

Seigal's Conjecture on Gram Determinants of Real Binary Tensors Fails for $n \geq 4$

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October 2, 2026

Abstract

For a real tensor of format $2 \times \cdots \times 2$ with n factors, let d_i be the determinant of the Gram matrix $M_i M_i^T$ of its i -th flattening M_i . Seigal called the set of tuples $(d_1, \dots, d_n)$ of tensors in the Frobenius unit ball the Gram locus, described it for $n = 3$, and conjectured that for $n \geq 4$ it is cut out of the cube $[0, 1/4]^n$ by a single polynomial inequality $Q_1 \geq Q_2$ (Linear Algebra Appl. 2018, Conjecture 1.5; Oberwolfach Reports 2017). We point out that this conjecture is false for every $n \geq 4$. For instance, $(9/100, 9/100, 9/100, 6/25)$ satisfies $Q_1 > 500 Q_2$ but is not the Gram tuple of any real or complex tensor in the unit ball, and the counterexamples form a set with nonempty interior. This is a short corollary of known results: by theorems of Domanov, Stegeman and De Lathauwer (2017), and of Higuchi, Sudbery and Szulc (2003) for complex tensors, plus an elementary step from the unit sphere to the unit ball, the Gram locus is the set of $d \in [0, 1/4]^n$ with $\sum_{j \neq i} \sqrt{1 - 4d_j} - \sqrt{1 - 4d_i} \leq n - 2$ for all i . From this we obtain the correct n -ary form of Seigal's theorem for $n = 3$, a union of two basic semialgebraic sets given by explicit integer polynomials, and we show that for no $n \geq 3$ is the Gram locus the nonnegativity set of a single polynomial inside the cube (for $n = 2$ it is). We also correct a misprinted inequality in Seigal's theorem for $n = 3$. This is an unrefereed note.

1 Introduction

Let $n \geq 2$ and let $A = (a_x)_{x \in \{0,1\}^n}$ be a real tensor of format $2 \times \cdots \times 2$ (n factors). Its i -th flattening M_i is the $2 \times 2^{n-1}$ matrix whose rows are indexed by x_i and whose columns are indexed by the remaining indices. Following Seigal [Sei18], put

$$d_i = \det(M_i M_i^T), \quad \mathcal{G}(A) = (d_1, \dots, d_n), \quad \mathcal{B} = \left\{ A : \|A\|^2 = \sum_x a_x^2 \leq 1 \right\}.$$

The set $\mathcal{G}(\mathcal{B}) \subset \mathbb{R}^n$ is the *Gram locus*. Each $G_i = M_i M_i^T$ has trace $\|A\|^2$, and its eigenvalues are the squared higher-order singular values of A in mode i ; so $\mathcal{G}(\mathcal{B})$ encodes the feasible higher-order singular values of binary tensors, which were asked for in [HU17].

Seigal proved that the convex hull of $\mathcal{G}(\mathcal{B})$ is the polytope $\{d \in [0, \frac{1}{4}]^n : d_i \leq \sum_{j \neq i} d_j \text{ for all } i\}$ [Sei18, Theorem 1.3], and introduced the polynomials

$$Q_1 = \prod_{i=1}^n \left(\sum_{j \neq i} d_j - d_i \right), \quad Q_2 = \frac{1}{2} \prod_{\varepsilon \in \{\pm 1\}^n} (\varepsilon_1 \sqrt{d_1} + \cdots + \varepsilon_n \sqrt{d_n}).$$

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The product defining Q_2 is invariant under every sign change of a $\sqrt{d_j}$, so Q_2 is a polynomial in d ; it is $\frac{1}{2}$ times a square. For $n = 3$ she showed [Sei18, Theorem 1.4] that inside the cube $\mathcal{G}(\mathcal{B})$ is the union of $\{Q_1 \geq Q_2\}$ and of the set where $Q_1 \leq Q_2$ and

$$(d_i - d_j)^2 + \frac{1}{2}(d_i + d_j) \leq \frac{3}{16} \quad \text{for all } \{i, j\} \subset \{1, 2, 3\}, \quad (1)$$

and she conjectured [Sei18, Conjecture 1.5] that the second set is not needed for larger n :

$$\mathcal{G}(\mathcal{B}) = \{d \in [0, \frac{1}{4}]^n : Q_1(d) \geq Q_2(d)\} \quad \text{for every } n \geq 4. \quad (2)$$

In her Oberwolfach abstract [Sei17] the general form of the Gram locus is described as “concisely expressed as the non-negativity of a single polynomial”, with a reference to this conjecture. The question appears as record OWR-15428-003 of the UnsolvedMath dataset [ula26], with the status “partially solved”. Seigal’s thesis [Sei19, Chapter 4] restates the conjecture unchanged as Conjecture 4.9, and Theorem 1.4 as Theorem 4.8; it cites [DSL17] only for necessary inequalities between the largest higher-order singular values.

