When Is the Set of Non-Completable Partial Probability Tensors Convex?

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Abstract

In 2017 Kahle, Kubjas, Kummer and Rosen asked whether, for every pattern of observed entries, the set of nonnegative partial tensors that cannot be completed to the joint distribution of independent discrete random variables is convex. The answer is no, and a negative answer is already implicit in examples of Kubjas and Rosen. We determine exactly when the answer is yes. For tensors of format $d_1 \times \cdots \times d_n$ with $n \geq 2$ and all $d_j \geq 2$, the set is convex if and only if no observed entry is *pinned*, that is, if through every observed entry there is a maximal slice containing no other observed entry. In that case the completable region is the sublevel set $\{G_E \leq 0\}$ of an explicit concave function G_E. Otherwise convexity fails, and under the standing assumptions of Kahle et al. it fails even inside the simplex $\{\sum_e x_e \leq 1\}$. For matrices the condition says that every component of the bipartite graph of observed entries is a star. Under these standing assumptions, to which the question refers, the answer is yes for all patterns only in the formats $2 \times m$ and $m \times 2$; under the two conditions stated with the question alone, only in the format 2×2. For $n \geq 3$ the convex cases under the standing assumptions are exactly the $\prod_j d_j$ "corner" patterns, which include the running example of Kahle et al. For corner patterns we give a one-variable completability criterion, extending the known cubic criterion for $2 \times 2 \times 2$ tensors, and we show that the irreducible boundary hypersurface has degree at most $n(n-1)$. This is an unrefereed note.

1 Introduction

Let $n \geq 1$ and $d_1, \ldots, d_n \geq 1$, and let $D = [d_1] \times \cdots \times [d_n]$, where $[d] = \{1, \ldots, d\}$. A partial tensor with pattern $E \subseteq D$ is a vector $x = (x_e)_{e \in E} \in \mathbb{R}^E$. It is *completable* if there are probability vectors $\theta_j \in \Delta_{d_j - 1}$ such that $x_e = \theta^e := \prod_{j=1}^n \theta_{j,e_j}$ for all $e \in E$, that is, if x is the restriction of the joint distribution of n independent random variables. Write $\Delta = \Delta_{d_1-1} \times \cdots \times \Delta_{d_n-1}$, $p_E(\theta) = (\theta^e)_{e \in E}$, and

$$\mathcal{C}_E = p_E(\Delta), \qquad \mathcal{N}_E = \mathbb{R}^E_{\geq 0} \setminus \mathcal{C}_E, \qquad \Delta_E = \Big\{ x \in \mathbb{R}^E_{\geq 0} : \sum_{e \in E} x_e \leq 1 \Big\}.$$

Thus $\mathcal{C}_E$ is the compact *completable region* and $\mathcal{N}_E$ is the set of nonnegative partial tensors that are not completable. Clearly $\mathcal{C}_E \subseteq \Delta_E$.

Kubjas and Rosen [KR17] described $\mathcal{C}_E$ for all matrix patterns and for diagonal tensors. Kahle, Kubjas, Kummer and Rosen [KKKR17] studied the algebraic boundary of $\mathcal{C}_E$ under the following standing assumptions [KKKR17, Section 3]:

$$|E| = \sum_{j=1}^n (d_j - 1), \qquad E \text{ meets every maximal slice of } D, \qquad \mathcal{C}_E \text{ has nonempty interior.} \tag{1}$$

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Here a *maximal slice* is a set $S_{j,k} = \{i \in D : \ i_j = k\}$. In Mario Kummer's abstract of the joint work in Oberwolfach Report 20/2017 [Kum18], the authors state that they focus on the case where $|E| = \sum_j (d_j - 1)$ and $\mathcal{C}_E$ is full-dimensional, that is, on the first and third conditions of (1); the slice condition is added in [KKKR17, Section 3]. The abstract ends with two questions. The first asks for the degree of the boundary hypersurface. The second concerns the complement of $\mathcal{C}_E$. They report that it held in every example they computed and in the diagonal case [KR17]. In our notation it reads as follows.

Problem 1.1 ([Kum18, p. 1271])**.** *Is the set $\mathcal{N}_E$ of nonnegative partial tensors that are not completable to the joint distribution of independent random variables always a convex set?*

The problem is record OWR-15428-015 of the UnsolvedMath dataset [ula26], where it is listed as partially solved. The answer is no. Our main result says exactly when it is yes.

Definition 1.2. Let $E \subseteq D$. A maximal slice $S_{j,k}$ is *private* to $e \in E$ if $S_{j,k} \cap E = \{e\}$. An element $e \in E$ is *pinned* if none of the n maximal slices through e is private to it, that is, if for every $j \in [n]$ there is $f \in E \setminus \{e\}$ with $f_j = e_j$.

Theorem 1.3. *Let $n \geq 2$, $d_1, \dots, d_n \geq 2$ and $\emptyset \neq E \subseteq D$.*

- *(a) If no element of E is pinned, then $\mathcal{C}_E = \{x \in \mathbb{R}^E_{\geq 0} : \ G_E(x) \leq 0\}$ for the concave function G_E of (2). Hence $\mathcal{N}_E$ and $\mathcal{N}_E \cap \Delta_E$ are convex. Moreover, $\mathcal{C}_E$ has nonempty interior.*
- *(b) If some element of E is pinned, then $\mathcal{N}_E$ is not convex. If, in addition, E meets every maximal slice and $E \neq D$, then $\mathcal{N}_E \cap \Delta_E$ is not convex either.*

Part (b) needs no assumption on the interior. Under the assumptions (1) it produces two non-completable points in Δ_E whose midpoint is completable. So the failure of convexity is not caused by entries larger than 1. The assumption $n \geq 2$ is needed: for $n = 1$ and $E = [d_1]$, no element is pinned, but $\mathcal{N}_E = \{x \geq 0 : \ \sum x_e \neq 1\}$. For matrices the condition of Theorem 1.3 is graph-theoretic. The *bipartite graph* $G(S)$ of a pattern $S \subseteq [d_1] \times [d_2]$ has the rows and columns as vertices and an edge rc for each $(r, c) \in S$.

Corollary 1.4. *Let $d_1, d_2 \geq 2$ and $\emptyset \neq S \subseteq [d_1] \times [d_2]$. Then $\mathcal{N}_S$ is convex if and only if every connected component of $G(S)$ with at least one edge is a star.*

Under the standard assumptions (1) the convex cases are very rare. For $s \in D$, $j \in [n]$ and $k \in [d_j]$ let $s[j \leftarrow k]$ be the index obtained from s by replacing its j-th entry by k. The *corner pattern* at s consists of the cells that differ from s in exactly one coordinate:

$$E^*(s) = \left\{ s[j \leftarrow k] : \ j \in [n], \ k \in [d_j] \setminus \{s_j\} \right\}.$$

The running example $E = \{211, 121, 112\}$ of [KKKR17, Examples 3.4, 3.16, 3.17] is $E^*(111)$. For matrices with $d_1, d_2 \geq 3$, $E^*(s)$ is a "hook": a row and a column through s, with s removed.

