

Partial Results on the Degree of the Algebraic Boundary in Rank-One Tensor Completion

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Abstract

Kahle, Kubjas, Kummer and Rosen showed that for a set E of observed entries of a $d_1 \times \cdots \times d_n$ tensor with $|E| = \sum_j (d_j - 1)$ that meets every maximal slice and has a full-dimensional completable region (we call such E admissible), the algebraic boundary of the set of partial tensors that are restrictions of joint distributions of independent random variables consists of coordinate hyperplanes and one irreducible hypersurface H . They asked for $\deg H$ as a function of $n, d_1, \dots, d_n$ and E . We give a partial answer. For the corner patterns, the cells that differ from a fixed cell in exactly one coordinate, H is defined by an explicit irreducible discriminant and $\deg H = n(n-1)$ in every format. In general $\deg H \leq m! / \prod_j (d_j - 1)!$ with $m = |E|$, with equality for some patterns. We call E saturated if the maximal minors of its incidence matrix are coprime; this is weaker than unimodularity of that matrix. If E is saturated, the complex completions of a generic partial tensor are the W roots of one Laurent polynomial equation $\Phi_x(\tau) = 1$, and we prove $q \deg H = W^* + \deg \text{Den}$, where $W^* \leq W$ counts the critical values of Φ_x that are not identically zero, $q \geq 1$, and Den is a constant multiple of the q -th power of the lowest-degree form of the equation of H . For $W = 2$ this gives an explicit formula; for matrices $\deg H = 2 + 2p$, where p is the number of observed entries that share their row and their column with other observed entries. For saturated patterns in binary formats we obtain $\deg H \leq W + \sum_e \mu_e$ with explicit tropical multiplicities. With exact computer certificates this determines $\deg H$ for all admissible patterns in the six formats $2 \times 2, 3 \times 3, 3 \times 4, 2 \times 2 \times 2, 2 \times 2 \times 3$ and $2 \times 2 \times 2 \times 2$. In a modular census of 2343 patterns, which is not a proof, all saturated patterns have $q = 1$ and $W^* = W$, and the tropical bound is sharp; we do not prove this in general. For patterns that are not saturated we have only the general bound and one certified example. This is an unrefereed note.

1 Introduction

Let $n \geq 2$ and $d_1, \dots, d_n \geq 2$ be integers, $[d] = \{1, \dots, d\}$ and $D = [d_1] \times \cdots \times [d_n]$. The *maximal slices* are the sets $S_{j,k} = \{\kappa \in D : \kappa_j = k\}$, indexed by $\mathcal{N} = \{(j, k) : j \in [n], k \in [d_j]\}$. For $\kappa \in D$ let $\kappa[j \leftarrow k]$ be obtained from κ by replacing its j -th entry by k . A vector $\theta \in \mathbb{C}^{\mathcal{N}}$ consists of blocks $\theta_j = (\theta_{j,1}, \dots, \theta_{j,d_j})$; we write $|\theta_j| = \sum_k \theta_{j,k}$ and $\theta^\kappa = \prod_j \theta_{j,\kappa_j}$. For a set $E \subseteq D$ of observed positions let $p_E(\theta) = (\theta^e)_{e \in E}$, let $\Delta = \Delta_{d_1-1} \times \cdots \times \Delta_{d_n-1}$ be the product of the probability simplices, and let $\mathcal{C}_E = p_E(\Delta) \subseteq \mathbb{R}^E$. So $\mathcal{C}_E$ consists of the partial tensors that are restrictions of the joint distribution of n independent discrete random variables. Kahle, Kubjas, Kummer and Rosen [KKKR17, Section 3] study $\mathcal{C}_E$ under the assumptions

$$|E| = m := \sum_{j=1}^n (d_j - 1), \quad E \text{ meets every maximal slice}, \quad \mathcal{C}_E \text{ has nonempty interior.} \quad (1)$$

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We call such E *admissible*. By [KKKR17, Theorem 3.13] the algebraic boundary of $\mathcal{C}_E$, that is, the Zariski closure of its Euclidean boundary, is the union of some coordinate hyperplanes and one irreducible hypersurface H . Its equation f generates the ideal obtained by eliminating θ from the ideal generated by the $x_e - \theta^e$ and a linear form $l_E(\theta)$ that divides the Jacobian determinant of p_E [KKKR17, Propositions 3.8 and 3.12]. That ideal is prime over $\mathbb{C}$, since its quotient ring is a polynomial ring (Lemma 2.2(b)), so f is irreducible over $\mathbb{C}$; we take $f \in \mathbb{Z}[x]$ primitive and regard $H = V(f)$ as a complex hypersurface in $\mathbb{C}^E$.

In Mario Kummer’s abstract of this work in Oberwolfach Report 20/2017 [Kum18], the first of two closing questions reads “What is the degree of the hypersurface H ?”; the authors expect it to be approachable using toric or tropical geometry. The abstract states only two of the conditions in (1), $|E| = \sum_j (d_j - 1)$ and full-dimensionality of $\mathcal{C}_E$; the condition that E meets every maximal slice is a standing assumption of [KKKR17, Section 3], as discussed in [Fer26]. Problem 3.15 of [KKKR17] asks for $\deg f$ as a function of $n, d_1, \dots, d_n$ and E . Under the assumptions (1), the only value computed there is $\deg f = 6$ for the pattern $\{211, 121, 112\}$ of $2 \times 2 \times 2$ tensors [KKKR17, Example 3.16]; outside them, degrees are known for diagonal patterns (see “Relation to earlier work” below). The question is part of record OWR-15428-014 of the UnsolvedMath dataset [ula26], which states both closing questions of [Kum18]; the second one, on convexity, is answered in the companion note [Fer26]. We give a partial answer to the first one. The degree genuinely depends on E : in the format $2 \times 2 \times 2$ the two orbits of admissible patterns have degrees 4 and 6, and in $2 \times 2 \times 2 \times 2$ the eleven orbits have degrees 6, 6, 6, 8, 8, 9, 11, 12, 12, 14, 18 (Proposition 7.1).

Corner patterns. For $s \in D$ the *corner pattern* $E^*(s) = \{s[j \leftarrow k] : j \in [n], k \in [d_j] \setminus \{s_j\}\}$ consists of the cells that differ from s in exactly one coordinate. It is admissible, and for $n \geq 3$ the corner patterns are exactly the admissible patterns for which the complement of $\mathcal{C}_E$ is convex [Fer26, Theorem 1.5]. The running example of [KKKR17] is $E^*(111)$.

Theorem 1.1. *Let $s \in D$ and $E = E^*(s)$. For $x \in \mathbb{C}^E$ and $j \in [n]$ let $X_j = \sum_{k \neq s_j} x_{s[j \leftarrow k]}$ be the sum of the observed entries on the j -th axis through s , and let*

$$\mathcal{D}_n(X_1, \dots, X_n) = \text{Disc}_A \left(\prod_{j=1}^n (A + X_j) - A^{n-1} \right).$$

Then $f = c \mathcal{D}_n(X_1, \dots, X_n)$ for some $c \in \mathbb{C}^$, and $\deg f = n(n-1)$, whatever $d_1, \dots, d_n$ are. The homogeneous components of $\mathcal{D}_n$ of the largest and the smallest degree are $\prod_{i < j} (X_i - X_j)^2$, of degree $n(n-1)$, and $\pm(n-1)^{n-1} (X_1 \cdots X_n)^{n-2}$, of degree $n(n-2)$.*

For $n = 2$ the polynomial is $(X_1 + X_2 - 1)^2 - 4X_1X_2$, which describes the criterion $\sqrt{X_1} + \sqrt{X_2} \leq 1$ of [KR17, Theorem 4]; for $d = (2, 2, 2)$ it is the 38-term sextic of [KKKR17, Example 3.16]. The companion note showed that f divides $\mathcal{D}_n(X)$, so $\deg f \leq n(n-1)$ [Fer26, Proposition 6.3]. The new content of Theorem 1.1 is that $\mathcal{D}_n(X_1, \dots, X_n)$ is irreducible, which gives equality for every n and every format.

General patterns. Let $B \in \{0, 1\}^{E \times \mathcal{N}}$ be the incidence matrix, $B_{e, (j, k)} = 1$ if and only if $e_j = k$. We show in Section 2 that for admissible E the kernel of B is the direct sum of the $(n-1)$ -dimensional space of vectors that are constant on each block with block values summing to 0 and the line spanned by one integer vector w . The *cell weights* $w(\kappa) = \sum_j w_{j, \kappa_j}$ are integers that vanish on E and generate $\mathbb{Z}$, and we put

$$W = \max_{\kappa \in D} w(\kappa) - \min_{\kappa \in D} w(\kappa).$$

We call E *saturated* if $B\mathbb{Z}^N = \mathbb{Z}^E$. For admissible E the matrix B has rank $m = |E|$ (Lemma 2.1), and E is saturated if and only if the maximal minors of B have greatest common divisor 1, that is, if and only if the lattice spanned by the rows of B is saturated in $\mathbb{Z}^N$. These rows are the columns of the matrix $A_E = B^T$ of [KKKR17], so this is the saturation condition of [KKKR17, Corollary 2.10]. We avoid the word “unimodular”: a saturated B need not be unimodular in the usual sense, where all maximal minors lie in $\{0, \pm 1\}$. For example, the incidence matrices of the corner patterns of $2 \times 2 \times 2$ and of $2 \times 2 \times 2 \times 2$ have maximal minors ± 2 and ± 3 , respectively. All corner patterns and all admissible matrix patterns are saturated (Lemma 2.4). For saturated E fix an integer matrix $R \in \mathbb{Z}^{N \times E}$ with $BR = I$ and let $r_\kappa = \sum_j R_{(j, \kappa_j)} \in \mathbb{Z}^E$, where $R_{(j, k)}$ is the row of R indexed by (j, k) . The *fibre polynomial* is the Laurent polynomial

$$\Phi_x(\tau) = \sum_{\kappa \in D} x^{r_\kappa} \tau^{w(\kappa)}.$$

Its critical values are the values $\gamma_1(x), \dots, \gamma_W(x)$ of Φ_x at the W zeros of $\tau d\Phi_x/d\tau$ in $\mathbb{C}^*$. For $j \in [n]$ let U_j^+ and U_j^- be the sums of the Laurent monomials $x^{R_{(j, k)}}$ over the $k \in [d_j]$ at which $w_{j, k}$ is maximal and minimal, respectively. We write f_{low} for the homogeneous component of f of the smallest degree.