We show that (2) is false for every $n \geq 4$, and that no single polynomial inequality can describe $\mathcal{G}(\mathcal{B})$ inside the cube when $n \geq 3$. The starting point is the following description of the Gram locus. It is a reformulation, in Seigal’s coordinates, of theorems of Domanov, Stegeman and De Lathauwer [DSL17, Theorems 4 and 5] (for real and complex tensors, with $I = 2$) and of Higuchi, Sudbery and Szulc [HSS03] (for complex tensors, in the language of quantum states; see also [Bra04]). We include short proofs in Section 2. Throughout, for $d \in [0, \frac{1}{4}]^n$ we write

$$r_j = \sqrt{1 - 4d_j} \in [0, 1], \quad \rho_i(d) = \sum_{j \neq i} r_j - r_i.$$

Theorem 1.1 (after Domanov–Stegeman–De Lathauwer and Higuchi–Sudbery–Szulc). *For every $n \geq 2$ we have $\mathcal{G}(\mathcal{B}) = \mathcal{G}(\partial\mathcal{B})$ and*

$$\mathcal{G}(\mathcal{B}) = \{d \in [0, \frac{1}{4}]^n : \rho_i(d) \leq n - 2 \text{ for } i = 1, \dots, n\}.$$

Every point of $\mathcal{G}(\mathcal{B})$ is the Gram tuple of a tensor of norm one with nonnegative entries.

Corollary 1.2. *For every $n \geq 4$, the set of points $d \in [0, \frac{1}{4}]^n$ with $Q_1(d) > Q_2(d)$ and $d \notin \mathcal{G}(\mathcal{B})$ has nonempty interior, so (2) is false. Explicit such points are:*

- (a) *for $n = 4$, $d' = (\frac{9}{100}, \frac{9}{100}, \frac{9}{100}, \frac{6}{25})$, where $Q_1(d') = 107811 \cdot 10^{-8} > 500 Q_2(d')$;*
- (b) *for $n \geq 5$, the point $(d', 0, \dots, 0)$;*
- (c) *for $n = 4$ and $0 < x \leq \frac{1}{12}$, the point $(x, x, x, 3x - x^2)$. These points tend to 0, and for every constant $c > 0$ they satisfy $Q_1 > c Q_2$ once x is small enough.*

No real or complex tensor of norm at most one has these Gram tuples. They violate the inclusion $\{Q_1 \geq Q_2\} \subseteq \mathcal{G}(\mathcal{B})$, while the random test of [Sei18, Section 3] probes only the opposite inclusion (Remark 3.7).

For the correct analogue of [Sei18, Theorem 1.4] put, for $n \geq 3$ and $d \in [0, \frac{1}{4}]^n$,

$$N_n = \prod_{\varepsilon \in \{\pm 1\}^n} \left(n - 2 - \sum_{j=1}^n \varepsilon_j r_j \right), \quad M_i = \prod_{\varepsilon \in \{\pm 1\}^{n-1}} \left(n - 2 - \sum_{j \neq i} \varepsilon_j r_j \right) \quad (1 \leq i \leq n).$$

Both products are invariant under each sign change $r_j \mapsto -r_j$, so they are polynomials in $r_1^2, \dots, r_n^2$ with integer coefficients. Substituting $r_j^2 = 1 - 4d_j$ makes N_n and M_i polynomials in d with integer coefficients, of degrees 2^{n-1} and 2^{n-2} .