Theorem 1.5. *Let $n \geq 2$, $d_1, \dots, d_n \geq 2$, and let $E \subseteq D$ satisfy $|E| = \sum_j (d_j - 1)$ and meet every maximal slice. Then E has no pinned element if and only if*

- *(i) $E = E^*(s)$ for some $s \in D$, or*
- *(ii) $n = 2$, $d_i = 2$ for some $i \in \{1, 2\}$, and every maximal slice $S_{3-i,k}$ contains exactly one element of E.*

In these cases $\mathcal{C}_E$ has nonempty interior and $\mathcal{N}_E$ is convex. In all other cases $\mathcal{N}_E$ and $\mathcal{N}_E \cap \Delta_E$ are not convex. If $n \geq 3$, or if $n = 2$ and $d_1, d_2 \geq 3$, then the $\prod_j d_j$ patterns $E^*(s)$ are pairwise distinct, and there are patterns satisfying all of (1) *with a pinned element. If $n = 2$ and $\min(d_1, d_2) = 2$, then every such E is of type (ii).*

So under the standing assumptions (1) of [KKKR17], to which the question refers, the answer to Problem 1.1 is "yes" exactly for the formats $2 \times m$ and $m \times 2$. For every other format it is "no". For $n \geq 3$, and for matrices with $d_1, d_2 \geq 3$, it is "yes" for exactly $\prod_j d_j$ patterns, which form one orbit under relabelling of the indices. This explains the evidence in [Kum18]: the diagonal patterns and the running example of [KKKR17] have no pinned element. The slice condition matters here. Under the two conditions stated in [Kum18] alone, the answer is "yes" for all patterns only in the format 2×2: in $2 \times m$ with $m \geq 3$ the pattern $\{11, 12, 21\} \cup \{(1, k) : \ 4 \leq k \leq m\}$ has full-dimensional $\mathcal{C}_E$ and the pinned entry 11 (Example 7.4). Explicit counterexamples in the smallest formats, 3×3 and $2 \times 2 \times 2$, are given in Section 7.

Relation to earlier work. The bare negative answer to Problem 1.1 is implicit in the paper [KR17] of Kubjas and Rosen, two of the proposers. We cite [KR17] by the numbering of the journal version; in arXiv:1407.3254v2 Theorem 4, Example 4, Theorem 7 and Example 9 are Theorem 2.12, Example 2.13, Theorem 4.1 and Example 4.2. Example 4 of [KR17] treats the 3×3 pattern $\{11, 12, 21, 33\}$, which satisfies (1). Its criterion $\sqrt{m_{11} + m_{12} + m_{21} + m_{12}m_{21}/m_{11}} + \sqrt{m_{33}} \leq 1$ is not monotone in m_{11}, and non-convexity follows in a few lines (Example 7.1). Likewise, [KR17, Example 9, 3(b)] gives the criterion for the $2 \times 2 \times 2$ pattern $\{000, 001, 110\}$ (indices in $\{0, 1\}$), which lies in the orbit of $\{111, 112, 222\}$ and satisfies (1). Part 3(a) of that example gives a non-monotone criterion for $\{000, 001, 010\}$, a pattern that misses a maximal slice. We found this consequence written down nowhere. For the corner pattern of $2 \times 2 \times 2$ tensors, the cubic criterion appears in [KR17, Example 9, 3(c)] and in [KKKR17, Example 3.17]. The latter also notes that the numerator of the last Sturm constant is a scalar multiple of the degree-6 boundary polynomial of [KKKR17, Example 3.16]. The function G_E below is modelled on the function f in the proof of [KR17, Theorem 7]. What we add is the characterization (Theorem 1.3 and Corollary 1.4), the classification under the standing assumptions (1) (Theorem 1.5), and the one-variable criterion and the degree bound for corner patterns of every format (Propositions 6.1 and 6.3).

2 Preliminaries

From now on, $n \geq 2$ and $d_1, \ldots, d_n \geq 2$. We write $\mathcal{P} = \{(j, k) : \ j \in [n], \ k \in [d_j]\}$ for the index set of the parameters $\theta_{j,k}$, $|v| = \sum_k v_k$ for a vector $v \geq 0$, ε_e for the unit vectors of $\mathbb{R}^E$, and Δ° for the set of $\theta \in \Delta$ with all entries positive. All points of $p_E(\Delta^\circ)$ have positive entries.

Lemma 2.1 (zeros)**.** *Let $e \in E$ be pinned, and for each j let $f^{(j)} \in E \setminus \{e\}$ satisfy $f^{(j)}_j = e_j$. If $y \in \mathbb{R}^E_{\geq 0}$ has $y_e = 0$ and $y_{f^{(j)}} > 0$ for all j, then $y \notin \mathcal{C}_E$.*

Proof. If $\theta \in \Delta$ completes y, then $\theta^e = 0$, so $\theta_{j,e_j} = 0$ for some j. This factor also divides $\theta^{f^{(j)}} = y_{f^{(j)}} > 0$, a contradiction. □

Lemma 2.2 (positivity)**.** *Suppose that E meets every maximal slice and $E \neq D$. If $x \in \mathcal{C}_E$ has positive entries, then every $\theta \in \Delta$ with $p_E(\theta) = x$ lies in Δ°, and $\sum_{e \in E} x_e < 1$.*

Proof. Each $\theta_{j,k}$ divides $\theta^f = x_f > 0$ for some $f \in E \cap S_{j,k}$, so $\theta > 0$. Then all entries of the probability tensor $\theta_1 \otimes \cdots \otimes \theta_n$ are positive, and the entries indexed by $D \setminus E \neq \emptyset$ carry positive mass. □

Lemma 2.3 (normalization)**.** *A point $x \in \mathbb{R}^E_{\geq 0}$ lies in $\mathcal{C}_E$ if and only if there are vectors $\theta_j \in \mathbb{R}^{d_j}_{\geq 0}$ with $\theta^e = x_e$ for all $e \in E$ and $\prod_j |\theta_j| = 1$.*

Proof. If $\prod_j |\theta_j| = 1$, all $|\theta_j|$ are positive, and the vectors $\theta_j/|\theta_j|$ lie in Δ_{d_j-1} and have the same products θ^e. The converse is trivial. □

Lemma 2.4 (finite fibres)**.** *Let $|E| = \sum_j (d_j - 1)$. If every $x \in p_E(\Delta^\circ)$ has only finitely many preimages in Δ°, then $\mathcal{C}_E$ has nonempty interior.*

Proof. Parametrize Δ° by the coordinates $\theta_{j,k}$ with $k < d_j$. It becomes an open subset U of $\mathbb{R}^m$, $m = |E|$, and p_E becomes a polynomial map $U \to \mathbb{R}^m$. Let r be the maximal rank of its Jacobian on U. The set where the rank equals r is open and nonempty. If $r < m$, the constant rank theorem [Lee13, Theorem 4.12] shows that near a point of this set the fibres of p_E are submanifolds of positive dimension $m - r$, hence infinite. So $r = m$, and by the inverse function theorem p_E maps a small neighbourhood of such a point onto an open subset of $\mathbb{R}^E$, which lies in $\mathcal{C}_E$. □

3 Pinned elements destroy convexity

Proposition 3.1. *If some $e \in E$ is pinned, then $\mathcal{N}_E$ is not convex.*