Theorem 1.2. *Let E be admissible and saturated.*

- (a) *$W \geq 2$. For x in a dense open subset of $\mathbb{C}^E$, the complex completions of x , i.e. the $\theta \in \mathbb{C}^N$ with $|\theta_j| = 1$ for all j and $p_E(\theta) = x$, correspond bijectively to the W roots of $\Phi_x(\tau) = 1$ in $\mathbb{C}^*$. A point $x \in (0, \infty)^E$ lies in $\mathcal{C}_E$ if and only if $\min_{\tau > 0} \Phi_x(\tau) \leq 1$.*
- (b) *Let W^* be the number of critical values that are not identically zero, and let Den be the least common multiple of the reduced denominators of the elementary symmetric functions of $\gamma_1, \dots, \gamma_W$. Then $\text{Den} \cdot \prod_i (1 - \gamma_i) = cf^q$ for some $c \in \mathbb{C}^*$ and some integer $q \geq 1$, $\text{Den} = c' f_{\text{low}}^q$ with $c' \in \mathbb{C}^*$, and*

$$q \deg f = W^* + \deg \text{Den}, \quad \text{equivalently} \quad \deg f - \deg f_{\text{low}} = W^*/q.$$

Here W^/q is the number of distinct nonzero critical values of Φ_x for generic x .*

- (c) *Every irreducible factor of Den is a coordinate x_e or divides the numerator of the Laurent polynomial U_j^+ or of U_j^- for some j such that $w_j = (w_{j, k})_k$ is not constant. In particular Den is a monomial if all $d_j = 2$.*

If $\deg \text{Den} > 0$, then $0 \in H$ and $V(f_{\text{low}})$ is the tangent cone of H at the origin; if Den is constant, then $0 \notin H$. We do not know whether $q = 1$ and $W^* = W$ always hold, that is, whether the W critical values are nonzero and distinct for generic x (Conjecture 7.2). If they are, $\deg f = W + \deg f_{\text{low}}$. For corner patterns and for the patterns of the next theorem they are.

Theorem 1.3. *Let E be admissible and saturated with $W = 2$. Then there are coordinates $j_1 \neq j_2$ and partitions $[d_{j_1}] = K_1^+ \sqcup K_1^-$ and $[d_{j_2}] = K_2^+ \sqcup K_2^-$ into nonempty sets such that $E \subseteq D^{(1)} \cup D^{(2)}$, where $D^{(1)} = \{\kappa : \kappa_{j_1} \in K_1^+, \kappa_{j_2} \in K_2^-\}$ and $D^{(2)} = \{\kappa : \kappa_{j_1} \in K_1^-, \kappa_{j_2} \in K_2^+\}$. Every completion θ of $x \in (\mathbb{C}^*)^E$ satisfies $\theta^\kappa = x^{r_\kappa}$ for $\kappa \in D^{(1)} \cup D^{(2)}$. Let $\beta_l = \sum_{\kappa \in D^{(l)}} x^{r_\kappa}$ for $l = 1, 2$. Then a point $x \in (0, \infty)^E$ is completable if and only if $\sqrt{\beta_1} + \sqrt{\beta_2} \leq 1$, and*

$$f = cx^M((1 - \beta_1 - \beta_2)^2 - 4\beta_1\beta_2), \quad \deg f = 2 + \sum_{e \in E} M_e,$$

$$M_e = \max\{0, -\text{ord}_e(\beta_1 + \beta_2), -2\text{ord}_e(\beta_1 - \beta_2)\},$$

where $\text{ord}_e L$ is the smallest exponent of x_e in a Laurent polynomial L . If the terms of lowest degree in x_e of β_1 and β_2 do not cancel in $\beta_1 - \beta_2$, then $M_e = 2 \max\{0, -\text{ord}_e \beta_1, -\text{ord}_e \beta_2\}$.

An entry $e \in E$ is *pinned* if each of the n maximal slices through e contains another element of E [Fer26]. Let $\text{pin}(E)$ be the number of pinned entries.

Corollary 1.4. *Let $n = 2$ and $d_1 \leq d_2$. Every admissible E is saturated with $W = 2$, and*

$$\deg f = 2 + 2 \text{pin}(E).$$

As E runs over the admissible patterns, $\text{pin}(E)$ takes exactly the values $0, 1, \dots, P(d_1, d_2)$, where $P(d_1, d_2) = \max\{0, \min\{d_1 + d_2 - 5, 2d_1 - 4\}\}$. So the largest degree is $2 + 2P(d_1, d_2)$; it equals 2 if $d_1 = d_2 = 2$, $2(d_1 + d_2 - 4)$ if $3 \leq d_1 \leq d_2 \leq d_1 + 1$, and $4d_1 - 6$ if $d_2 \geq d_1 + 1$. For example it is 6 for 3×5 matrices.

The completability criterion for matrices is [KR17, Theorem 4]; given it, Corollary 1.4 needs only the irreducibility statement ($q = 1$) of Theorem 1.3 and a computation of denominators. The tensor case of Theorem 1.3 is the generalization.

Theorem 1.5. *For every admissible E , including patterns that are not saturated,*

$$\deg f \leq \frac{m!}{\prod_{j=1}^n (d_j - 1)!},$$

the degree of the Segre embedding of $\mathbb{P}^{d_1-1} \times \dots \times \mathbb{P}^{d_n-1}$. Equality holds for $\{12, 21\}$ in 2×2 , for the corner patterns of $2 \times 2 \times 2$, and for $\{111, 122, 212, 223\}$ in $2 \times 2 \times 3$, where $\deg f = 12$.

Tropical bound and computations. For saturated patterns in binary formats we prove $\deg f \leq W + \sum_e \mu_e$ with explicit Newton-polygon multiplicities μ_e (Corollary 4.4). With exact certificates this determines $\deg f$ for all orbits in six small formats (Proposition 7.1). A modular census, correct with high probability but not a proof, covers 2343 patterns in 25 formats (Section 7.2).

Relation to earlier work. The setting, the hypersurface H and the question are due to [KKKR17, Kum18]; the degree 6 of the $2 \times 2 \times 2$ corner pattern is [KKKR17, Example 3.16], and [KKKR17, Example 3.17] recovers its boundary from a Sturm sequence. The completability criteria behind several of our formulas are due to Kubjas and Rosen; we cite [KR17] by the numbering of the journal version; in arXiv:1407.3254v2, Theorem 4, Example 4 and Example 9 are Theorem 2.12, Example 2.13 and Example 4.2. Their Theorem 4 is the matrix criterion $\sum_l \sqrt{b_l} \leq 1$, where b_l are the sums of the completed blocks; Example 4 gives the criterion for the 3×3 pattern $\{11, 12, 21, 33\}$; Example 9, part 3(b), gives the criterion $\sqrt{t_{000} + t_{001}} + \sqrt{t_{110} + t_{001}t_{110}/t_{000}} \leq 1$ for the $2 \times 2 \times 2$ pattern $\{000, 001, 110\}$, which is our pinned orbit $\{111, 112, 221\}$, and part 3(c) the cubic criterion for the corner pattern. The companion note [Fer26] introduced pinned entries, classified the admissible patterns without them, and proved $\deg f \leq n(n-1)$ for corner patterns.

Cai, Recke and Yahl [CRY24] study completions in log-linear models under the hypothesis $|E| = \text{rank } A_E = \text{rank } A - 1$; we use the numbering of arXiv:2312.15154v2. For the independence model $\text{rank } A = m+1$ and $A_E = B^T$, so the hypothesis says $|E| = \text{rank } B = m$, and by Lemma 2.1(b) it holds for every admissible E . Their vector $\nu \in \text{im } A^T \cap \ker \pi_E$ is, up to a scalar, our vector of cell weights $(w(\kappa))_{\kappa \in D}$. Their Proposition 4.4 and Corollary 4.5 describe the singular locus of the projection as $\{p : \nu^T p = 0\}$ and its image under the moment map as $\text{im } A_E \cap P_A$; this is the positive real form of Lemma 2.2(a) and of the parametrization in Lemma 2.2(c). Their Proposition 4.8, by which two completions differ by factors $e^{\alpha\nu\kappa}$, is the positive real form of Lemma 4.1. Their Theorems 4.11, 4.13 and 4.16 and Proposition 4.14 describe the interior and the boundary of $\mathcal{C}_E$ and show, since E meets every maximal slice and so lies in no proper facial set, that a point of $\mathcal{C}_E$ with nonzero coordinates

has two completions in Δ if it is an interior point and one if it is a boundary point; this contains the statements on completions in Δ in Lemma 4.2 and Theorem 1.2(a). They also give a procedure for the algebraic boundary, but they do not discuss its degree. What we claim as new is the count of complex completions in Theorem 1.2(a), parts (b) and (c) of Theorem 1.2, and the results on degrees.

Outside the assumptions (1), boundary degrees are known for diagonal patterns. For $E = \{(i, \dots, i) : i \in [d]\}$ in the format $d \times \dots \times d$ with n factors, Kubjas and Rosen found an irreducible polynomial of degree n^{d-1} that vanishes on the boundary of $\mathcal{C}_E$ [KR17, Proposition 6] (Proposition 5.2 in arXiv:1407.3254v2, where the letters n and d are exchanged), and [KKKR17, Section 4] constructs a polynomial $\tilde{Q}_{n,d}$ of the same degree n^{d-1} and describes $\mathcal{C}_E$ by polynomial inequalities. Here $|E| = d$ while $m = n(d-1)$, so a diagonal pattern satisfies (1) only for $n = d = 2$; then it is a corner pattern of 2×2 , and the degree 2 agrees with Theorem 1.1. Under the assumptions (1) we found no other work on the degree of H .

What remains open. A formula for $\deg f$ for all patterns. In particular we do not prove $q = 1$ and $W^* = W$ for saturated patterns, nor that the tropical bound is sharp (Conjectures 7.2 and 7.3), and for patterns that are not saturated we have only Theorem 1.5 and the certified example in $2 \times 2 \times 2 \times 2$.

2 Preliminaries

Let $\mathcal{A} = \{\theta \in \mathbb{C}^{\mathcal{N}} : |\theta_j| = 1 \text{ for all } j\}$, an affine space of dimension m , and $\mathcal{A}^\circ = \mathcal{A} \cap (\mathbb{C}^*)^{\mathcal{N}}$. A *complex completion* of $x \in \mathbb{C}^E$ is a point $\theta \in \mathcal{A}$ with $p_E(\theta) = x$. We use the bilinear form $\langle u, v \rangle = \sum_{(j,k)} u_{j,k} v_{j,k}$ on $\mathbb{C}^{\mathcal{N}}$, write $|\lambda| = \sum_e \lambda_e$ for $\lambda \in \mathbb{C}^E$, and let $\mathcal{S}$ be the space of vectors $\sigma \in \mathbb{C}^{\mathcal{N}}$ with $\sigma_{j,k} = c_j$ for all j, k and $\sum_j c_j = 0$. Clearly $\mathcal{S} \subseteq \ker B$.

