barlak–Szabó, [2, Corollary 2.8]). *Let F be a finite strongly connected aperiodic graph with at least two edges. Every outer finite abelian action on $C^*(F)$ by diagonal quasi-free automorphisms is strongly approximately inner.*

Here aperiodicity means primitivity of the graph matrix; equivalently, all sufficiently large path lengths occur between every pair of vertices. Edge scalings are among the diagonal quasi-free automorphisms of this theorem. The question in [1, p. 2621, Question 1] asks for the conclusion on all simple purely infinite Cuntz–Krieger algebras. The theorem stated immediately after that question, and the published results [2, Corollaries 2.7–2.8], assume both aperiodicity and outerness. The purpose of this note is to supply the reductions that remove these hypotheses in the setting of Theorem 1.1. We retain explicit control of the fixed-point algebra throughout; ordinary approximate innerness would not suffice.

2 Fixed full corners

We first give two elementary corner facts, with the lifting formulas included. A projection q is *full* in a unital algebra B if the closed ideal generated by q is B . In that case there are finitely many $x_i \in Bq$ with

$$\sum_i x_i x_i^* = 1. \quad (1)$$

Indeed, fullness gives a finite sum $\sum_i y_i y_i^*$, $y_i \in Bq$, that is invertible; multiplication on the left by the inverse square root normalizes the sum. One way to obtain an invertible sum is to approximate 1 by an element $\sum_i a_i q b_i$ within distance less than one and use the operator matrix inequality $YY^* \leq \|[b_i]^T\|^2 \sum_i a_i q a_i^*$ for $Y = \sum_i a_i q b_i$.

Lemma 2.1. *Suppose q is full in a unital C^* -algebra $\mathcal{A}$, $\theta(q) = q$, and $\theta|_{q\mathcal{A}q}$ is inner. Then θ is inner.*

Proof. Let $u \in \mathcal{U}(q\mathcal{A}q)$ implement the restriction and choose $x_i \in \mathcal{A}q$ satisfying (1). Put

$$v = \sum_i \theta(x_i) u x_i^*.$$

Since $x_i^* x_j \in q\mathcal{A}q$ and $\theta(x_i^* x_j) = u x_i^* x_j u^*$, multiplication gives $v^* v = v v^* = 1$. For example,

$$v^* v = \sum_{i,j} x_i u^* \theta(x_i^* x_j) u x_j^* = \sum_{i,j} x_i x_i^* x_j x_j^* = 1.$$

Similarly, for $a \in \mathcal{A}$,

$$v a v^* = \sum_{i,j} \theta(x_i) \theta(x_i^* a x_j) \theta(x_j)^* = \theta(a).$$

□

Lemma 2.2. *Let $\alpha : G \curvearrowright \mathcal{A}$ be an action and let q be a projection full in $\mathcal{A}^\alpha$. If the restricted action on $q\mathcal{A}q$ is strongly approximately inner, then so is α .*

Proof. Choose a frame (1) with $x_i \in \mathcal{A}^\alpha q$. Fix g and choose $u_n \in \mathcal{U}(q\mathcal{A}^\alpha q)$ implementing α_g on $q\mathcal{A}q$. Since all $x_i^* x_j$ are fixed,

$$\|[u_n, x_i^* x_j]\| \longrightarrow 0.$$

Set $w_n = \sum_i x_i u_n x_i^* \in \mathcal{A}^\alpha$. Then

$$w_n w_n^* \longrightarrow 1, \quad w_n^* w_n \longrightarrow 1.$$

For the first limit, expand as $\sum_{i,j} x_i u_n (x_i^* x_j) u_n^* x_j^*$ and use the commutator limit; the second is identical with u_n^* . For every $a \in \mathcal{A}$,

$$\begin{aligned} w_n a w_n^* &= \sum_{i,j} x_i u_n (x_i^* a x_j) u_n^* x_j^* \\ &\longrightarrow \sum_{i,j} x_i \alpha_g(x_i^* a x_j) x_j^* = \alpha_g(a). \end{aligned}$$

For large n , w_n is invertible. Its polar unitary $v_n = w_n (w_n^* w_n)^{-1/2}$ lies in $\mathcal{A}^\alpha$ and satisfies $\|v_n - w_n\| \rightarrow 0$, so it has the required property. $\square$

The fullness assumption in the second lemma is in the *fixed algebra*, not merely in $\mathcal{A}$. When an action of a finite group on a simple unital algebra is outer, its fixed algebra is simple. This follows from Kishimoto's crossed-product simplicity theorem [6], since the fixed algebra is the corner determined by the averaging projection. See also [4, Section 2 and the proof of Proposition 4.2]. Thus every nonzero fixed projection is full in the fixed algebra in this situation.