Proof. Choose $\theta^0 \in \Delta^\circ$ with $\theta^0_{j,e_j} = 1 - \frac{1}{2n}$ for every j; this is possible since $d_j \geq 2$. Put $x^0 = p_E(\theta^0)$. By Bernoulli's inequality, strict for $n \geq 2$, $x^0_e = (1 - \frac{1}{2n})^n > \frac{1}{2}$. Let $y^\pm = x^0 \pm x^0_e \varepsilon_e \in \mathbb{R}^E_{\geq 0}$. Then $y^- \notin \mathcal{C}_E$ by Lemma 2.1, since all other entries of y^- are positive, and $y^+ \notin \mathcal{C}_E$ because $y^+_e = 2x^0_e > 1$. Their midpoint x^0 lies in $\mathcal{C}_E$. □

Proposition 3.2. *Suppose that E meets every maximal slice, $E \neq D$, $\mathcal{C}_E$ has nonempty interior and some $e \in E$ is pinned. Then $\mathcal{N}_E \cap \Delta_E$ is not convex.*

Proof. Let x^0 be an interior point of $\mathcal{C}_E$. A neighbourhood of x^0 lies in $\mathcal{C}_E \subseteq \mathbb{R}^E_{\geq 0}$, so x^0 has positive entries. By Lemma 2.2, $\sigma := \sum_f x^0_f < 1$. For $0 \leq t \leq T := x^0_e + 1 - \sigma$ put $x(t) = x^0 + (t - x^0_e)\varepsilon_e$. These points lie in Δ_E, and $T > x^0_e$. The point $x(0)$ is not completable by Lemma 2.1. The point $x(T)$ has positive entries summing to 1, so it is not completable by Lemma 2.2. The set $I = \{t \in [0,T] :\ x(t) \in \mathcal{C}_E\}$ is closed and contains a neighbourhood of x^0_e. Let

$$\alpha = \inf\{t :\ [t, x^0_e] \subseteq I\}, \qquad \beta = \sup\{t :\ [x^0_e, t] \subseteq I\}.$$

Then $[\alpha, \beta] \subseteq I$, so $0 < \alpha < x^0_e < \beta < T$. If an interval $(\alpha - \eta, \alpha)$ with $\eta > 0$ were contained in I, then $[\alpha - \eta, x^0_e] \subseteq I$, contradicting the definition of α; similarly for $(\beta, \beta + \eta)$. So for $\eta = \min\{\alpha, T - \beta, \beta - \alpha\}$ there are $a \in (\alpha - \eta, \alpha)$ and $b \in (\beta, \beta + \eta)$ outside I. Then $x(a), x(b) \in \mathcal{N}_E \cap \Delta_E$, and $\alpha < (a+b)/2 < \beta$, so their midpoint $x((a+b)/2)$ lies in $\mathcal{C}_E$. □

Proposition 3.3. *If $E \neq D$ and $\mathcal{C}_E$ has empty interior, then $\mathcal{N}_E \cap \Delta_E$ is not convex.*

Proof. Let $x^0 = p_E(\theta^0)$ with $\theta^0 \in \Delta^\circ$. Its entries are positive and, since $E \neq D$, their sum is less than 1. So there is an open ball $B \subseteq \Delta_E$ centred at x^0. The closed sets $\mathcal{C}_E$ and $2x^0 - \mathcal{C}_E$ have empty interior, hence so does their union. Pick $y \in B$ outside both sets. Then y and $2x^0 - y$ lie in $\mathcal{N}_E \cap \Delta_E$, and their midpoint x^0 is completable. □

Proof of Theorem 1.3(b). The first statement is Proposition 3.1. The second follows from Proposition 3.2 if $\mathcal{C}_E$ has nonempty interior and from Proposition 3.3 otherwise. □

4 Without pinned elements the complement is convex

In this section E has no pinned element. For every $e \in E$ fix a coordinate $j(e)$ such that $S_{j(e),e_{j(e)}}$ is private to e, and put $\pi(e) = (j(e), e_{j(e)}) \in \mathcal{P}$. A private slice contains only one element of E, so π is injective. Let $\mathcal{R} = \mathcal{P} \setminus \pi(E)$ and $E_j = \{e \in E : \ j(e) = j\}$.

Lemma 4.1. *If $e \in E$ and $j \neq j(e)$, then $(j, e_j) \in \mathcal{R}$.*

Proof. Suppose $(j, e_j) = \pi(f)$. If $f = e$ then $j = j(e)$. If $f \neq e$, the slice $S_{j,e_j} = S_{j(f),f_{j(f)}}$ contains both e and f, so it is not private to f. $\square$

For $r = (r_q)_{q\in\mathcal{R}} \in (0,\infty)^{\mathcal{R}}$ and $e \in E$ put $M_e(r) = \prod_{j\neq j(e)} r_{(j,e_j)} > 0$; this is well defined by Lemma 4.1. For $x \in \mathbb{R}^E_{\geq 0}$ define

$$\Phi_x(r) = \prod_{j=1}^{n} \Big(\sum_{k:\,(j,k)\in\mathcal{R}} r_{(j,k)} + \sum_{e\in E_j} \frac{x_e}{M_e(r)} \Big), \qquad G_E(x) = \inf_{r\in(0,\infty)^{\mathcal{R}}} \log \Phi_x(r) \in [-\infty, \infty), \tag{2}$$

with $\log 0 = -\infty$. The meaning is as follows. Let $\theta(r,x)$ be the vectors $\theta_j \in \mathbb{R}^{d_j}_{\geq 0}$ with $\theta_q = r_q$ for $q \in \mathcal{R}$ and $\theta_{\pi(e)} = x_e/M_e(r)$ for $e \in E$. Then $\theta(r,x)^e = \theta_{\pi(e)} M_e(r) = x_e$ for every $e \in E$, and $\Phi_x(r) = \prod_j |\theta_j(r,x)|$.

Theorem 4.2. *Let E have no pinned element. Then:*

(i) G_E is concave on $\mathbb{R}^E_{\geq 0}$;

(ii) $G_E(cx) = G_E(x) + \log c$ for all $x \in \mathbb{R}^E_{\geq 0}$ and $c > 0$;

(iii) $\mathcal{C}_E = \{x \in \mathbb{R}^E_{\geq 0} : \ G_E(x) \leq 0\}$;

(iv) $G_E(x) \geq \log \sum_e x_e$, so G_E is finite and continuous on $(0,\infty)^E$, and $\mathcal{C}_E$ has nonempty interior.

In particular $\mathcal{N}_E = \{x \in \mathbb{R}^E_{\geq 0} : \ G_E(x) > 0\}$ and $\mathcal{N}_E \cap \Delta_E$ are convex. This proves Theorem 1.3(a).

Proof. (i) For fixed r, each factor in (2) is an affine function of x with nonnegative coefficients and nonnegative constant term. Its logarithm is concave as a function $\mathbb{R}^E_{\geq 0} \to [-\infty, \infty)$. So $\log \Phi_x(r)$ is concave in x, and so is the pointwise infimum G_E.

(ii) Let σ_c multiply every coordinate $r_{(1,k)}$ with $(1,k) \in \mathcal{R}$ by c and fix the other coordinates; this is a bijection of $(0,\infty)^{\mathcal{R}}$. If $j(e) = 1$, then M_e has no factor from coordinate 1, so $M_e(\sigma_c r) = M_e(r)$. If $j(e) \neq 1$, then M_e has exactly one such factor, $r_{(1,e_1)}$, which lies in $\mathcal{R}$ by Lemma 4.1; so $M_e(\sigma_c r) = cM_e(r)$. Hence the first factor of $\Phi_{cx}(\sigma_c r)$ is c times the first factor of $\Phi_x(r)$, and all other factors agree. So $\Phi_{cx}(\sigma_c r) = c\,\Phi_x(r)$, and (ii) follows. (If coordinate 1 has no parameter in $\mathcal{R}$, then σ_c is the identity and, by Lemma 4.1, $E_1 = E$; the argument is the same.)