Lemma 2.1. *Let $|E| = m$ and let E meet every maximal slice.*

- (a) *If $\text{rank } B = m$, then $\ker B = \mathcal{S} \oplus \mathbb{C}w$ for some $w \in \mathbb{Q}^{\mathcal{N}}$. No such w is constant on every block, and no such w lies in $\mathbb{C}\delta_{(j,k)} + \mathcal{S}$ for a unit vector $\delta_{(j,k)}$.*
- (b) *$\mathcal{C}_E$ has nonempty interior if and only if $\text{rank } B = m$. So E is admissible if and only if $|E| = m$, E meets every maximal slice and $\text{rank } B = m$.*

Proof. (a) Here $\dim \ker B = |\mathcal{N}| - m = n > \dim \mathcal{S}$, and B has rational entries. If $w_{j,k} = c_j$ for all j, k , then $(Bw)_e = \sum_j c_j = 0$, so $w \in \mathcal{S}$. If $w = c\delta_{(j,k)} + \sigma$ with $\sigma \in \mathcal{S}$, then $cB\delta_{(j,k)} = 0$; since $S_{j,k}$ meets E , $B\delta_{(j,k)} \neq 0$, so $c = 0$ and $w \in \mathcal{S}$. (b) If $\text{rank } B < m$, then $p_E(\mathcal{A}^\circ)$ lies in the image of the monomial map $t \mapsto (t^{B_e})_{e \in E}$ of $(\mathbb{C}^*)^{\mathcal{N}}$, where $t^{B_e} = \prod_j t_{j,e_j}$, and the Zariski closure of this image has dimension $\text{rank } B < m$. Since $\mathcal{A}^\circ$ is dense in $\mathcal{A}$, $p_E(\mathcal{A})$ lies in a proper algebraic subset of $\mathbb{C}^E$, and so does $\mathcal{C}_E$, which then has empty interior. If $\text{rank } B = m$, then by Lemma 2.2(a) below the differential of $p_E|_{\mathcal{A}}$ is invertible at every $\theta \in \mathcal{A}^\circ$ with $\langle w, \theta \rangle \neq 0$. By (a) the affine function $\theta \mapsto \langle w, \theta \rangle$ is not constant on $\mathcal{A}$, so it does not vanish identically on the nonempty open subset Δ° of the real points of $\mathcal{A}$, where Δ° is the set of $\theta \in \Delta$ with positive entries. By the inverse function theorem, $p_E(\Delta^\circ)$ contains an open set. $\square$

Lemma 2.2. *Let $|E| = m$, let E meet every maximal slice, let $\text{rank } B = m$, and let w be as in Lemma 2.1.*

- (a) *For $\theta \in \mathcal{A}^\circ$ the differential of $p_E|_{\mathcal{A}}$ at θ is singular if and only if $\theta \in \text{Im } B^T$. Moreover $\text{Im } B^T \cap \mathcal{A} = \{\theta \in \mathcal{A} : \langle w, \theta \rangle = 0\} = \{B^T \lambda / |\lambda| : \lambda \in \mathbb{C}^E, |\lambda| \neq 0\}$.*
- (b) *On $\mathcal{A}$, the linear form l_E of [KKKR17] is a nonzero constant multiple of $\theta \mapsto \langle w, \theta \rangle$.*

(c) H is the Zariski closure of $p_E(\{\theta \in \mathcal{A}^\circ : \langle w, \theta \rangle = 0\})$, and also of the set of points

$$x(\lambda) = \left(|\lambda|^{-n} \prod_{j=1}^n L_{j,e_j}(\lambda) \right)_{e \in E}, \quad L_{j,k}(\lambda) = \sum_{e \in E: e_j=k} \lambda_e, \quad \lambda \in \mathbb{C}^E, \quad |\lambda| \neq 0.$$

Proof. (a) Let $\theta \in \mathcal{A}^\circ$ and $x = p_E(\theta)$, so all $x_e \neq 0$. The tangent space of $\mathcal{A}$ consists of the v with $\sum_k v_{j,k} = 0$ for all j , and $dp_e(v) = x_e \sum_j v_{j,e_j} / \theta_{j,e_j}$. The differential is singular if and only if there is $\lambda \neq 0$ with $\sum_e \lambda_e dp_e = 0$ on the tangent space. With $\lambda'_e = \lambda_e x_e$ we have $\sum_e \lambda_e dp_e(v) = \sum_{(j,k)} v_{j,k} (B^T \lambda')_{j,k} / \theta_{j,k}$, and this vanishes on the tangent space if and only if for every j the numbers $(B^T \lambda')_{j,k} / \theta_{j,k}$, $k \in [d_j]$, are equal to some μ_j . Summing $(B^T \lambda')_{j,k} = \mu_j \theta_{j,k}$ over k gives $\mu_j = |\lambda'|$ for all j , so $B^T \lambda' = |\lambda'| \theta$. If $|\lambda'| = 0$, then $B^T \lambda' = 0$ and $\lambda' = 0$, because B^T is injective. So the differential is singular if and only if $\theta = B^T \lambda' / |\lambda'|$ for some λ' ; conversely every $\theta = B^T \mu \in \mathcal{A}$ has $|\mu| = |\theta_1| = 1$ and $\mu \neq 0$. This proves the first claim and the last description of $\text{Im } B^T \cap \mathcal{A}$, because the block sums of $B^T \lambda$ equal $|\lambda|$. Finally $\text{Im } B^T = (\ker B)^\perp$, and $\langle \sigma, \theta \rangle = \sum_j c_j = 0$ for $\sigma \in \mathcal{S}$ and $\theta \in \mathcal{A}$; so $\theta \in \mathcal{A}$ lies in $\text{Im } B^T$ if and only if $\langle w, \theta \rangle = 0$.

(b) By [KKKR17, Proposition 3.8], the Jacobian determinant of p_E with respect to the coordinates $\theta_{j,k}$, $k < d_j$, of $\mathcal{A}$ is $l_E \prod_{(j,k)} \theta_{j,k}^{\alpha(j,k)-1}$ with $\alpha(j,k) = |E \cap S_{j,k}| \geq 1$, and $l_E \neq 0$ has degree at most 1. By (a), l_E and $\langle w, \theta \rangle$ have the same zeros on $\mathcal{A}^\circ$. The zero set V_w of $\langle w, \theta \rangle$ in $\mathcal{A}$ is an affine hyperplane, by Lemma 2.1(a). It is not contained in a coordinate hyperplane $\{\theta_{j,k} = 0\}$: otherwise $\langle w, \theta \rangle$ and $\theta_{j,k}$ would be proportional on $\mathcal{A}$, that is, $w - c \delta_{(j,k)}$ would be constant on every block with block values summing to 0, which Lemma 2.1(a) excludes. So $V_w \cap \mathcal{A}^\circ$ is dense in V_w , and l_E vanishes on V_w . A nonzero polynomial of degree at most one that vanishes on an affine hyperplane is a multiple of its equation.

(c) The elimination ideal defines the Zariski closure of $p_E(V(l_E) \cap \mathcal{A})$, which by (b) is the closure of $p_E(V_w)$ and, as $V_w \cap \mathcal{A}^\circ$ is dense in V_w , of $p_E(V_w \cap \mathcal{A}^\circ)$. By (a), $V_w = \{B^T \lambda / |\lambda|\}$, and $p_E(B^T \lambda / |\lambda|) = x(\lambda)$. $\square$

Lemma 2.2(c) is a unirational parametrization of H by forms of degree n ; it is convenient for computations and was pointed out by the first independent verifier. Over the positive reals, Lemma 2.2(a) and the parametrization in (c) are the case of the independence model of [CRY24, Proposition 4.4 and Corollary 4.5]. Indeed, for $p = (\theta^\kappa)_{\kappa \in D}$ with $\theta \in \mathcal{A}$ we have $\sum_\kappa w(\kappa) \theta^\kappa = \langle w, \theta \rangle$ (see the proof of Theorem 1.5), so their singular locus $\{\nu^T p = 0\}$ corresponds to $\{\langle w, \theta \rangle = 0\}$; and the entries of p in the slice $S_{j,k}$ sum to $\theta_{j,k}$, so their condition that the image of p under the moment map lies in the span of the columns of A_E becomes $\theta \in \text{Im } B^T$. Likewise, the implication from $\text{rank } B = m$ to nonempty interior in Lemma 2.1(b) is a case of [CRY24, Corollary 4.7]. From now on E is admissible.

Lemma 2.3. *The vector w can be chosen in $\mathbb{Z}^N$ so that the cell weights $w(\kappa) = \sum_j w_{j,\kappa_j}$ are integers generating $\mathbb{Z}$. They vanish on E and are determined by E up to sign. With $a_- = \min_\kappa w(\kappa)$ and $a_+ = \max_\kappa w(\kappa)$ we have $a_- < 0 < a_+$, and $W = a_+ - a_- = \sum_j (\max_k w_{j,k} - \min_k w_{j,k}) \geq 2$.*

Proof. Adding an element of $\mathcal{S}$ to w does not change the cell weights, and $w(e) = (Bw)_e = 0$ for $e \in E$; since $\ker B = \mathcal{S} \oplus \mathbb{C}w$, the cell weights are determined up to a scalar. Scale the rational vector w so that the cell weights are coprime integers. A difference of two cell weights is a sum of differences $w_{j,k} - w_{j,k'}$, and every such difference is a difference of two cell weights; so these differences are integers. Fix $k_0(j) \in [d_j]$ and integers c_j with $\sum_j c_j = w(\kappa_0)$, where $\kappa_0 = (k_0(j))_j$, and replace $w_{j,k}$ by $w_{j,k} - w_{j,k_0(j)} + c_j$; this adds an element of $\mathcal{S}$ and makes w integral. If $a_- = 0$, then every $e \in E$ has e_j in the set where w_j is minimal, for every j . Taking j with w_j non-constant (Lemma 2.1(a)) and k outside this set, $S_{j,k}$ would not meet E . So $a_- < 0$, and similarly $a_+ > 0$. $\square$

Lemma 2.4. *Corner patterns and admissible patterns with $n = 2$ are admissible and saturated.*

Proof. For $E = E^*(s)$ the columns (j, k) with $k \neq s_j$ form an identity submatrix of B , and E meets every slice; so $\text{rank } B = m$ and E is admissible and saturated. For $n = 2$, B is the incidence matrix of the bipartite graph $G(E)$ with the rows and columns as vertices and the elements of E as edges. If $G(E)$ contains a cycle, the alternating sum of the rows of B along the cycle vanishes. So for admissible E the graph is a forest; with $|E| = d_1 + d_2 - 2$ edges and no isolated vertex it has exactly two trees. Removing leaves one at a time orders the edges as $e_1, \dots, e_m$ with vertices $v_i \in e_i$ that lie in no $e_{i'}$ with $i' > i$; the maximal minor of B on the columns $v_1, \dots, v_m$ is then triangular with unit diagonal. $\square$

3 Corner patterns

Proof of Theorem 1.1. We may assume $s = (1, \dots, 1)$, and we write $a_j = \theta_{j,1}$.