3 Removing the graph period

Proposition 3.1. *Theorem 1.1 holds whenever the action is outer, with no aperiodicity assumption on E .*

Proof. Let d be the period of the strongly connected graph E , that is, the greatest common divisor of closed-walk lengths. Its cyclic decomposition is

$$E^0 = V_0 \sqcup \cdots \sqcup V_{d-1}, \quad s(e) \in V_j \implies r(e) \in V_{j+1 \bmod d}.$$

Let $q = \sum_{v \in V_0} p_v$. Define F to have vertices V_0 and one edge for each length- d path in E starting in V_0 , with the inherited initial and final vertices. The return graph F is strongly connected and aperiodic. For completeness, at any vertex the closed-walk lengths have gcd d . Finitely many such lengths already have this gcd, and the semigroup generated by their quotients by d contains every sufficiently large integer. Concatenating these closed walks with fixed connecting paths shows that, between any two vertices of V_0 , every sufficiently large multiple of d occurs in E . These are precisely path lengths in F multiplied by d .

The family

$$\{p_v : v \in V_0\} \cup \{s_\mu : |\mu| = d, s(\mu) \in V_0\}$$

satisfies the graph relations for F in $q\mathcal{A}q$. It generates this corner. To see surjectivity explicitly, consider a spanning monomial $s_\mu s_\nu^*$ of $q\mathcal{A}q$. Both initial vertices lie in V_0 , and the final vertices agree. Hence $|\mu| \equiv |\nu| \pmod{d}$. Choose $0 \leq t < d$ so that a length- t path from the common final vertex ends in V_0 . The graph relations yield

$$s_\mu s_\nu^* = \sum_{\substack{|\tau|=t \\ s(\tau)=r(\mu)}} s_{\mu\tau} s_{\nu\tau}^*. \quad (2)$$

Both extended paths have lengths divisible by d and decompose into edges of F .

The resulting homomorphism $C^*(F) \rightarrow q\mathcal{A}q$ is injective by gauge-invariant uniqueness. Indeed, if $\gamma_z(s_e) = z s_e$ is the original gauge action, every spanning monomial of $q\mathcal{A}q$ has degree divisible by d . Consequently

$$\eta_w = \gamma_z|_{q\mathcal{A}q} \quad \text{for } z^d = w$$

is a well-defined continuous circle action and $\eta_w(s_\mu) = w s_\mu$ for $|\mu| = d$. All vertex projections are nonzero. We have therefore identified $q\mathcal{A}q \cong C^*(F)$ equivariantly. The elementary gauge-invariant uniqueness theorem used here follows by averaging over the circle and checking injectivity on the finite-dimensional path-matrix blocks of the fixed core.

The algebra $\mathcal{A}$ is simple and purely infinite [7, Corollary 3.11]. In particular $q\mathcal{A}q$ is simple, so F cannot be the single-loop graph. Thus F has at least two edges. The restricted action is still diagonal, with the product of the edge characters along each length- d path. It is outer by Lemma 2.1, since q is full in $\mathcal{A}$. Barlak–Szabó’s theorem now applies to $C^*(F)$. Finally $\mathcal{A}^\alpha$ is simple, so q is full in it; Lemma 2.2 gives the assertion on $\mathcal{A}$. $\square$

4 Removing the inner kernel

We next characterize those edge scalings that are inner. Let D_E be the C^* -algebra generated by all path-range projections $s_\mu s_\mu^*$. Since E has no sinks and every cycle has an exit, D_E is a maximal abelian subalgebra and identifies canonically with $C(E^\infty)$; see [3, Section 2.1]. Strong connectivity and the exclusion of a directed cycle ensure the exit condition.