$\mathcal{C}_E \subseteq \{G_E \leq 0\}$. Let $\theta \in \Delta$ complete x, and for $\epsilon > 0$ let $r^\epsilon_q = \theta_q + \epsilon$ for $q \in \mathcal{R}$. If $x_e > 0$, then $x_e = \theta_{\pi(e)} \prod_{j\neq j(e)} \theta_{j,e_j}$, so $x_e/M_e(r^\epsilon) \leq \theta_{\pi(e)}$; if $x_e = 0$ this is trivial. The parameters of coordinate j are split into those in $\mathcal{R}$ and the $\pi(e)$ with $e \in E_j$. So the j-th factor of $\Phi_x(r^\epsilon)$ is at most $|\theta_j| + d_j\epsilon = 1 + d_j\epsilon$, and $G_E(x) \leq \log \prod_j (1 + d_j\epsilon) \to 0$.

Positive x with $G_E(x) < 0$ lie in $\mathcal{C}_E$. Pick r^0 with $\Phi_x(r^0) < 1$. Fix $e \in E$ and a coordinate $i \neq j(e)$, which exists as $n \geq 2$. By Lemma 4.1, $q = (i, e_i) \in \mathcal{R}$. For $t \in (0,1]$ let r^t agree with r^0 except that $r^t_q = t\,r^0_q$, and let $\varphi_j(t)$ be the j-th factor of $\Phi_x(r^t)$. Each $M_f(r^t)$ is either $M_f(r^0)$ or $tM_f(r^0)$.

- For $j \neq i$, the parameters $r_{(j,k)}$ do not change and each $x_f/M_f(r^t)$ with $f \in E_j$ can only grow as t decreases, so $\varphi_j(t) \geq \varphi_j(1) > 0$ (each of the d_j terms of $\varphi_j(1)$ is positive). For $j = j(e)$ we even have $\varphi_{j(e)}(t) \geq x_e/(tM_e(r^0)) \to \infty$ as $t \to 0$.
- Coordinate i has a parameter $q' \neq q$, since $d_i \geq 2$. If $q' \in \mathcal{R}$, then $\varphi_i(t) \geq r^0_{q'}$. Otherwise $q' = \pi(f)$ with $j(f) = i$. Then M_f has no factor from coordinate i, so $\varphi_i(t) \geq x_f/M_f(r^0) > 0$.

So $\Phi_x(r^t) \to \infty$ as $t \to 0$. By continuity there is t with $\Phi_x(r^t) = 1$, and Lemma 2.3 applied to $\theta(r^t, x)$ gives $x \in \mathcal{C}_E$.

$\{G_E \leq 0\} \subseteq \mathcal{C}_E$. If $G_E(x) < 0$, pick r^0 with $\Phi_x(r^0) < 1$. For small $\delta > 0$ we still have $\Phi_{x+\delta\mathbf{1}}(r^0) < 1$, so $x + \delta\mathbf{1} \in \mathcal{C}_E$ by the previous step. Letting $\delta \to 0$ and using that $\mathcal{C}_E$ is closed gives $x \in \mathcal{C}_E$. If $G_E(x) = 0$, then $G_E(cx) = \log c < 0$ for $0 < c < 1$ by (ii), so $cx \in \mathcal{C}_E$; letting $c \to 1$ gives $x \in \mathcal{C}_E$. This proves (iii).

(iv) $\Phi_x(r) = \prod_j |\theta_j(r,x)|$ is the sum of all entries of $\theta_1 \otimes \cdots \otimes \theta_n$, and the entries indexed by E alone sum to $\sum_e x_e$. So $\Phi_x(r) \geq \sum_e x_e$. A concave function that is finite on an open convex set is continuous there [Roc70, Theorem 10.1]. Let $x^0 = p_E(\theta^0)$ with $\theta^0 \in \Delta^\circ$. Then $x^0 \in (0,\infty)^E$, and $G_E(x^0) \leq 0$ by (iii). By (ii), $G_E(x^0/2) \leq -\log 2 < 0$. By continuity $G_E < 0$ on a neighbourhood of $x^0/2$, and this neighbourhood lies in $\mathcal{C}_E$ by (iii).

Finally, if $G_E(y) > 0$ and $G_E(z) > 0$, then for $0 \leq \lambda \leq 1$ concavity gives $G_E(\lambda y + (1-\lambda)z) \geq \lambda G_E(y) + (1-\lambda)G_E(z) > 0$. So $\mathcal{N}_E = \{G_E > 0\}$ is convex, and so is its intersection with the convex set Δ_E. □

Remark 4.3. (1) With $r = e^u$, each factor in (2) is a sum of exponentials of affine functions of u with positive coefficients. So $\log \Phi_x(e^u)$ is convex in u, and $G_E(x)$ is the value of a geometric program [BKVH07]; completability of x can be decided by convex optimization. (2) For the diagonal pattern $E = \{(i,\ldots,i) : i \in [d]\}$ in $[d]^n$, choose $j(e) = 1$. Then Φ_x is unchanged if each vector $(r_{(j,i)})_i$, $j \geq 2$, is rescaled, and minimizing over probability vectors is exactly the minimization in the proof of [KR17, Theorem 7]. For $x \in (0,\infty)^E$ it gives $G_E(x) = n \log \sum_i x_{i\cdots i}^{1/n}$, so Theorem 4.2 recovers the criterion $\sum_i x_{i\cdots i}^{1/n} \leq 1$ of [KR17, Theorem 7] and [KKKR17, Theorem 4.1]. (3) The hypothesis $n \geq 2$ is used only in the step with the coordinate $i \neq j(e)$; the example $n = 1$, $E = [d_1]$ in the introduction shows that it cannot be dropped.

5 Matrices and the standing assumptions

Proof of Corollary 1.4. The entry $(r,c) \in S$ is pinned if and only if row r and column c both contain another element of S, that is, if both endpoints of the edge rc have degree at least 2 in $G(S)$. In a star no edge has this property. Conversely, let K be a component with at least one edge in which no edge has it. If some vertex v of K has degree at least 2, all its neighbours have degree 1, so K is the star centred at v; otherwise K is a single edge. Now apply Theorem 1.3. □

Proof of Theorem 1.5. Suppose that E has no pinned element. Every maximal slice meets E. Call it *private* if it contains exactly one element of E and *shared* otherwise. Every element of E lies in a private slice, and different elements lie in different private slices. So there are at least $|E| = \sum_j d_j - n$ private slices among the $\sum_j d_j$ maximal slices, and at most n shared ones.

If some coordinate j has no shared slice, the d_j slices $S_{j,k}$ partition E into singletons, so $|E| = d_j$ and $\sum_{i \neq j}(d_i - 1) = 1$. This is impossible for $n \geq 3$. For $n = 2$ it says that the other coordinate has $d_i = 2$, which is case (ii).