Step 1: the critical hyperplane. The vector with $w_{j,1} = 1$ and $w_{j,k} = 1 - n$ for $k \geq 2$ lies in $\ker B$, since $w(s[j \leftarrow k]) = (1 - n) + (n - 1) = 0$, and it is not constant on blocks. On $\mathcal{A}$, $\langle w, \theta \rangle = \sum_j (a_j + (1 - n)(1 - a_j)) = n(\sum_j a_j - (n - 1))$. (For $n = 3$ this is $-3l_E$, a constant multiple of the form $l_E = 2 - a_1 - a_2 - a_3$ of [KKKR17, Example 3.4].)

Step 2: reduction to n variables. By Lemma 2.2(c), H is the closure of $p_E(V)$ with $V = \{\theta \in \mathcal{A} : \sum_j a_j = n - 1\}$. For $\theta \in \mathcal{A}$,

$$X_j(p_E(\theta)) = \sum_{k \geq 2} \theta_{j,k} \prod_{i \neq j} a_i = (1 - a_j) \prod_{i \neq j} a_i =: \xi_j(a).$$

Let $\Lambda = \{a \in \mathbb{C}^n : \sum_j a_j = n - 1\}$ and let Z be the closure of $\xi(\Lambda)$ in $\mathbb{C}^n$. The map $X = (X_1, \dots, X_n) : \mathbb{C}^E \rightarrow \mathbb{C}^n$ is a linear surjection, and $p_E(V) \subseteq X^{-1}(Z)$; hence $H \subseteq X^{-1}(Z)$.

Step 3: $Z = V(\mathcal{D}_n)$, and Z is irreducible. Write $h_X(A) = \prod_j (A + X_j) - A^{n-1}$, so that $\mathcal{D}_n(X) = 0$ if and only if h_X has a multiple root A_* . Suppose $A_* \neq 0$. Then $A_* + X_j \neq 0$ for all j , since otherwise $h_X(A_*) = -A_*^{n-1} \neq 0$. Put $a_j = A_*/(A_* + X_j)$. From $h_X(A_*) = 0$ we get $\prod_j a_j = A_*^n/A_*^{n-1} = A_*$, hence $X_j = A_*(1 - a_j)/a_j = \xi_j(a)$, and

$$h'_X(A_*) = \sum_j \prod_{i \neq j} (A_* + X_i) - (n - 1)A_*^{n-2} = A_*^{n-2} \left(\sum_j a_j - (n - 1) \right),$$

so $a \in \Lambda$. Conversely, if $a \in \Lambda$ has no zero entry and $A_* = \prod_j a_j$, then $A_* + \xi_j(a) = A_*/a_j$, so $h_{\xi(a)}(A_*) = 0$ and $h'_{\xi(a)}(A_*) = 0$ by the same computation. The root $A_* = 0$ is multiple if and only if $\prod_j X_j = 0$ and $e_{n-1}(X) = [n = 2]$, where e_k is the k -th elementary symmetric function; this happens on a set Y_0 of codimension 2. By Step 4, $\mathcal{D}_n$ is not constant, so $V(\mathcal{D}_n)$ is a hypersurface and has no component inside Y_0 . Therefore $V(\mathcal{D}_n)$ is the closure of $\xi(\Lambda \cap (\mathbb{C}^*)^n)$, which is Z , and it is irreducible.

Step 4: degrees. The coefficient c_k of A^{n-k} in h_X is $e_k(X) - [k = 1]$, a polynomial of degree k with leading form $e_k(X)$. The discriminant of a monic polynomial of degree n is isobaric of weight $n(n - 1)$ when c_k has weight k . So $\deg \mathcal{D}_n \leq n(n - 1)$, and the component of degree $n(n - 1)$ is $\text{Disc } \prod_j (A + X_j) = \prod_{i < j} (X_i - X_j)^2 \neq 0$. For the lowest component let $X = tY$ with $e_n(Y) \neq 0$ and $t \rightarrow 0$. In the Newton polygon of h_{tY} with respect to t the coefficient $t^k e_k(Y)$ of A^{n-k} has valuation at least k for $k \geq 2$, with equality for $k = n$, and the coefficients of A^n and A^{n-1} have valuation 0. The points $(n - k, k)$ with $2 \leq k \leq n - 1$ lie strictly above the segment from $(0, n)$ to $(n - 1, 0)$. So one root tends to 1, and the other $n - 1$ roots are $\zeta e_n(Y)^{1/(n-1)} t^{n/(n-1)} (1 + o(1))$, where ζ runs over the $(n - 1)$ -st roots of unity. Taking the product of $(A_i - A_{i'})^2$ over pairs of roots gives $\mathcal{D}_n(tY) = t^{n(n-2)} \text{Disc}_A(A^{n-1} - e_n(Y))(1 + o(1))$. Hence the lowest component of $\mathcal{D}_n$ has degree $n(n - 2)$ and equals $\text{Disc}_A(A^{n-1} - e_n) = \pm(n - 1)^{n-1} e_n^{n-2}$.

Step 5: $\mathcal{D}_n$ is irreducible. Let $\text{disc}_n(c)$ be the discriminant of the generic monic polynomial $A^n + c_1 A^{n-1} + \dots + c_n$, so that $\mathcal{D}_n = \text{disc}_n \circ \gamma$ with $\gamma(X) = (e_1(X) - 1, e_2(X), \dots, e_n(X))$. The

map γ is finite, and its Jacobian determinant is $\pm \prod_{i < j} (X_i - X_j)$. We first show that $\nabla \text{disc}_n \neq 0$ at every monic g with exactly one multiple root, which is double. Write $g = g_1 g_2$ with g_1 quadratic and $\gcd(g_1, g_2) = 1$. The multiplication map $(g_1, g_2) \mapsto g_1 g_2$ has Jacobian determinant $\pm \text{Res}(g_1, g_2) \neq 0$, and $\text{Disc}(g_1 g_2) = \text{Disc}(g_1) \text{Disc}(g_2) \text{Res}(g_1, g_2)^2$ [GKZ94, Chapter 12]. At our point $\text{Disc}(g_1) = 0$ while $\text{Disc}(g_2) \text{Res}(g_1, g_2)^2 \neq 0$, and $\text{Disc}(A^2 + bA + c) = b^2 - 4c$ has nonzero gradient; by the chain rule $\nabla \text{disc}_n(g) \neq 0$. The monic polynomials with a triple root or two multiple roots form an algebraic set B_2 of dimension at most $n - 2$, so $\gamma^{-1}(B_2)$ has dimension at most $n - 2$ and does not contain the irreducible $(n - 1)$ -dimensional set Z . Nor is Z contained in a hyperplane $X_i = X_j$, because $\xi_i(a) - \xi_j(a) = (a_j - a_i) \prod_{l \neq i, j} a_l$. So there is $X^0 \in Z$ with distinct coordinates and $\gamma(X^0) \notin B_2$, and there $\nabla \mathcal{D}_n(X^0) = \nabla \text{disc}_n(\gamma(X^0)) J_\gamma(X^0) \neq 0$. Since $V(\mathcal{D}_n) = Z$ is irreducible, $\mathcal{D}_n = c g^r$ with g irreducible; if $r \geq 2$, then $\nabla \mathcal{D}_n$ would vanish on Z . So $\mathcal{D}_n$ is irreducible.

Step 6: conclusion. In linear coordinates on $\mathbb{C}^E$ that contain $X_1, \dots, X_n$, the polynomial $\mathcal{D}_n(X)$ is irreducible, and $X^{-1}(Z) = V(\mathcal{D}_n(X))$ is an irreducible hypersurface containing H . Both are irreducible of dimension $m - 1$, so they are equal, and $f = c \mathcal{D}_n(X)$. The top form $\prod_{i < j} (X_i - X_j)^2$ is nonzero as a polynomial in x , so $\deg f = n(n - 1)$. $\square$

4 Saturated patterns

In this section E is admissible and saturated, w is as in Lemma 2.3, $R \in \mathbb{Z}^{N \times E}$ satisfies $BR = I$, and we use the notation of Section 1. Put $u_{j,k}(x) = x^{R_{(j,k)}}$, a Laurent monomial. Then $\prod_j u_{j,\kappa_j} = x^{r_\kappa}$, and r_e is the unit vector of e for $e \in E$, because the rows of BR are the unit vectors. Let $\Phi_{x,j}(\tau) = \sum_k u_{j,k}(x) \tau^{w_{j,k}}$, so that

$$\Phi_x(\tau) = \prod_{j=1}^n \Phi_{x,j}(\tau) = \sum_a C_a(x) \tau^a, \quad C_a(x) = \sum_{\kappa: w(\kappa)=a} x^{r_\kappa},$$

and Φ_x depends on w only through the cell weights. The Laurent polynomials of Section 1 are

$$U_j^+ = \sum_{k: w_{j,k} = \max w_j} u_{j,k}, \quad U_j^- = \sum_{k: w_{j,k} = \min w_j} u_{j,k},$$

and $C_{a_+} = \prod_j U_j^+$ and $C_{a_-} = \prod_j U_j^-$ are nonzero Laurent polynomials with positive coefficients. Let $T = (\mathbb{C}^*)^E$ and $U_0 = \{x \in T : C_{a_+}(x) C_{a_-}(x) \neq 0\}$.