Theorem 1.3. *Let $n \geq 3$. Then*

$$\mathcal{G}(\mathcal{B}) = \{d \in [0, \frac{1}{4}]^n : N_n(d) \leq 0\} \cup \{d \in [0, \frac{1}{4}]^n : N_n(d) \geq 0, M_1(d) \geq 0, \dots, M_n(d) \geq 0\}.$$

For $n = 3$ one has $N_3 = -512(Q_1 - Q_2)$ and $M_k = 16((d_i - d_j)^2 + \frac{1}{2}(d_i + d_j) - \frac{3}{16})$ for $\{i, j, k\} = \{1, 2, 3\}$. Hence [Sei18, Theorem 1.4] holds with “ $\geq \frac{3}{16}$ ” in place of “ $\leq \frac{3}{16}$ ” in (1).

The inequality “ $\leq \frac{3}{16}$ ” in (1) is a misprint (Remark 4.2): Seigal’s proof yields “ $\geq$ ”, and with “ $\leq$ ” the statement is false. The union in Theorem 1.3 cannot be replaced by a single inequality. Let U be the interior of the convex hull of $\mathcal{G}(\mathcal{B})$, that is, the set of $d \in (0, \frac{1}{4})^n$ with $d_i < \sum_{j \neq i} d_j$ for all i .

Theorem 1.4. *Let $n \geq 3$. There is no polynomial $F \in \mathbb{R}[d_1, \dots, d_n]$ such that*

$$\mathcal{G}(\mathcal{B}) \cap U = \{d \in U : F(d) \geq 0\}.$$

In particular, inside the cube $\mathcal{G}(\mathcal{B})$ is not the nonnegativity set of a single polynomial, not even together with the linear inequalities $d_i \leq \sum_{j \neq i} d_j$. For $n = 2$, $\mathcal{G}(\mathcal{B}) = \{d \in [0, \frac{1}{4}]^2 : -(d_1 - d_2)^2 \geq 0\}$.

What is new. The exact locus is known in substance: for $I = 2$, [DSL17, Theorems 4 and 5] describe exactly which squared multilinear singular values occur, [HSS03] does the same for complex tensors of norm one, and Corollary 1.2 follows in a few lines. Yet we found no source that states the locus in Seigal’s coordinates. On the contrary, the authors of [DSL17] remark that their result was independently proved for real $2 \times \dots \times 2$ tensors in Seigal’s preprint, and Krämer [Krä19, Section 1.4] repeats this, although [Sei18, Theorem 1.3] gives only the convex hull of $\mathcal{G}(\mathcal{B})$ and (2) contradicts [DSL17, Theorem 4]. The explicit refutation, Theorems 1.3 and 1.4 and the correction of the misprint are, as far as we found, not in the literature.

2 The Gram locus

For $c \in [0, \frac{1}{4}]$ let $f(c) = \frac{1}{2}(1 - \sqrt{1 - 4c})$, the smaller eigenvalue of a positive semidefinite 2×2 matrix with trace 1 and determinant c . It is an increasing bijection $[0, \frac{1}{4}] \rightarrow [0, \frac{1}{2}]$ with inverse $h(\lambda) = \lambda(1 - \lambda)$, and $r(c) = \sqrt{1 - 4c} = 1 - 2f(c)$. More generally, a positive semidefinite 2×2 matrix with trace $t > 0$ and determinant $c \leq t^2/4$ has smaller eigenvalue

$$\mu(c, t) = \frac{1}{2}(t - \sqrt{t^2 - 4c}) = \frac{2c}{t + \sqrt{t^2 - 4c}},$$

and $\mu(c, 1) = f(c)$. Let P_n be the polytope of all $\lambda \in [0, \frac{1}{2}]^n$ with $\lambda_i \leq \sum_{j \neq i} \lambda_j$ for all i . For a tensor A we write μ_i for the smaller eigenvalue of $G_i = M_i M_i^T$.

Lemma 2.1 ([DSL17, Theorem 4], [HSS03]). *For every real tensor A and every i , $\mu_i \leq \sum_{j \neq i} \mu_j$. The same holds for complex tensors, with $G_i = M_i M_i^*$.*

In the notation of [DSL17], with $t = \|A\|^2$, the largest squared multilinear singular value of mode i is $\sigma_i^2 = t - \mu_i$, and their inequality $\sum_{k \neq i} \sigma_k^2 \leq (n - 2)t + \sigma_i^2$ is exactly $\mu_i \leq \sum_{k \neq i} \mu_k$. We give a short direct proof.