Otherwise every coordinate j has exactly one shared slice, S_{j,s_j} say, and there are exactly $|E|$ private slices, so each element e lies in exactly one private slice, $S_{j(e),e_{j(e)}}$. For $i \neq j(e)$ the slice

S_{i,e_i} is shared, so $e_i = s_i$; and $e_{j(e)} \neq s_{j(e)}$. Hence $e \in E^*(s)$ for $s = (s_1, \dots, s_n)$. Since $|E^*(s)| = \sum_j (d_j - 1) = |E|$, we get $E = E^*(s)$.

Conversely, the only element of $E^*(s)$ in a slice $S_{j,k}$ with $k \neq s_j$ is $s[j \leftarrow k]$, so $E^*(s)$ has no pinned element, and it meets every maximal slice. In case (ii) every element lies in a private slice of coordinate $3-i$. In both cases Theorem 1.3(a) gives the interior and the convexity. In all other cases E has a pinned element, and $E \neq D$ because $|E| < |D|$; so Theorem 1.3(b) applies.

If $n \geq 3$ or $d_1, d_2 \geq 3$, the slice S_{j,s_j} contains $\sum_{i \neq j}(d_i - 1) \geq 2$ elements of $E^*(s)$, while the other slices of coordinate j contain one each. So s is determined by $E^*(s)$. Patterns with a pinned element are given in Proposition 5.1. If $n = 2$ and $d_1 = 2$ (say), then $|E| = d_2$ and E meets each of the d_2 slices $S_{2,k}$, so each contains exactly one element of E. □

For $n = 2$ and $d_1 = 2$ the patterns of type (ii) are the $2^{d_2} - 2$ ways of placing one entry in each column without leaving a row empty. They include the corner patterns.

Proposition 5.1. *(a) Let $n \geq 3$ and $\mathbf{1} = (1, \dots, 1)$. The pattern $E' = \big(E^*(\mathbf{1}) \setminus \{\mathbf{1}[1 \leftarrow 2]\}\big) \cup \{(2, 2, 1, \dots, 1)\}$ satisfies* (1)*, and* $(1, 2, 1, \dots, 1)$ *is pinned.*

(b) Let $n = 2$ and $d_1, d_2 \geq 3$. The pattern $S = \{11, 12, 21, 33\} \cup \{(1, k) : \ 4 \leq k \leq d_2\} \cup \{(k, 1) : 4 \leq k \leq d_1\}$ satisfies (1)*, and* $(1,1)$ *is pinned.*

Proof. (a) Clearly $|E'| = |E^*(\mathbf{1})| = \sum_j (d_j - 1)$. The slice $S_{1,2}$ contains $(2, 2, 1, \dots, 1)$, the slices $S_{1,1}$ and $S_{2,1}$ contain $(1, 1, 2, 1, \dots, 1)$, and all other slices contain an element of $E^*(\mathbf{1})$ that was kept. The element $e = (1, 2, 1, \dots, 1)$ is pinned: $S_{1,1}$ contains $(1, 1, 2, 1, \dots, 1)$, and $S_{2,2}$ and $S_{j,1}$ for $j \geq 3$ contain $(2, 2, 1, \dots, 1)$. For the interior we use Lemma 2.4. Let $\theta \in \Delta^\circ$ complete x, and write $a_j = \theta_{j,1}$, $A = \prod_j a_j$, $\rho = x_{(2,2,1,\dots,1)}/x_{(1,2,1,\dots,1)}$, $X_j = \sum_{k \geq 2} x_{\mathbf{1}[j \leftarrow k]}$ for $j \geq 2$ and $X_1' = \sum_{k \geq 3} x_{\mathbf{1}[1 \leftarrow k]}$. From $x_{\mathbf{1}[j \leftarrow k]} = \theta_{j,k} A / a_j$ we get $\theta_{j,k} = x_{\mathbf{1}[j \leftarrow k]} a_j / A$ for the observed axis cells, and the entries at $(2, 2, 1, \dots, 1)$ and $(1, 2, 1, \dots, 1)$ give $\theta_{1,2} = \rho a_1$. The normalizations $|\theta_j| = 1$ then give $a_j = A/(A + X_j)$ for $j \geq 2$ and $a_1 = A/(A(1+\rho) + X_1')$, and $A = \prod_j a_j$ becomes

$$\big(A(1+\rho) + X_1'\big) \prod_{j \geq 2} (A + X_j) = A^{n-1}.$$

This is a polynomial equation of degree n in A with leading coefficient $1 + \rho \neq 0$. So A takes at most n values, and A determines θ.

(b) The count and the slice condition are immediate, and $(1,1)$ is pinned by $(1,2)$ and $(2,1)$. Let $(a, b) \in \Delta^\circ$ complete x and $t = a_1$. Row 1 and column 1 are observed except in position 3, so $b_k = x_{1k}/t$ and $a_k = t x_{k1}/x_{11}$ for $k \neq 3$. Hence $a_3 = 1 - t\alpha$ and $b_3 = 1 - \beta/t$ with $\alpha = \sum_{k \neq 3} x_{k1}/x_{11}$ and $\beta = \sum_{k \neq 3} x_{1k}$, and $a_3 b_3 = x_{33}$ becomes $-\alpha t^2 + (1 + \alpha\beta - x_{33})t - \beta = 0$, a nonzero quadratic. So there are at most two completions in Δ°, and Lemma 2.4 applies. (Nonempty interior also follows from [KR17, Theorem 4].) □

6 Corner patterns

Fix $s \in D$ and $E = E^*(s)$. For $x \in \mathbb{R}^E_{\geq 0}$ let $X_j = \sum_{k \neq s_j} x_{s[j \leftarrow k]}$ be the observed mass on the j-th axis through s, and for $A > 0$ put

$$h_X(A) = A \prod_{j=1}^{n} \Big(1 + \frac{X_j}{A}\Big) = \frac{\prod_j (A + X_j)}{A^{n-1}}, \qquad P_X(A) = \prod_{j=1}^{n} (A + X_j) - A^{n-1} = A^{n-1}\big(h_X(A) - 1\big).$$

Proposition 6.1. *We have $x \in \mathcal{C}_{E^*(s)}$ if and only if $\inf_{A>0} h_X(A) \le 1$. Suppose that at least two X_j are positive. Then $x \in \mathcal{C}_{E^*(s)}$ if and only if P_X has a root $A \in (0,1]$. For every such root,*

$$\theta_{j,s_j} = \frac{A}{A+X_j}, \qquad \theta_{j,k} = \frac{x_{s[j\leftarrow k]}}{A+X_j} \quad (k \ne s_j)$$

is a completion of x with $\theta^s = A$, and every completion of x arises in this way, from the root $A = \theta^s$.