Lemma 4.1. *Let $x \in T$. The map $\tau \mapsto \theta(\tau)$, $\theta(\tau)_{j,k} = u_{j,k}(x) \tau^{w_{j,k}} / \Phi_{x,j}(\tau)$, is a bijection from $\{\tau \in \mathbb{C}^* : \Phi_x(\tau) = 1\}$ onto the set of complex completions of x . For such τ , $\theta(\tau)^\kappa = x^{r_\kappa} \tau^{w(\kappa)}$ for all $\kappa \in D$, and $\langle w, \theta(\tau) \rangle = \tau \Phi'_x(\tau)$. In particular $\theta(\tau)$ is a critical point of $p_E|_{\mathcal{A}}$ if and only if τ is a multiple root of $\Phi_x - 1$.*

Proof. Let θ be a complex completion of x . Every $\theta_{j,k}$ divides some $x_e \neq 0$, so $\theta \in \mathcal{A}^\circ$, and $t = \theta/u(x)$ (entrywise) satisfies $t^{B_e} = 1$ for all e . Write $t = \exp(v)$ with $v \in \mathbb{C}^N$. Then $Bv \in 2\pi i \mathbb{Z}^E$, and since $B\mathbb{Z}^N = \mathbb{Z}^E$ there is $z \in \mathbb{Z}^N$ with $Bz = Bv/2\pi i$. So $v' = v - 2\pi iz \in \ker B = \mathcal{S} \oplus \mathbb{C}w$, say $v' = \sigma + \zeta w$, and $t = \exp(v') = c \tau^w$ with $\tau = e^\zeta$ and $c = \exp(\sigma)$ constant on blocks with $\prod_j c_j = 1$. The conditions $|\theta_j| = 1$ give $c_j = 1/\Phi_{x,j}(\tau)$, and $\prod_j c_j = 1$ gives $\Phi_x(\tau) = 1$. Conversely, if $\Phi_x(\tau) = 1$, then all $\Phi_{x,j}(\tau) \neq 0$, $\theta(\tau) \in \mathcal{A}$, and $\theta(\tau)^\kappa = x^{r_\kappa} \tau^{w(\kappa)} / \Phi_x(\tau)$, which for $\kappa = e \in E$ is x_e . If $\theta(\tau) = \theta(\tau')$, then $(\tau/\tau')^{w_{j,k} - w_{j,k'}} = 1$ for all j, k, k' ; these exponents generate $\mathbb{Z}$ (Lemma 2.3), so $\tau = \tau'$. Finally $\langle w, \theta(\tau) \rangle = \sum_j \tau \Phi'_{x,j}(\tau) / \Phi_{x,j}(\tau) = \tau \Phi'_x(\tau) / \Phi_x(\tau)$, and Lemma 2.2(a) applies. $\square$

For $x \in (0, \infty)^E$ and $\tau, \tau' > 0$, Lemma 4.1 gives $\theta(\tau')^\kappa = \theta(\tau)^\kappa (\tau'/\tau)^{w(\kappa)}$; this statement about completions in Δ is the case of the independence model of [CRY24, Proposition 4.8]. The complex statement uses the saturation of E to lift $Bv \in 2\pi i \mathbb{Z}^E$.

Lemma 4.2. *A point $x \in (0, \infty)^E$ lies in $\mathcal{C}_E$ if and only if $\min_{\tau > 0} \Phi_x(\tau) \leq 1$; its completions in Δ are the $\theta(\tau)$ with $\tau > 0$ and $\Phi_x(\tau) = 1$, and there are at most two. Moreover $\Phi_{sx}(\tau) = s \Phi_x(s^\alpha \tau)$ for all $s \in \mathbb{C}^*$, for some $\alpha \in \mathbb{Z}$, and each C_a is homogeneous of degree $1 + \alpha a$.*

Proof. A completion $\theta \in \Delta$ of $x > 0$ has positive entries, so $t = \theta/u(x)$ has positive entries, $v = \log t$ is real, and $Bv = 0$. Hence $v \in \mathcal{S}_{\mathbb{R}} \oplus \mathbb{R}w$ and $\theta = \theta(\tau)$ with $\tau > 0$, as in the proof of Lemma 4.1; conversely $\theta(\tau) \in \Delta$ for $\tau > 0$ with $\Phi_x(\tau) = 1$. On $(0, \infty)$ the function Φ_x is positive, continuous and tends to ∞ at 0 and at ∞ , because $a_- < 0 < a_+$ and $C_{a_{\pm}}(x) > 0$; and $v \mapsto \Phi_x(e^v)$ is strictly convex, being a sum of exponentials with positive coefficients. This proves the first two claims. Let $\bar{\rho} = R\mathbf{1}_E \in \mathbb{Z}^N$. Then $B\bar{\rho} = \mathbf{1}_E = B(\mathbf{1}_N/n)$, so $\bar{\rho} = \mathbf{1}_N/n + \sigma + \alpha w$ with $\sigma \in \mathcal{S}$ and $\alpha \in \mathbb{Q}$, and $\sum_e (r_\kappa)_e = \sum_j \bar{\rho}_{(j, \kappa_j)} = 1 + \alpha w(\kappa)$. This is an integer for every κ , and the cell weights generate $\mathbb{Z}$, so $\alpha \in \mathbb{Z}$. $\square$

The statement on completions in Δ is contained in [CRY24, Theorems 4.13 and 4.16 and Proposition 4.14], which show more precisely that x has two completions in Δ if it is an interior point of $\mathcal{C}_E$ and one if it is a boundary point; by the convexity in the proof, these are the cases $\min_{\tau > 0} \Phi_x(\tau) < 1$ and $\min_{\tau > 0} \Phi_x(\tau) = 1$.

Proof of Theorem 1.2. (a) By Lemma 2.3, $W \geq 2$. For $x \in U_0$ the Laurent polynomials $\tau^{-a_-}(\Phi_x - 1)$ and $\tau^{-a_-}\tau\Phi'_x$ are polynomials of degree W with nonzero constant terms, since $0 < -a_- < W$. So $\Phi_x - 1$ has W roots and there are W critical points $\rho_1(x), \dots, \rho_W(x)$ in $\mathbb{C}^*$, counted with multiplicity. If $\Phi_x - 1$ has a multiple root, then $x \in H$ by Lemma 4.1 and Lemma 2.2(c). So for $x \in U_0 \setminus H$ there are exactly W completions. The last claim is Lemma 4.2.

(b) Let $\gamma_i = \Phi_x(\rho_i)$ and $\chi_x(z) = \prod_i (z - \gamma_i)$. With $\Phi_x = \tau^{a_-}\tilde{\Phi}$ and $Q = \tau^{-a_-}\tau\Phi'_x$ we have $\prod_i (z - \gamma_i) = \prod_i \rho_i^{a_-} \prod_i (z\rho_i^{-a_-} - \tilde{\Phi}(\rho_i))$. The second product is $\text{Res}_\tau(Q, z\tau^{-a_-} - \tilde{\Phi})$ divided by a power of the leading coefficient $a_+C_{a_+}$ of Q , and $\prod_i \rho_i = \pm a_-C_{a_-}/(a_+C_{a_+})$. So the coefficients of χ_x lie in $\mathbb{Q}[x^{\pm 1}][C_{a_+}^{-1}, C_{a_-}^{-1}]$. By Lemma 4.2 the critical points of Φ_{sx} are $s^{-\alpha}\rho_i$ and its critical values are $s\gamma_i$; so the k -th elementary symmetric function $e_k(\gamma)$ is a homogeneous rational function of degree k . Hence its reduced denominator and Den are homogeneous polynomials, their irreducible factors divide $\prod_e x_e \cdot \text{num}(C_{a_+}) \text{num}(C_{a_-})$, where $\text{num}(L)$ denotes the numerator of a Laurent polynomial L , and Den has no zero on U_0 . Let

$$F = \text{Den} \cdot \chi_x(1) = \sum_{k=0}^W (-1)^k F_k, \quad F_k = \text{Den} \cdot e_k(\gamma),$$

a polynomial whose homogeneous component of degree $\deg \text{Den} + k$ is $(-1)^k F_k$. Let $\Sigma = p_E(V_w \cap \mathcal{A}^\circ) \subseteq T$, so $H = \bar{\Sigma}$ by Lemma 2.2(c). By Lemma 4.1, $\Sigma \cap U_0 = \{x \in U_0 : \text{some } \gamma_i(x) = 1\} = V(F) \cap U_0$. This set is closed in U_0 , and the closure of $\Sigma \setminus U_0$ does not meet U_0 ; hence $H \cap U_0 = V(F) \cap U_0$.

$H \cap U_0 \neq \emptyset$. Choose $x^0 \in U_0 \cap (0, \infty)^E$ and let $s_0 = 1/\min_{\tau > 0} \Phi_{x^0}(\tau)$. By Lemma 4.2, $\min_{\tau > 0} \Phi_{s_0 x^0} = 1$, so $\Phi_{s_0 x^0} - 1 \geq 0$ on $(0, \infty)$ with a zero, which is a multiple root. Thus $s_0 x^0 \in \Sigma$, and $s_0 x^0 \in U_0$ because the C_a are homogeneous.

$F = cf^q$. The set $H \cap U_0$ is a nonempty open subset of the irreducible H , hence dense, and F vanishes on it; so $f \mid F$. Let h be an irreducible factor of F . If h is homogeneous, it divides every homogeneous component of F , in particular $F_0 = \text{Den}$, and then $(\text{Den}/h)e_k(\gamma) = F_k/h$ is a polynomial for all k , contradicting the definition of Den as a least common multiple. If h is not homogeneous, $V(h)$ is not contained in the union of homogeneous hypersurfaces $\mathbb{C}^E \setminus U_0$, so $V(h) \cap U_0$ is dense in $V(h)$ and contained in $V(F) \cap U_0 \subseteq H$; hence $V(h) = H$ and h is a multiple of f . So $F = cf^q$ with $q \geq 1$. (We have $F \neq 0$: otherwise every $x \in U_0$ would lie in H .)

Degrees. Let $W^* = W - \text{ord}_{z=0} \chi_x(z)$, the multiplicity being taken over $\mathbb{C}(x)$. Then $e_k(\gamma) = 0$ for $k > W^*$ and $e_{W^*}(\gamma) \neq 0$, so the components of largest and smallest degree of F are $\pm F_{W^*}$, of

degree $\deg \text{Den} + W^*$, and $F_0 = \text{Den}$. Comparing with $F = cf^q$ gives $q \deg f = W^* + \deg \text{Den}$ and $\text{Den} = cf_{\text{low}}^q$.

Distinct critical values. H is not a cone: for $y \in H \cap U_0$ the punctured line $\{sy : s \in \mathbb{C}^*\}$ lies in U_0 and meets H only where $\prod_i (1 - s\gamma_i(y)) = 0$, at finitely many points. Let $\mathcal{E}f = \sum_e x_e \partial f / \partial x_e$. If $H \subseteq V(\mathcal{E}f)$, then $f \mid \mathcal{E}f$, so $\mathcal{E}f = \lambda f$ for a constant λ and f would be homogeneous. Hence $H \cap V(\mathcal{E}f)$ has dimension at most $m - 2$. If $s_0 \neq 0$ is a multiple root of $s \mapsto f(sx)$, then $s_0 x \in H \cap V(\mathcal{E}f)$, so x lies in the cone over $H \cap V(\mathcal{E}f)$, a set of dimension less than m . So for generic x the nonzero roots of $f(sx)$ are simple. Since $F(sx) = s^{\deg \text{Den}} \text{Den}(x) \prod_i (1 - s\gamma_i(x)) = cf(sx)^q$, each nonzero value $\gamma_i(x)$ occurs exactly q times, and the number of distinct nonzero critical values is $W^*/q = \deg f - \deg f_{\text{low}}$.