Proof. Take $i = 1$ and let $t = \|A\|^2$. For $j \geq 2$ choose $O_j \in O(2)$ with $O_j G_j O_j^T = \text{diag}(t - \mu_j, \mu_j)$, and replace A by $(I \otimes O_2 \otimes \dots \otimes O_n)A$. This multiplies M_1 on the right by an orthogonal matrix, so G_1 does not change, and it conjugates G_j by O_j for $j \geq 2$. So we may assume that $\sum_{x: x_j=1} a_x^2 = \mu_j$ for every $j \geq 2$. The rows of M_1 are $v = (a_{0y})_y$ and $w = (a_{1y})_y$, $y \in \{0, 1\}^{n-1}$. Let $u = (u_0, u_1)$ be

a unit vector orthogonal to $(a_{00\dots 0}, a_{10\dots 0})$. Then, by the Rayleigh principle, the choice of u and the Cauchy–Schwarz inequality,

$$\mu_1 \leq u^T G_1 u = \sum_y (u_0 a_{0y} + u_1 a_{1y})^2 = \sum_{y \neq 0} (u_0 a_{0y} + u_1 a_{1y})^2 \leq \sum_{y \neq 0} (a_{0y}^2 + a_{1y}^2) \leq \sum_{j=2}^n \sum_{x: x_j=1} a_x^2 = \sum_{j=2}^n \mu_j,$$

where the last inequality holds because every x with $(x_2, \dots, x_n) \neq 0$ has $x_j = 1$ for some $j \geq 2$. For complex tensors the same argument works with unitary O_j and $|\cdot|^2$. $\square$

Lemma 2.2 (case $I = 2$ of the construction in the proof of [DSL17, Theorem 5]). *For every $\lambda \in P_n$ there is a real tensor A with nonnegative entries, $\|A\| = 1$ and $G_j = \text{diag}(1 - \lambda_j, \lambda_j)$ for all j . In particular $\mathcal{G}(A) = (h(\lambda_1), \dots, h(\lambda_n))$.*

Proof. First, $2\lambda = \sum_K w_K \mathbf{1}_K$ for weights $w_K \geq 0$ on sets $K \subseteq \{1, \dots, n\}$ with $|K| \geq 2$ and $\sum_K w_K = 2 \max_j \lambda_j \leq 1$. To see this, put $y = 2\lambda$, let m be an index with y_m maximal, and $S = \sum_{j \neq m} y_j \geq y_m$. If $y_m = S$, take $w_{\{m,j\}} = y_j$ for $j \neq m$. Otherwise the support K of y has at least three elements (if $|K| \leq 2$, the inequality $y_m \leq S$ forces $y_m = S$). Subtract $w \mathbf{1}_K$ with $w = \min\{\min_{j \in K} y_j, (S - y_m)/(|K| - 2)\} > 0$. The new vector is nonnegative, still has its maximum at m , and still satisfies $y_m \leq S$, because this inequality decreases by $(|K| - 2)w \leq S - y_m$. Either the support shrinks or equality $y_m = S$ is reached, so the process ends after at most n steps. Each step lowers the maximum by the weight it uses, so the total weight is the initial maximum $2 \max_j \lambda_j$.

For $|K| \geq 2$ let U_K be the uniform distribution on the strings $x \in \{0, 1\}^n$ of even weight that are supported in K . Each $j \in K$ has $x_j = 1$ for exactly half of them. Hence $p = (1 - \sum_K w_K) \delta_0 + \sum_K w_K U_K$ is a probability distribution on strings of even weight with $\Pr_p(x_j = 1) = \lambda_j$ for all j . Put $a_x = \sqrt{p(x)}$. Then $\|A\| = 1$ and $(G_j)_{11} = \Pr_p(x_j = 1) = \lambda_j$. The off-diagonal entry of G_j is a sum of products $a_x a_{x'}$ over pairs x, x' that differ only in coordinate j ; one of the two has odd weight, so every product vanishes. $\square$

The tensors of [DSL17] are likewise supported on a parity class. The only ingredient not stated in [DSL17, HSS03] is the passage from the unit sphere to the unit ball.