Proof. Suppose first that at least two X_j are positive. Let $\theta \in \Delta$ complete x, and put $a_j = \theta_{j,s_j}$ and $A = \theta^s = \prod_j a_j$. If $a_i = 0$ for some i, every observed entry off the i-th axis vanishes, so $X_j = 0$ for all $j \ne i$, which is excluded. So $A > 0$. From $x_{s[j\leftarrow k]} = \theta_{j,k} A/a_j$ and $|\theta_j| = 1$ we get $a_j(1 + X_j/A) = 1$, which gives the displayed formulas. Then $A = \prod_j a_j = A^n / \prod_j (A + X_j)$ says $P_X(A) = 0$, and $A \le 1$. Conversely, if $P_X(A) = 0$ with $A > 0$, the displayed vectors are probability vectors, and

$$\theta^{s[j\leftarrow k]} = \frac{x_{s[j\leftarrow k]}}{A+X_j} \prod_{i\ne j} \frac{A}{A+X_i} = x_{s[j\leftarrow k]} \frac{A^{n-1}}{\prod_i (A+X_i)} = x_{s[j\leftarrow k]}, \qquad \theta^s = \frac{A^n}{\prod_i (A+X_i)} = A.$$

The function h_X is continuous, $h_X(A) \ge A$, and $h_X(A) \ge X_i X_j / A$ for two positive X_i, X_j. So $h_X \to \infty$ at 0 and at ∞, and $\inf h_X \le 1$ holds if and only if $h_X(A) = 1$ for some $A > 0$, that is, $P_X(A) = 0$; such a root satisfies $A \le h_X(A) = 1$.

If $X_j = 0$ for all $j \ne i$, then $h_X(A) = A + X_i$ has infimum X_i. A completion must satisfy $X_i = \sum_{k \ne s_i} \theta_{i,k} \prod_{l \ne i} a_l \le 1$; conversely, if $X_i \le 1$, then $\theta_{l,s_l} = 1$ for $l \ne i$, $\theta_{i,k} = x_{s[i\leftarrow k]}$ for $k \ne s_i$ and $\theta_{i,s_i} = 1 - X_i$ is a completion. In this degenerate case a completion need not come from a root of P_X in $(0,1]$: for $n = 2$, $X_1 = 1$ and $X_2 = 0$, every completion has $\theta_{1,s_1} = 0$, so $\theta^s = 0$, while $P_X(A) = A^2$. $\square$

Remark 6.2. (1) The function $G_{E^*(s)}$ of (2), with $j(e) = j$ for e on the j-th axis, is $\log \inf_A h_X(A)$: here $\mathcal{R} = \{(j, s_j)\}$ and $\Phi_x(r) = h_X(\prod_j r_{(j,s_j)})$. (2) For $n = 2$, $\inf_A h_X(A) = (\sqrt{X_1} + \sqrt{X_2})^2$, and the criterion $\sqrt{X_1} + \sqrt{X_2} \le 1$ is [KR17, Theorem 4] for the hook. (3) For $2 \times 2 \times 2$ tensors, $P_X(A) = A^3 + (e_1 - 1)A^2 + e_2 A + e_3$, where e_k are the elementary symmetric functions of the three observed entries. This is the criterion of [KR17, Example 9, 3(c)] and the cubic of [KKKR17, Example 3.17]. (4) In general $P_X(A) = A^n + (e_1(X) - 1)A^{n-1} + \sum_{k\ge 2} e_k(X) A^{n-k}$, and all coefficients except possibly that of A^{n-1} are nonnegative. By Descartes' rule of signs P_X has at most two positive roots, so if at least two X_j are positive, x has at most two completions. Compare [CRY24]: for log-linear models and suitable sets of observed entries, points with nonzero coordinates in the completable region have one or two completions.

The first question of [Kum18] asks for the degree of the irreducible boundary hypersurface H [KKKR17, Problem 3.15]. Record OWR-15428-014 of [ula26] states both questions of [Kum18], the degree of H and the convexity question. Its convexity half is answered by Theorem 1.3; its degree half remains open. For corner patterns we obtain an upper bound.

Proposition 6.3. *Let $E = E^*(s)$.*

(a) If $x \in (0,\infty)^E$ is a boundary point of $\mathcal{C}_E$, then P_X has a multiple root in $(0,1]$, so $\mathrm{Disc}_A(P_X) = 0$.

(b) As a polynomial in x, $\mathrm{Disc}_A(P_X)$ *has degree exactly $n(n-1)$; its homogeneous part of top degree is $\prod_{i<j}(X_i - X_j)^2$.*

(c) The irreducible polynomial f defining H in [KKKR17, Theorem 3.13] divides $\mathrm{Disc}_A(P_X)$*, so $\deg f \le n(n-1)$. For $d = (2,2,2)$,* $\mathrm{Disc}_A(P_X)$ *is exactly the degree-*6 *polynomial of [KKKR17, Example 3.16], and equality holds.*

Proof. (a) Let $\mu(X) = \inf_A h_X(A)$. The set $\{\mu < 1\} = \bigcup_A \{h_X(A) < 1\}$ is open and lies in $\mathcal{C}_E$ by Proposition 6.1. A boundary point x lies in the closed set $\mathcal{C}_E$ but not in this open set, so $\mu(X) = 1$. All X_j are positive because $x \in (0,\infty)^E$, so the infimum is attained at some $A_* > 0$. Then $P_X \geq 0$ on $(0,\infty)$ and $P_X(A_*) = 0$, so A_* is a multiple root, and $A_* \leq h_X(A_*) = 1$.

(b) $P_X(A) = A^n + \sum_{k=1}^n c_k A^{n-k}$ with $c_1 = e_1(X) - 1$ and $c_k = e_k(X)$ for $k \geq 2$. The discriminant of a monic polynomial of degree n is isobaric of weight $n(n-1)$ in its coefficients, c_k having weight k. Since c_k has degree k in X with top part $e_k(X)$, the part of degree $n(n-1)$ of $\mathrm{Disc}_A(P_X)$ is $\mathrm{Disc}_A \prod_j (A + X_j) = \prod_{i<j}(X_i - X_j)^2$, and there is no part of higher degree. The X_j are linear forms in disjoint sets of variables, so this is a nonzero polynomial of degree $n(n-1)$ in x.

(c) The pattern $E^*(s)$ satisfies (1) by Theorem 1.5. Let Z be the Zariski closure of the boundary $\partial\mathcal{C}_E$. By [KKKR17, Proposition 3.3, Theorem 3.13], Z has pure codimension one and consists of $V(f)$ and some coordinate hyperplanes. Here f is irreducible in $\mathbb{R}[x]$; in [KKKR17, Section 3] all ideals are ideals of real polynomials. Let $B_+ = \partial\mathcal{C}_E \cap (0,\infty)^E$. By Theorem 4.2(iii),(iv), G_E is finite and continuous on $(0,\infty)^E$, and $\mathcal{C}_E \cap (0,\infty)^E = \{x \in (0,\infty)^E : \; G_E(x) \leq 0\}$. Hence $B_+ = \{x \in (0,\infty)^E : \; G_E(x) = 0\}$: points with $G_E < 0$ are interior points of $\mathcal{C}_E$, points with $G_E > 0$ are not in $\mathcal{C}_E$, and if $G_E(x) = 0$, then $x \in \mathcal{C}_E$ while $G_E(cx) = \log c > 0$ for $c > 1$ by Theorem 4.2(ii). By Theorem 4.2(ii) again, B_+ is the radial graph $u \mapsto e^{-G_E(u)}u$ of the continuous function G_E over the open $(|E|-1)$-dimensional set $U = \{u \in (0,\infty)^E : \; \sum_e u_e = 1\}$ of directions; that is, the semialgebraic map $x \mapsto x/\sum_e x_e$ maps B_+ bijectively onto U. Since $\mathcal{C}_E$ is semialgebraic, so is B_+, and $\dim B_+ \geq \dim U = |E|-1$, because semialgebraic maps do not increase dimension [BCR98, Section 2.8]. The set B_+ avoids the coordinate hyperplanes, so $B_+ \subseteq V(f)$. Hence $|E|-1 \leq \dim B_+ \leq \dim V(f) \leq |E|-1$, the last inequality because $f \neq 0$. By [BCR98, Theorem 4.5.1], applied to the irreducible polynomial $f \in \mathbb{R}[x]$, the ideal (f) is real. By the real Nullstellensatz [BCR98, Section 4.1] it is the ideal of $V(f)$, so $V(f)$ is irreducible. Now $V(f) \subseteq Z \subseteq \overline{B_+} \cup \bigcup_e V(x_e)$, where the bar is the Zariski closure. Since $V(f)$ is irreducible and meets $(0,\infty)^E$, it is contained in $\overline{B_+}$, which lies in $V(\mathrm{Disc}_A(P_X))$ by (a). So $\mathrm{Disc}_A(P_X) \in (f)$. The statement for $d = (2,2,2)$ was checked by exact computation (Section 8); it re-confirms the observation in [KKKR17, Example 3.17] quoted in the introduction. $\square$