(c) By the proof of (b), every irreducible factor of Den divides $\prod_e x_e \cdot \prod_j \text{num}(U_j^+) \text{num}(U_j^-)$. Let J be the set of j with w_j constant, say $w_{j,k} = c_j$. For $j \in J$ we have $\Phi_{x,j} = \tau^{c_j} U_j^+$, so $\Phi_x = (\prod_{j \in J} U_j^+) \Psi$ with $\Psi = \tau^{\sum_{j \in J} c_j} \prod_{j \notin J} \Phi_{x,j}$. Hence Φ_x and Ψ have the same critical points, and the critical values of Φ_x are those of Ψ times the Laurent polynomial $\prod_{j \in J} U_j^+$. Applying the argument of (b) to Ψ , whose extreme coefficients are $\prod_{j \notin J} U_j^\pm$, shows that every $e_k(\gamma)$ is regular on the set of $x \in T$ with $U_j^\pm(x) \neq 0$ for all $j \notin J$. This proves the first claim. Finally $U_j^\pm$ is a monomial if the extreme value of w_j is attained only once, which is the case for non-constant w_j when $d_j = 2$. $\square$

4.1 A tropical bound

Fix $e \in E$. For every value a of the cell weights let $v_a = \min\{(r_\kappa)_e : w(\kappa) = a\}$. Let Γ_e be the lower convex hull of the points (a, v_a) with $a \neq 0$. For an edge ϵ of Γ_e with endpoints at $a_1 < a_2$ and slope σ_ϵ put $\nu_\epsilon = \min_a(v_a - \sigma_\epsilon a)$, the minimum taken over all values a including $a = 0$, and let

$$\mu_e = \sum_{\epsilon} (a_2 - a_1) \max\{0, -\nu_\epsilon\}.$$

A different choice of R changes r_κ by $w(\kappa)\beta$ for some $\beta \in \mathbb{Z}^E$; this shears Γ_e and leaves μ_e unchanged.

Proposition 4.3. *The multiplicity of x_e in Den is at most μ_e .*

Proof. Put $x_e = t$ and give the other coordinates generic complex values. The coefficient of the lowest power of t in C_a is a nonzero Laurent polynomial with positive coefficients in the other coordinates, so C_a has t -adic valuation exactly v_a . By the Newton polygon theorem over the field of Puiseux series [MS15], applied to $\sum_{a \neq 0} a C_a \tau^a$, an edge ϵ carries $a_2 - a_1$ critical points ρ of valuation $-\sigma_\epsilon$, and then $\text{val } \Phi_x(\rho) \geq \min_a(v_a - \sigma_\epsilon a) = \nu_\epsilon$. For generic values of the other coordinates, the multiplicity of x_e in the reduced denominator of $e_k(\gamma)$ is $\max\{0, -\text{val } e_k(\gamma)\}$, and $\text{val } e_k(\gamma) \geq \sum_i \min\{0, \text{val } \gamma_i\}$. Hence the multiplicity of x_e in Den is at most $\sum_i \max\{0, -\text{val } \gamma_i\} \leq \mu_e$. $\square$

Corollary 4.4. *If Den is a monomial, for instance if $d_1 = \dots = d_n = 2$ or, more generally, if every non-constant w_j attains its maximum and its minimum only once, then $\deg f \leq (W^* + \sum_e \mu_e)/q \leq W + \sum_{e \in E} \mu_e$.*

Proof. Combine Theorem 1.2(b),(c) with Proposition 4.3. $\square$

5 Patterns with $W = 2$ and matrices

Proof of Theorem 1.3. The numbers $\max_k w_{j,k} - \min_k w_{j,k}$ are nonnegative integers with sum $W = 2$. If a single w_j had range 2, all other blocks being constant, then every $e \in E$ would have e_j in one level set of w_j , and the slices of the other levels would not meet E . So there are $j_1 \neq j_2$ with w_{j_1}, w_{j_2} of range 1, all other w_j being constant. Let $K_i^\pm$ be the sets where w_{j_i} is maximal and minimal. Since

only the cell weights matter, we may take $w_{j_1,k} = [k \in K_1^+]$, $w_{j_2,k} = -[k \in K_2^-]$ and $w_j = 0$ otherwise. Then $w(\kappa) = [\kappa_{j_1} \in K_1^+] - [\kappa_{j_2} \in K_2^-]$, the set of cells of weight 0 is $D^{(1)} \cup D^{(2)}$, and it contains E . The claim on completions is Lemma 4.1. With $G = \prod_{j \neq j_1, j_2} \Phi_{x,j}$, which does not depend on τ ,

$$\Phi_x(\tau) = G(U_1^- + U_1^+ \tau)(U_2^+ + U_2^- \tau^{-1}), \quad C_0 = \beta_1 + \beta_2, \quad C_1 C_{-1} = \beta_1 \beta_2,$$

where $U_i^\pm$ is the sum of the $u_{j_i,k}$ over $k \in K_i^\pm$, $\beta_1 = GU_1^+ U_2^-$ and $\beta_2 = GU_1^- U_2^+$. The critical points satisfy $\rho^2 = C_{-1}/C_1$, and the critical values are $\gamma_\pm = (\sqrt{\beta_1} \pm \sqrt{\beta_2})^2$. For $x > 0$ the minimum of Φ_x on $(0, \infty)$ is γ_+ , so the criterion follows from Lemma 4.2. Next $\chi_x(z) = z^2 - 2(\beta_1 + \beta_2)z + (\beta_1 - \beta_2)^2$ has Laurent polynomial coefficients. The reduced denominator of a Laurent polynomial L is $\prod_e x_e^{\max\{0, -\text{ord}_e L\}}$, so $\text{Den} = x^M$ with M as stated, and $\chi_x(1) = (1 - \beta_1 - \beta_2)^2 - 4\beta_1\beta_2$. At $x = p_E(\theta)$ we have $\beta_1 = ab$ and $\beta_2 = (1-a)(1-b)$ with a the sum of the $\theta_{j_1,k}$ over $k \in K_1^+$ and b the sum of the $\theta_{j_2,k}$ over $k \in K_2^-$; so $\beta_1 \neq \beta_2$ as functions, and γ_+, γ_- are nonzero and distinct. By Theorem 1.2(b), $q = 1$ and $W^* = 2$, which gives the formula for f and $\deg f = 2 + \deg \text{Den}$. Since β_1, β_2 have positive coefficients, $\text{ord}_e(\beta_1 + \beta_2) = \min\{\text{ord}_e \beta_1, \text{ord}_e \beta_2\}$, and without cancellation the same holds for $\beta_1 - \beta_2$. $\square$

Remark 5.1. For the $2 \times 2 \times 2$ pattern $E = \{111, 112, 221\}$ one finds $\beta_1 = x_{111} + x_{112}$ and $\beta_2 = x_{221}(x_{111} + x_{112})/x_{111}$, which is the criterion of [KR17, Example 9, 3(b)], and $M = (2, 0, 0)$. So $f = ((x_{111} + x_{112})(x_{111} + x_{221}) - x_{111})^2 - 4x_{111}x_{221}(x_{111} + x_{112})^2$ has degree 4, and $f_{\text{low}} = x_{111}^2$.

Call an edge of a tree *internal* if both its ends have degree at least 2.

Lemma 5.2. *Let T be a tree with p row vertices and p' column vertices, $p \leq p'$ and $p + p' \geq 3$. Then T has at most $\min\{p + p' - 3, 2p - 2\}$ internal edges. If $p \geq 2$, every number t with $1 \leq t \leq \min\{p + p' - 3, 2p - 2\}$ is the number of internal edges of such a tree.*

Proof. The column degrees sum to $p + p' - 1$, so at least $p' - p + 1$ columns are leaves, and T has at least $\max\{2, p' - p + 1\}$ leaves. Deleting the leaves leaves a tree whose edges are the internal edges, so there are at most $p + p' - 1 - \max\{2, p' - p + 1\}$ of them. For the second claim take a path $v_0 v_1 \cdots v_{t+2}$ starting at a column; it uses $\lfloor (t+3)/2 \rfloor \leq p$ rows and $\lceil (t+3)/2 \rceil \leq p'$ columns, the former because $t+3 \leq 2p+1$ and the latter because $t+3 \leq p+p' \leq 2p'$. Attach the remaining rows as leaves to the interior column v_2 and the remaining columns as leaves to the interior row v_1 . The internal edges are $v_1 v_2, \dots, v_t v_{t+1}$. $\square$

Proof of Corollary 1.4. By Lemma 2.4, $G(E)$ is a spanning forest of the complete bipartite graph with two trees T_1, T_2 , and E is saturated. Solving $Bw = 0$ on each tree gives $w_{1,r} = [r \in T_1]$ and $w_{2,c} = -[c \in T_1]$ after normalization, so $W = 2$, $D^{(l)}$ is the set of cells whose row and column lie in T_l , and β_l is the sum of the entries of the completed block of T_l . For a row r and a column c of T_l , let $e_1, \dots, e_{2s+1}$ be the edges of the path from r to c . The completed entry is $x^{r(c)} = x_{e_1} x_{e_3} \cdots x_{e_{2s+1}} / (x_{e_2} \cdots x_{e_{2s}})$, since both sides are Laurent monomials and agree at every completion, where $\theta_{1,r} \theta_{2,c}$ telescopes along the path; the edges $e_2, e_4, \dots$ are internal. Conversely, an internal edge uv occurs in even position in the path $u'uvv'$ for neighbours $u' \neq v$ of u and $v' \neq u$ of v . No edge occurs twice in a path, so the reduced denominator of β_l is the product of the internal edges of T_l . The Laurent polynomials β_1 and β_2 involve disjoint sets of variables, so no cancellation occurs, and $M_e = 2$ for internal edges and $M_e = 0$ otherwise. An edge (r, c) is internal if and only if row r and column c contain other elements of E , that is, if and only if it is pinned. So $\deg f = 2 + 2 \text{pin}(E)$.