Lemma 4.1. *For $\lambda_e \in \mathbb{T}$, let $\theta(p_v) = p_v$ and $\theta(s_e) = \lambda_e s_e$. Then θ is inner if and only if there are $a_v \in \mathbb{T}$ with*

$$\lambda_e = a_{s(e)} \overline{a_{r(e)}} \quad (e \in E^1). \quad (3)$$

In that case $\sum_v a_v p_v$ implements θ . Fixing $a_{v_0} = 1$ at one base vertex makes the a_v unique.

Proof. Sufficiency follows directly from the graph relations. For necessity write $\theta = \text{Ad}(u)$. As θ fixes D_E pointwise, maximal abelianness gives $u \in D_E$. View u as a continuous function $f : E^\infty \rightarrow \mathbb{T}$. The identity $u s_e = \lambda_e s_e u$ says

$$f(ex) = \lambda_e f(x) \quad (s(x) = r(e)).$$

For any nonempty closed walk μ , evaluating at the periodic path $\mu\mu\cdots$ gives $\lambda_\mu := \prod_{e \text{ along } \mu} \lambda_e = 1$, with multiplicities in the product.

Choose a directed path ρ_v from v to v_0 , taking ρ_{v_0} empty, and put $a_v = \lambda_{\rho_v}$. Any two paths from v to v_0 have the same product: append the same path from v_0 back to v and use the closed-walk identity. Comparing $e\rho_{r(e)}$ with $\rho_{s(e)}$ proves (3). The ratio of any two vertex solutions is constant along edges and hence constant on the strongly connected graph. This gives uniqueness. $\square$

Proof of Theorem 1.1. Let

$$N = \{g \in G : \alpha_g \text{ is inner}\}.$$

It is a subgroup. For $n \in N$, Lemma 4.1 gives a unique normalized vertex unitary

$$U_n = \sum_v a_v(n) p_v, \quad a_{v_0}(n) = 1, \quad \alpha_n = \text{Ad}(U_n).$$

The uniqueness implies $U_{nm} = U_n U_m$, so each $a_v : N \rightarrow \mathbb{T}$ is a character. Every character of a subgroup of a finite abelian group extends to the group. To see this without assuming that N is a

direct factor, extend one generator at a time: if $g^k \in H$ and k is minimal, choose $z^k = a(g^k)$ and define $a(hg^j) = a(h)z^j$ on $\langle H, g \rangle$. This is well-defined and multiplicative.

Choose extensions $b_v : G \rightarrow \mathbb{T}$, with $b_{v_0} = 1$, and put $V_g = \sum_v b_v(g)p_v$. These unitaries form a representation, are fixed by every α_h , and have $V_n = U_n$ on N . Consequently

$$\beta_g = \text{Ad}(V_g^*) \circ \alpha_g$$

is an action, trivial on N . Its edge characters are

$$\chi'_e(g) = \chi_e(g) \overline{b_{s(e)}(g)} b_{r(e)}(g). \quad (4)$$

The induced action of G/N is outer: β_g is inner exactly when α_g is inner, that is, when $g \in N$. Proposition 3.1 supplies strong approximate innerness of this quotient action. The trivial quotient causes no difficulty; it can simply be implemented by 1.

Fix $g \in G$ and choose $u_n \in \mathcal{U}(\mathcal{A}^\beta)$ implementing β_g . Since $\beta_g(p_v) = p_v$, we have $\|u_n, p_v\| \rightarrow 0$ for every v . Therefore

$$z_n = \sum_v p_v u_n p_v \in \mathcal{A}^\beta \cap \{p_v : v \in E^0\}' \quad \text{satisfies} \quad \|z_n - u_n\| \rightarrow 0.$$

Its polar unitary $t_n = z_n(z_n^* z_n)^{-1/2}$ is defined for large n , lies in the same algebra, and still implements β_g in the point-norm limit. For every h ,

$$\alpha_h(t_n) = V_h \beta_h(t_n) V_h^* = t_n,$$

because t_n commutes with all vertex projections. Also $V_g \in \mathcal{A}^\alpha$. Thus $V_g t_n \in \mathcal{U}(\mathcal{A}^\alpha)$ and $\text{Ad}(V_g t_n) \rightarrow \alpha_g$, as required. $\square$

Remark 4.2. The last block-diagonal correction is necessary for this argument: a unitary fixed by β need not be fixed by α . Likewise, one must not apply simplicity of the fixed algebra directly to a nonouter action. We apply that fact only after removing N .