7 Explicit examples

Examples 7.1 and 7.2 satisfy (1). In both, all points have positive entries and coordinate sum less than 1. The midpoint is an interior point of $\mathcal{C}_E$, because the criterion is continuous on $(0,\infty)^E$ and holds strictly there. Both examples are direct consequences of the criteria of Kubjas and Rosen. Example 7.4 drops the slice condition.

Example 7.1 (3×3; [KR17, Example 4])**.** Let $E = \{11, 12, 21, 33\}$; the entry 11 is pinned. If (a,b) completes $x > 0$, then $a_2 b_2 = x_{12}x_{21}/x_{11}$, so $(a_1 + a_2)(b_1 + b_2) = B := x_{11} + x_{12} + x_{21} + x_{12}x_{21}/x_{11}$, and the Cauchy–Schwarz inequality gives $\sqrt{(a_1+a_2)(b_1+b_2)} + \sqrt{a_3 b_3} \leq 1$, that is, $\sqrt{B} + \sqrt{x_{33}} \leq 1$. By [KR17, Theorem 4] this condition is also sufficient. Fix $x_{12} = \frac{3}{16}$, $x_{21} = \frac{1}{8}$, $x_{33} = \frac{1}{32}$.

- $x_{11} = \frac{1}{4}$: $x = p_E(\theta)$ for $\theta_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $\theta_2 = (\frac{1}{2}, \frac{3}{8}, \frac{1}{8})$; here $B = \frac{21}{32}$ and $\sqrt{B} + \sqrt{x_{33}} < 1$.
- $x_{11} = \frac{1}{20}$ and $x_{11} = \frac{9}{20}$: $B = \frac{133}{160}$ and $B = \frac{391}{480}$, and in both cases $\sqrt{B} + \sqrt{x_{33}} > 1$. The coordinate sums are $\frac{63}{160}$ and $\frac{127}{160}$.

Since $\frac{1}{4}$ is the midpoint of $\frac{1}{20}$ and $\frac{9}{20}$, the set $\mathcal{N}_E \cap \Delta_E$ is not convex. On this line the completable values of x_{11} form the interval between the roots $0.0830\ldots$ and $0.2821\ldots$ of $x_{11}^2 - (\frac{23}{32} - \frac{\sqrt{2}}{4})x_{11} + \frac{3}{128}$.

Example 7.2 ($2 \times 2 \times 2$; [KR17, Example 9, 3(b)])**.** Let $E = \{111, 112, 222\}$; the entry 112 is pinned. If θ completes $x > 0$, then $\theta_{1,1}\theta_{2,1} = P := x_{111} + x_{112}$, $\theta_{3,2} = x_{112}/P$ and $\theta_{1,2}\theta_{2,2} = Q := x_{222}P/x_{112}$, and the Cauchy–Schwarz inequality gives $\sqrt{P}+\sqrt{Q}\le 1$. Flipping the third index maps E to the pattern $\{000,001,110\}$ of [KR17, Example 9, 3(b)], whose criterion shows that this condition is also sufficient. Put $x_{111}=x_{222}=\frac{3}{32}$.

- $x_{112}=\frac{9}{32}$: $x=p_E(\theta)$ for $\theta_1=(\frac{1}{2},\frac{1}{2})$, $\theta_2=(\frac{3}{4},\frac{1}{4})$, $\theta_3=(\frac{1}{4},\frac{3}{4})$; here $P=\frac{3}{8}$, $Q=\frac{1}{8}$ and $\sqrt{P}+\sqrt{Q}<1$.
- $x_{112}=\frac{1}{100}$ and $x_{112}=\frac{221}{400}$: $(P,Q)=(\frac{83}{800},\frac{249}{256})$ and $(\frac{517}{800},\frac{1551}{14144})$, and in both cases $\sqrt{P}+\sqrt{Q}>1$. The coordinate sums are $\frac{79}{400}$ and $\frac{37}{50}$.

Since $\frac{9}{32}$ is the midpoint of $\frac{1}{100}$ and $\frac{221}{400}$, the set $\mathcal{N}_E\cap\Delta_E$ is not convex.

Remark 7.3. In Example 7.1 the unobserved entry $x_{22}=x_{12}x_{21}/x_{11}$ is determined by the observed ones, and in Example 7.2 so is $x_{221}=x_{111}x_{222}/x_{112}$. Nothing in (1) excludes such patterns, and Theorem 1.3 does not depend on it. Pinned patterns without such entries also exist. The notion of a finitely completable entry is from [KKKR17, Section 2 (Propositions 2.8 and 2.9, Example 2.11)]; for generic data these entries form the closure of E in a matroid, which may be taken to be the linear matroid of the exponent vectors. Accordingly, we call $i\in D\setminus E$ *finitely completable* if the exponent vector of the monomial θ^i lies in the rational span of those of the θ^e, $e\in E$, that is, if i lies in the closure of E in this linear matroid. An exact computation shows that in the format $2\times2\times3$, exactly 12 of the 228 pinned patterns satisfying (1) have no finitely completable entry. An example is $\{111,122,212,223\}$, in which 122 and 212 are pinned. In the formats $2\times2\times2\times2$ and $2\times3\times3$ the counts are 64 of 1240 and 72 of 4032 pinned patterns certified to satisfy (1); in $2\times2\times2$ there are none.

Example 7.4 ($2\times m$ without the slice condition)**.** Let $d=(2,m)$ with $m\ge3$ and $E=\{11,12,21\}\cup\{(1,k):\ 4\le k\le m\}$. Then $|E|=m=(d_1-1)+(d_2-1)$, but E misses the slice $S_{2,3}$, and 11 is pinned. Let $x\in\mathbb{R}^E_{\ge0}$ with $x_{11},x_{21}>0$, and put

$$B=x_{11}+x_{21}+\Big(x_{12}+\sum_{k\ge4}x_{1k}\Big)\frac{x_{11}+x_{21}}{x_{11}}.$$

If (a,b) completes x, then $a_1+a_2=1$ gives $b_1=x_{11}+x_{21}$ and $a_1=x_{11}/b_1$; then $b_k=x_{1k}/a_1$ for $k\ne1,3$, and $b_3=1-B$. Conversely, if $B\le1$ these formulas give a completion. So x has at most one completion, and it is completable if and only if $B\le1$. In particular Lemma 2.4 applies, and $\mathcal{C}_E$ has nonempty interior. Thus E satisfies the two conditions stated in [Kum18]. Put $x_{12}=x_{21}=\frac{1}{10}$ and $x_{1k}=0$ for $k\ge4$, so that $B=x_{11}+x_{12}+x_{21}+x_{12}x_{21}/x_{11}$.