For the range, let T_l have p_l rows and p'_l columns. A tree with a single edge has no internal edges. If T_l has at least 3 vertices, Lemma 5.2, applied with rows and columns exchanged when $p_l > p'_l$, shows that T_l has at most $p_l + p'_l - 3$ internal edges and at most $2 \min\{p_l, p'_l\} - 2 \leq 2p_l - 2$ of them. Hence in all cases T_l has at most $2p_l - 2$ internal edges, and at most $p_l + p'_l - 3$ if it has at least 3

vertices. Summing over $l = 1, 2$ gives $\text{pin}(E) \leq 2d_1 - 4$, and $\text{pin}(E) \leq d_1 + d_2 - 5$ unless $d_1 = d_2 = 2$. The value 0 is attained by the corner patterns. If $1 \leq t \leq P(d_1, d_2)$, let T_2 be a single edge and T_1 a tree on the remaining $d_1 - 1$ rows and $d_2 - 1$ columns with t internal edges, which exists by Lemma 5.2. $\square$

6 The general upper bound

Proof of Theorem 1.5. Let $\mathbb{P} = \mathbb{P}^{d_1-1} \times \dots \times \mathbb{P}^{d_n-1}$, let $\sigma : \mathbb{P} \rightarrow \mathbb{P}^{|D|-1}$, $[\theta] \mapsto [\theta^\kappa]_{\kappa \in D}$, be the Segre embedding, and let $\Sigma = \sigma(\mathbb{P})$. Its Hilbert function is $t \mapsto \prod_j \binom{t+d_j-1}{d_j-1}$, so $\deg \Sigma = m! / \prod_j (d_j - 1)!$ [Har92, Lecture 18]. Consider the linear forms $\ell_0 = \sum_\kappa X_\kappa$, $\ell_e = X_e$ for $e \in E$, and $\ell_w = \sum_\kappa w(\kappa) X_\kappa$ on $\mathbb{P}^{|D|-1}$, and the linear projection $\pi = [\ell_0 : (\ell_e)_{e \in E}]$ to $\mathbb{P}^m$. Since $\ell_0(\sigma(\theta)) = \prod_j |\theta_j|$ and $\sum_\kappa w(\kappa) \theta^\kappa = \sum_j \langle w_j, \theta_j \rangle \prod_{i \neq j} |\theta_i|$, the open set $\Sigma \cap \{\ell_0 \neq 0\}$ is identified with $\mathcal{A}$, on which $\pi \circ \sigma = [1 : p_E]$ and $\ell_w / \ell_0 = \langle w, \theta \rangle$. Let V_0 be the closure of $\sigma(V_w)$. The hyperplane section $\Sigma \cap V(\ell_w)$ may be reducible (if w_j is constant, $\ell_w \circ \sigma$ is divisible by $|\theta_j|$), but its part in $\{\ell_0 \neq 0\}$ is $\sigma(V_w)$, so all its other components lie in $\{\ell_0 = 0\}$. By Lemma 2.2(c) the projective closure $\bar{H}$ of H , whose degree is $\deg f$, is the closure of $\pi(V_0 \cap \{\ell_0 \neq 0\})$.

Let $N = \deg f$. The constructible set $\pi(V_0 \cap \{\ell_0 \neq 0\})$ is dense in $\bar{H}$, and $\dim V_0 = \dim \bar{H}$; so it contains a dense open subset O of $\bar{H}$ over which π has finite fibres on V_0 . Let $L \subseteq \mathbb{P}^m$ be a general linear subspace of codimension $m - 1$, cut out by linear forms $g_1, \dots, g_{m-1}$. Then $L \cap \bar{H}$ consists of N points, and they lie in O , because $\bar{H} \setminus O$ has dimension at most $m - 2$. Choose preimages $z_1, \dots, z_N \in V_0 \cap \{\ell_0 \neq 0\}$. The set of points of $V_0 \cap \{\ell_0 \neq 0\}$ where all $g_i \circ \pi$ vanish is finite, and near z_i the intersection $\Sigma \cap V(\ell_w) \cap V(g_1 \circ \pi, \dots, g_{m-1} \circ \pi)$ coincides with it. So $z_1, \dots, z_N$ are isolated points of the intersection of Σ with m hyperplanes. By the refined Bézout theorem [Ful98, Example 8.4.6], the number of irreducible components of this intersection is at most $\deg \Sigma$, so $N \leq \deg \Sigma$.

The equality cases: $\{12, 21\}$ is the corner pattern $E^*(11)$ of 2×2 , with $\deg f = 2$ by Theorem 1.1; the $2 \times 2 \times 2$ corner patterns have $\deg f = 6 = 3!$; and for $\{111, 122, 212, 223\}$, $\deg f = 12 = 4!/2!$ by Proposition 7.1. $\square$

7 Computations

7.1 Exact certificates

We use four kinds of certificates for $\deg f = N$.

- (L) *Lower bound.* If a nonzero polynomial of degree at most $N - 1$ vanished on H , then so would a primitive one $g \in \mathbb{Z}[x]$, and $|\lambda|^{n \deg g} g(x(\lambda))$ would be the zero polynomial in $\mathbb{Z}[\lambda]$; reducing modulo a prime ℓ , the nonzero coefficient vector of $g \bmod \ell$ would be a kernel vector of the matrix of all monomials of degree at most $N - 1$ evaluated at points $x(\lambda)$, $\lambda \in \mathbb{F}_\ell^E$ with $|\lambda| \neq 0$, computed in $\mathbb{F}_\ell$. So full column rank of such a matrix modulo ℓ proves $\deg f \geq N$. The same holds for images of rational points of V_w reduced modulo ℓ .
- (C) *Upper bound by an explicit polynomial.* A polynomial $F \in \mathbb{Z}[x]$ of degree N is obtained from kernels modulo several primes and rational reconstruction. Its vanishing on H is then proved exactly: $F \circ p_E$ restricted to V_w , in affine coordinates $s \in \mathbb{C}^{m-1}$, has degree at most nN in each variable, so its vanishing on the grid $\{0, \dots, nN\}^{m-1}$ implies that it is zero [Alo99, Lemma 2.1]; the values on the grid are computed in exact rational arithmetic. Equivalently one checks that $|\lambda|^{nN} F(x(\lambda))$ vanishes on such a grid, modulo enough primes to exceed twice an a priori bound on its values. Then $f \mid F$ and $\deg f \leq N$.

- (T) *Theorems.* Theorem 1.1, and Theorem 1.3 with M computed in exact integer Laurent polynomial arithmetic.
- (B) *Bounds.* Theorem 1.5, or Corollary 4.4 with W and the μ_e computed in exact rational arithmetic; combined with (L).

Admissibility is decided exactly by Lemma 2.1(b), saturation by the greatest common divisor of the maximal minors of B , and W by an exact kernel computation. (The programs in the release and their outputs call saturated patterns “unimodular” and write “n.u.” for patterns that are not saturated.)

Proposition 7.1. *The degrees in Table 1 are correct. In the matrix formats, the orbits of admissible patterns have the following degrees, by Corollary 1.4: 2 in 2×2 ; $2, 4$ in 3×3 ; $2, 4, 4, 6$ in 3×4 .*

E	W	$\deg f$	cert.	E	W	$\deg f$	cert.
$2 \times 2 \times 2$				$2 \times 2 \times 2 \times 2$			
$\{111, 112, 221\}$	2	4	T, C, B	$\{1111, 1112, 1121, 2211\}$	2	6	T
$\{111, 122, 212\}^c$	3	6	T, C, B	$\{1111, 1112, 1121, 2212\}$	2	6	T
$2 \times 2 \times 3$				$\{1111, 1112, 1221, 2221\}$	2	6	T
$\{111, 112, 113, 221\}$	2	4	T, C	$\{1111, 1112, 1121, 2222\}$	2	8	T
$\{111, 112, 121, 213\}$	2	4	T, C	$\{1111, 1112, 1221, 2222\}$	2	8	T
$\{111, 112, 123, 223\}$	2	4	T, C	$\{1111, 1112, 1221, 2121\}$	3	9	B
$\{111, 112, 121, 223\}$	2	6	T, C	$\{1111, 1112, 1221, 2122\}$	3	11	B
$\{111, 112, 123, 213\}^c$	3	6	T, C	$\{1111, 1122, 1212, 2112\}^c$	4	12	T, B
$\{111, 112, 221, 223\}$	2	6	T, C	$\{1111, 1122, 2212, 2221\}^*$	2	12	C
$\{111, 121, 212, 223\}$	2	6	T, C	$\{1111, 1122, 1212, 2121\}$	4	14	B
$\{111, 112, 123, 221\}$	3	8	C	$\{1111, 1122, 1212, 2221\}$	5	18	B
$\{111, 122, 212, 223\}$	4	12	C, B				

Table 1: Degrees of H for all orbits of admissible patterns in the formats $2 \times 2 \times 2$, $2 \times 2 \times 3$ and $2 \times 2 \times 2 \times 2$ (one pattern per orbit under relabelling within coordinates and permuting coordinates of equal size). c : corner pattern; * : not saturated. Certificates as in Section 7; “B” combines an upper bound with a lower bound (L).

Proof. The matrix formats are covered by Corollary 1.4. In Table 1, every orbit marked T is covered by Theorem 1.1 or by Theorem 1.3; for the latter the exact values of M are computed by three independent programs. Every orbit of $2 \times 2 \times 2$ and $2 \times 2 \times 3$ has a certificate of type (C) from the independent verifier’s program, which reconstructs F and proves its vanishing exactly; the finder’s program gives a second one for all these orbits except the last. For $\{111, 122, 212, 223\}$ the certified F has 217 terms, and Theorem 1.5 with (L) gives a second proof. For the binary orbits marked B, Corollary 4.4 gives $\deg f \leq W + \sum_e \mu_e$, which equals the listed degree, and (L) gives $\deg f \geq$ the listed degree; each rank computation was done by the author’s program and by at least one of the two verifiers’ programs. For the pattern that is not saturated, F has 78 terms, lowest form $(x_{1111}x_{1122}x_{2212}x_{2221})^2$, and its vanishing on H and the lower bound (L) were proved. $\square$

In the example that is not saturated, the elementary divisors of B are $1, 1, 1, 2$, so the torus $\{t : t^{B_e} = 1\}$ has two components; a numerical count finds $4 = 2W$ complex completions of a generic point, and $\deg f - \deg f_{\text{low}} = 4$.