5 Cuntz–Krieger algebras and the source question

Corollary 5.1. *Let M be a finite zero-one matrix with no zero rows or columns and suppose $\mathcal{O}_M$ is simple and purely infinite. Every finite abelian action on $\mathcal{O}_M$ by diagonal quasi-free automorphisms is strongly approximately inner.*

Proof. First use the definition in [1, p. 2621]: the action preserves the span of the canonical generators T_i and fixes each $p_i = T_i T_i^*$. Multiplying the expansion of $\alpha_g(T_i)$ by p_i shows that $\alpha_g(T_i) = \chi_i(g) T_i$ for a character χ_i .

Realize $\mathcal{O}_M$ as the algebra of the graph with vertices i and edges (i, j) whenever $M_{ij} = 1$. The dictionary is

$$p_i = T_i T_i^*, \quad s_{ij} = T_i p_j, \quad T_i = \sum_{j: M_{ij}=1} s_{ij}.$$

The defining relations verify the two inverse homomorphisms. The action fixes vertices and scales s_{ij} by $\chi_i(g)$.

This graph has no sources or sinks. Simplicity forces it to be strongly connected. Here is a direct explanation of the latter finite-graph fact. If the condensation graph has more than one component, take a source component C . Since the original graph has no sources, C contains a cycle. Choose an

infinite periodic path x in that cycle and let X be its tail-equivalence class, allowing a shift in both paths. On $\ell^2(X)$ use the usual path representation: p_v selects paths starting at v , and s_e prepends e when composable. Removing the first edge gives the adjoint, and partitioning paths by their first edge proves the graph relations. This representation is nonzero. No path in X can start outside C , because there is no path from another component into C . Consequently every p_v outside C is in its kernel. These projections are nonzero in the universal graph algebra, as is seen in the path representation on all infinite paths. This contradicts simplicity.

Pure infiniteness excludes a directed cycle, whose graph algebra is a matrix algebra over $C(\mathbb{T})$. Theorem 1.1 now applies.

For the broader definition in [2, Definition 1.7], which fixes the domain projections instead, first use [2, Proposition 1.8(2)] to conjugate the abelian action to one that scales the canonical generators. Strong approximate innerness is preserved under conjugacy, so the same conclusion follows. $\square$

6 A nonsplit example and verification scope

Consider the graph with two edges $a, b : 0 \rightarrow 1$ and one edge $c : 1 \rightarrow 0$. It has period two; the corner at p_0 is generated by $s_a s_c$ and $s_b s_c$ and is a copy of $\mathcal{O}_2$. Let a generator g of $\mathbb{Z}/4\mathbb{Z}$ multiply all three edges by i . The inner kernel is $N = \{1, g^2\}$: the two length-two cycles have phase -1 under g , whereas $\alpha_{g^2} = \text{Ad}(p_0 - p_1)$. This subgroup is not a direct factor of $\mathbb{Z}/4\mathbb{Z}$. Taking $V_g = p_0 + ip_1$, the untwisted action multiplies a, b by -1 and fixes c . It factors through an outer $\mathbb{Z}/2\mathbb{Z}$ -action, as the proof requires.

The accompanying standard-library Python script checks the return-graph period calculation for all 149 strongly connected zero-one graphs on at most three vertices. It also checks the character-extension and kernel identities for 3,194 finite actions on specified small graphs, including actions of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the exact frame intertwining formula on 16 matrix units. These are finite checks of the algebraic reductions, not a verification of the C^* -norm arguments or a substitute for the proofs.

This note is an unrefereed, AI-assisted manuscript. AI tools were used in developing and checking the reductions, searching the literature, and preparing the text and supplementary computations. The aperiodic outer-action theorem is due to Barlak and Szabó; the standard graph and fixed-point results are cited above. We make no claim of formal verification, independent human review, or absolute priority.

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