- $x_{11}=\frac{2}{5}$: $B=\frac{5}{8}$, and $a=(\frac{4}{5},\frac{1}{5})$, $b=(\frac{1}{2},\frac{1}{8},\frac{3}{8},0,\dots,0)$ is a completion.
- $x_{11}=\frac{1}{200}$ and $x_{11}=\frac{159}{200}$: $B=\frac{441}{200}$ and $B=\frac{32041}{31800}$, both larger than 1. The coordinate sums are $\frac{41}{200}$ and $\frac{199}{200}$.

Since $\frac{2}{5}$ is the midpoint of $\frac{1}{200}$ and $\frac{159}{200}$, the set $\mathcal{N}_E\cap\Delta_E$ is not convex. This example is due to the second independent referee (Section 8). The patterns of Proposition 5.1 satisfy (1) and have a pinned element, and in the format 2×2 no pattern of size 2 has a pinned element, since a pinned entry needs another entry in its row and another in its column. So under the two conditions $|E|=\sum_j(d_j-1)$ and $\operatorname{int}\mathcal{C}_E\ne\emptyset$ alone, the answer to Problem 1.1 is "yes" for all such patterns only in the format 2×2, both for $\mathcal{N}_E$ and for $\mathcal{N}_E\cap\Delta_E$.

8 Concluding remarks

Problem 1.1 has a negative answer, as could already be read off from [KR17], and the convex cases are exactly the patterns without pinned elements. Under the assumptions (1) with $n \geq 3$ these are only the corner patterns, the case studied in [KKKR17, Examples 3.4, 3.16, 3.17]. The degree question [KKKR17, Problem 3.15], which is the other half of record OWR-15428-014 of [ula26], remains open. Proposition 6.3 bounds the degree by $n(n-1)$ for corner patterns only.

Verification. The claims were checked as follows.

1. The finder's programs are included in the release. `witnesses_exact.py` checks Examples 7.1 and 7.2, a $3 \times 3 \times 2$ variant and the witness of Proposition 3.1 in rational arithmetic. `census.py` enumerates all patterns with $|E| = \sum_j (d_j - 1)$ that meet every maximal slice, in 17 formats up to $3 \times 3 \times 3$, $2 \times 3 \times 4$ and 2^5, and confirms Theorem 1.5 in all of them; nonempty interior is certified by a Jacobian determinant that is nonzero modulo a 61-bit prime at an integer point. For example, $2 \times 2 \times 2$ has 32 such patterns, 8 corner patterns and 24 pinned ones; in $3 \times 3 \times 3$, 141,723 of the 153,828 patterns are certified full-dimensional, and 27 of these are unpinned. `families_check.py` checks Proposition 5.1 in 20 formats, and `kkkr_boundary_check.py` checks in integer arithmetic that $\mathrm{Disc}_A(P_X)$ equals the polynomial of [KKKR17, Example 3.16] term by term. `numeric_tests.py` gives numerical evidence only: the sign of G_E agrees with a least-squares completion oracle for 10 unpinned patterns, random segments show no violation of convexity, and the witnesses of Proposition 3.2 are found for all 24 pinned $2 \times 2 \times 2$ patterns and for 10 random pinned $3 \times 3 \times 3$ patterns.

2. The author's program `audit_checks.py` (standard library, exact arithmetic, a few seconds) checks the numbers in Examples 7.1 and 7.2, the completion formula of Proposition 6.1 and the equations in the proof of Proposition 5.1 at random rational points. With its own exact test for nonempty interior it reproduces the numbers of pinned full-dimensional patterns in $2 \times 2 \times 2$, $2 \times 2 \times 3$, $2 \times 3 \times 3$ and 2^4, and it gives the counts of Remark 7.3. The author's program `revision2_checks.py` (standard library, exact) checks Example 7.4 for $3 \leq m \leq 8$, the midpoint in Example 7.2 and the degenerate example after Proposition 6.1.

3. Two independent verifiers checked the proofs of an earlier version by hand, reran the finder's programs and wrote separate code. (The proofs of Lemma 2.4 and Proposition 3.3, part (c) of Proposition 6.3 and Remark 7.3 were written afterwards; see item 4.) Their code comprises an exact completability oracle for Example 7.1; a census with a floating-point Jacobian rank in 8 formats, which matches `census.py`; an independent transcription of the polynomial of [KKKR17, Example 3.16], which equals $\mathrm{Disc}_A(P_X)$ at 200 random rational points; a test of Proposition 6.1 for $2 \times 2 \times 2$ against a least-squares oracle (150 agreements, no disagreement); and a search on random lines for completable points between two non-completable ones, which found none for 9 unpinned patterns and found such points for pinned control patterns.

4. A second independent referee checked the previous version of this note line by line, including Lemma 2.4, Proposition 3.3, part (c) of Proposition 6.3 and Remark 7.3, and found no error affecting a theorem. The referee's programs, written from the text before the finder's programs were read, are in the folder `referee2` of the release. They reproduce the census of Theorem 1.5 in 20 formats with separate Jacobian certificates, the counts of Remark 7.3, the numbers of Examples 7.1 and 7.2 in exact arithmetic, and the identity of $\mathrm{Disc}_A(P_X)$ with the polynomial of [KKKR17, Example 3.16] term by term. A numerical test of Theorem 4.2 gave 163 sign agreements and no disagreement. The referee pointed out the scope issue shown by Example 7.4

and an under-justified dimension step in part (c) of Proposition 6.3, which was rewritten as the referee suggested. That step, the text of Example 7.4, the degenerate example after Proposition 6.1 and Remark 6.2(4) were written after the referee's report and were checked by the author only.

Scope and priority. Theorems 1.3 and 1.5 answer Problem 1.1 as posed in [Kum18], both for $\mathcal{N}_E$ and inside the simplex Δ_E, and both for all patterns and within the standing assumptions (1) of [KKKR17]. Under the two conditions stated in [Kum18] without the slice condition, Example 7.4 shows that "always convex" holds only in the format 2×2. The negative answer itself is not new in substance: it follows in a few lines from [KR17, Example 4 and Example 9, 3(b)], which predate the question and are due to two of its proposers. We found it written down nowhere, and it may be folklore. The cubic criterion for the $2 \times 2 \times 2$ corner pattern and its relation to the degree-6 boundary polynomial are due to [KR17, KKKR17]. We did not find the characterization by pinned elements, the classification of Theorem 1.5, or the results of Section 6 for general corner patterns in the literature. We read [Kum18], arXiv:1605.01678v2 (whose numbering we use for [KKKR17]), arXiv:1407.3254v2 and the journal version of [KR17]. We searched arXiv, OpenAlex (all works citing [KKKR17] or [KR17], among them [CRY24]), zbMATH and the web, and found no work that answers or revisits Problem 1.1. The second referee repeated these searches independently and found no such work either. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples, the characterization and the proofs, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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