7.2 A census

For a random line $x(t) = P + tQ$ over $\mathbb{F}_\ell$ with $\ell \approx 2^{31}$, the finder's program computes the resultant of $\tau^{-a-}(\Phi_{x(t)} - 1)$ and its τ -derivative as a polynomial in t by interpolation, removes the factors coming from the coordinate hyperplanes, from the leading and constant coefficients and from common roots at $\tau = 0$, and counts the distinct remaining roots. For a generic line this number is $\deg f$. The computation was repeated with two or three independent lines and primes, which always agreed; the results are correct with high probability, but they are not proofs. For patterns that are not saturated, the same computation is done along a curve $x_e = y_e^c$, where $y = P + tQ$ is a random line and c is the least common multiple of the elementary divisors of B , on all components of the torus $\{t : t^{B_e} = 1\}$; the number of points of H on this curve is $c \deg f$, and the program divides it by c . Table 2 summarizes the census. The orbit counts in the exhaustive formats are exact (Lemma 2.1(b)); the numbers of admissible patterns are 2, 6, 14, 30, 45, 228, 975, 2176 in the matrix formats and 32, 240, 1408, 4050, 1256, 7360, 24720, 48624 in the others, in the order of Table 2.

format	orb.	n.s.	$W=2$	degrees	bd.	format	orb.	n.s.	$W=2$	degrees	bd.
2×2	1	0	1	2	2	$2 \times 2 \times 2$	2	0	1	4–6	6
2×3	1	0	1	2	3	$2 \times 2 \times 3$	9	0	6	4–12	12
2×4	2	0	2	2	4	$2 \times 2 \times 4$	16	0	10	4–13	20
2×5	2	0	2	2	5	$2 \times 3 \times 3$	39	1	24	4–20	30
3×3	2	0	2	2–4	6	$2 \times 2 \times 2 \times 2$	11	1	5	6–18	24
3×4	4	0	4	2–6	10	$2 \times 2 \times 5$	30	0	19	4–14	30
3×5	6	0	6	2–6	15	$2 \times 2 \times 2 \times 3$	122	5	58	6–31	60
4×4	7	0	7	2–8	20	$2 \times 3 \times 4$	229	4	142	4–30	60

Table 2: The exhaustive part of the census: exact numbers of orbits of admissible patterns, of orbits that are not saturated (n.s.), and of saturated orbits with $W = 2$, whose degrees are proved by Theorem 1.3; the range of the census degrees; and the bound of Theorem 1.5 (bd.).

The census also contains random samples of admissible patterns: 200 each in 4×5 and 5×5 , 600 in $3 \times 3 \times 3$, 200 in $3 \times 3 \times 4$, 300 in 2^5 , 200 in $2 \times 2 \times 3 \times 3$, 60 in $3 \times 4 \times 5$, 60 in 2^6 and 40 in 3^4 . The largest degrees found are 39 in $3 \times 3 \times 3$, 46 in 2^5 , 104 in 2^6 and 122 in 3^4 , far below the bounds 90, 120, 720 and 2520 of Theorem 1.5. Of the 2343 patterns, 72 are not saturated. In the 2271 saturated ones the census observed the following.

- $q = 1$ and $W^* = W$, so $\deg f = W + \deg \text{Den} = W + \deg f_{\text{low}}$. Odd degrees occur; in 2^4 the degrees 9 and 11 are proved in Proposition 7.1.
- For every $e \in E$ the multiplicity of x_e in Den equals μ_e . Where Den has other factors, they come from the Laurent polynomials in Theorem 1.2(c), and a heuristic extension of the tropical recipe to them (in the finder's program) also agreed with the census degree.
- Corner patterns always had degree $n(n-1)$, and matrices $2 + 2 \text{pin}(E)$.

Conjecture 7.2. *For every admissible saturated E , the W critical values of Φ_x are nonzero and pairwise distinct for generic x . Equivalently $q = 1$ and $W^* = W$, and then $\deg f = W + \deg f_{\text{low}}$.*

Conjecture 7.3. *For every admissible saturated E and every $e \in E$, the multiplicity of x_e in Den equals μ_e . Together with Conjecture 7.2, this gives $\deg f = W + \sum_e \mu_e$ whenever Den is a monomial, for instance in binary formats.*

Conjecture 7.2 holds for $W = 2$ (Theorem 1.3) and for corner patterns: there the cell weights are $1 - \#\{j : \kappa_j \neq s_j\}$, so $W = n$, and Theorem 1.1 gives $W^*/q = \deg f - \deg f_{\text{low}} = n$. Without Conjecture 7.2, Conjecture 7.3 and Theorem 1.2(b) would give only $\deg f = (W^* + \sum_e \mu_e)/q$ when Den is a monomial.

8 Concluding remarks

Theorems 1.1 and 1.3 and Corollary 1.4 answer Problem 3.15 of [KKKR17] for corner patterns, for all matrix patterns and for all saturated patterns with $W = 2$; Theorem 1.5 bounds the degree in general; and Theorem 1.2 reduces the saturated case to the behaviour of the critical values of one Laurent polynomial, in the spirit of the tropical approach suggested in [Kum18]. A complete answer would require Conjectures 7.2 and 7.3, a description of the non-monomial factors of Den, and an extension of Theorem 1.2 to patterns that are not saturated, where the fibre torus is disconnected.

Verification. The claims were checked as follows.

1. The finder's programs are included in the release. They implement the census of Section 7.2, give certificates (L)+(C) for 16 patterns (all orbits of 3×3 , 3×4 and $2 \times 2 \times 2$ and eight orbits of $2 \times 2 \times 3$) and lower bounds (L) for 15 patterns, check Theorem 1.1 in exact arithmetic (for $n \leq 8$, $\mathcal{D}_n$ restricted to an integer line has degree $n(n-1)$, the predicted leading coefficient, and is irreducible over $\mathbb{Q}$; $\mathcal{D}_n(\xi(a)) = 0$ at rational points for $n \leq 7$), and evaluate Theorem 1.3 exactly for the 290 saturated orbits with $W = 2$ of the exhaustive formats, with no cancellation in $\beta_1 - \beta_2$ in any of the 1602 pairs of an orbit and a coordinate.
2. The author's programs (exact arithmetic; the rank computations use numpy) recompute Table 2 with the exact test of Lemma 2.1(b), check that the orbit representatives agree with the census, recompute the degrees of the 290 orbits with $W = 2$, prove the lower bounds (L) used in Table 1, and verify the range in Corollary 1.4 by brute force in all matrix formats up to 5×5 , 4×6 , 3×7 and 2×8 . They also compute the bound of Corollary 4.4 and compare it with the census degrees of the 132 orbits that the census output lists individually: the bound equals the census degree for all 25 of these orbits, of any W , that satisfy the criterion of Corollary 4.4 (every non-constant w_j attains its maximum and its minimum only once). Over all exhaustive orbits the criterion holds for 31 orbits with $W \geq 3$ alone, so this comparison does not cover all of them.
3. A first independent verifier checked all proofs by hand and wrote separate code based on Lemma 2.2(c): an exact orbit enumeration; certificates (L)+(C) for all 18 orbits of 2×2 , 3×3 , 3×4 , $2 \times 2 \times 2$ and $2 \times 2 \times 3$, including a 217-term polynomial of degree 12; the discriminants $\mathcal{D}_n$ for $n \leq 4$ and a grid proof of $\mathcal{D}_n(\xi(a)) = 0$ for $n = 3, 4$; a certificate for the $2 \times 3 \times 3$ corner pattern; Theorem 1.3 for all 290 orbits; lower bounds (L) for census degrees up to 14; numerical counts of complex completions; and reruns of the finder's programs. The author ran this verifier's certificate program on the pattern of 2^4 that is not saturated.
4. A second independent verifier checked the source, the proofs and the literature, and wrote an implicitization based on the linear form of [KKKR17, Proposition 3.12]; it proves the lower bounds (L) for 17 degrees, among them 9, 11, 12, 14 and 18 in 2^4 .
5. A further independent referee checked the previous version of this note line by line and found no mathematical error. Its programs, written from the text before the other programs were read, are in the folder `referee2` of the release, next to those of the second verifier. They reproduce

the exact columns of Table 2; Theorem 1.3 for all 290 orbits, again without cancellation in the 1602 pairs; the bounds of Corollary 4.4 used in Table 1; all lower bounds (L) of Table 1; the certificates (C) for $2 \times 2 \times 2$, $2 \times 2 \times 3$ and the pattern of 2^4 that is not saturated, with the vanishing on H proved in exact integer arithmetic; the statements of Theorem 1.1 on random integer lines for $n \leq 7$, and the equality of $\mathcal{D}_3$ with the polynomial of [KKKR17, Example 3.16]; and the range in Corollary 1.4. They also match the 132 orbits listed individually in the census output with the referee’s own orbits, with no disagreement. The referee pointed out the overlap with [CRY24] and the known degrees for diagonal patterns, which are now credited in Section 1. The remarks relating Lemmas 2.2, 4.1 and 4.2 to [CRY24], the remarks on the word “saturated” and on diagonal patterns, and the rewritten steps in the proofs of Lemma 5.2 and Corollary 1.4 were written after the referee’s report and were checked by the author only; the author’s program `revision2_checks.py` checks their elementary statements in exact arithmetic.

Scope and priority. This note answers the degree question of [Kum18] and [KKKR17, Problem 3.15] only in part: for corner patterns in every format, for all admissible matrix patterns, for all saturated patterns with $W = 2$, and for all admissible patterns in the six small formats of Proposition 7.1; for all other patterns it proves the structural identity of Theorem 1.2 (saturated case) and the bound of Theorem 1.5. The census results are computational evidence, not proofs. The matrix criterion and the criteria for $\{11, 12, 21, 33\}$ and the $2 \times 2 \times 2$ patterns are due to Kubjas and Rosen [KR17], the degree 6 of the $2 \times 2 \times 2$ corner pattern to [KKKR17], and the bound $\deg f \leq n(n-1)$ for corner patterns to the companion note [Fer26]. The positive real forms of Lemma 2.2(a),(c) and Lemma 4.1, and the statements on completions in Δ in Lemma 4.2 and Theorem 1.2(a), are due to Cai, Recke and Yahl [CRY24], whose hypothesis holds for all admissible patterns; we claim as new only the count of complex completions in Theorem 1.2(a), parts (b) and (c) of Theorem 1.2, and the results on degrees. The degrees for diagonal patterns, which satisfy (1) only in 2×2 , are due to [KR17, KKKR17]. We read [Kum18], arXiv:1605.01678v2 (whose numbering we use for [KKKR17]), arXiv:1407.3254v2 and the journal version of [KR17], and the full text of arXiv:2312.15154v2 (whose numbering we use for [CRY24]). We searched arXiv, Crossref and zbMATH, the works citing [KKKR17] and [KR17] in OpenAlex and Semantic Scholar, and the web. Under the assumptions (1) we found no work that determines the degree of H beyond [KKKR17, Example 3.16] and [Fer26]. The OpenAlex lists of works citing [KKKR17] (10 works) and [KR17] (4 works) were last retrieved on September 30, 2026. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and the certificates, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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