Lemma 2.3. *If $\|A\| \leq 1$ and $d = \mathcal{G}(A)$, then $d \in [0, \frac{1}{4}]^n$ and $f(d_i) \leq \sum_{j \neq i} f(d_j)$ for every i .*

Proof. Let $t = \|A\|^2$; we may assume $t > 0$. Then $0 \leq d_i = \det G_i \leq t^2/4 \leq \frac{1}{4}$ and $\mu_i = \mu(d_i, t)$. Let m be an index with d_m maximal and assume $d_m > 0$ (otherwise $d = 0$). For $i \neq m$ the claimed inequality is clear, since $f(d_i) \leq f(d_m)$ and $f(d_m)$ is one of the summands. For $0 \leq c \leq d_m$ and $\tau \geq 2\sqrt{d_m}$ consider

$$R_c(\tau) = \frac{\mu(c, \tau)}{\mu(d_m, \tau)} = \frac{c}{d_m} \cdot \frac{\tau + \sqrt{\tau^2 - 4d_m}}{\tau + \sqrt{\tau^2 - 4c}}.$$

Since $\frac{d}{d\tau} \log(\tau + \sqrt{\tau^2 - 4c}) = 1/\sqrt{\tau^2 - 4c}$ is increasing in c , the function R_c is nondecreasing in τ . By Lemma 2.1, $1 \leq \sum_{j \neq m} R_{d_j}(t)$. As $2\sqrt{d_m} \leq t \leq 1$, also $1 \leq \sum_{j \neq m} R_{d_j}(1)$, which is the claim for $i = m$. $\square$

Proof of Theorem 1.1. Write $h(P_n)$ for the image of P_n under $\lambda \mapsto (h(\lambda_1), \dots, h(\lambda_n))$. Since f is increasing with inverse h , a point $d \in [0, \frac{1}{4}]^n$ lies in $h(P_n)$ if and only if $f(d_i) \leq \sum_{j \neq i} f(d_j)$ for all i . With $f(d_j) = (1 - r_j)/2$ this reads $\rho_i(d) \leq n - 2$. By Lemma 2.3, $\mathcal{G}(\mathcal{B}) \subseteq h(P_n)$. By Lemma 2.2, $h(P_n) \subseteq \mathcal{G}(\partial \mathcal{B}) \subseteq \mathcal{G}(\mathcal{B})$. $\square$

Remark 2.4. (a) By Lemma 2.1 the same argument applies to complex tensors in the unit ball, so they have the same Gram locus (with $G_i = M_i M_i^*$).

(b) In the coordinates $\lambda_i = f(d_i)$, the smaller eigenvalues of the G_i for tensors of norm one, the Gram locus is P_n , and Seigal's Q_1 describes it: $\mathcal{G}(\mathcal{B}) = \{d \in [0, \frac{1}{4}]^n : Q_1(f(d_1), \dots, f(d_n)) \geq 0\}$. Indeed, for $\lambda \geq 0$ any two factors $c_i = \sum_{j \neq i} \lambda_j - \lambda_i$ and c_k of $Q_1(\lambda)$ satisfy $c_i + c_k \geq 0$, so at most one factor is negative, and if one is, no other vanishes. The map $d \mapsto \lambda$ is not polynomial, and Theorem 1.4 shows that this cannot be repaired.

3 Counterexamples to Conjecture 1.5

For $d = (a, a, a, b)$, pair the two linear forms in the definition of Q_2 that differ only in ε_4 : with $k = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ their product is $(k\sqrt{a} + \sqrt{b})(k\sqrt{a} - \sqrt{b}) = k^2a - b$, and $k = \pm 3$ once each and $k = \pm 1$ three times each. This gives

$$Q_1(a, a, a, b) = (a + b)^3(3a - b), \quad Q_2(a, a, a, b) = \frac{1}{2}(9a - b)^2(a - b)^6. \quad (3)$$

Proposition 3.1. *The point $d' = (\frac{9}{100}, \frac{9}{100}, \frac{9}{100}, \frac{6}{25})$ lies in the open cube $(0, \frac{1}{4})^4$ and in the convex hull of $\mathcal{G}(\mathcal{B})$. It satisfies*

$$Q_1(d') = \left(\frac{33}{100}\right)^3 \frac{3}{100} = \frac{107811}{10^8}, \quad Q_2(d') = \frac{1}{2} \left(\frac{57}{100}\right)^2 \left(\frac{15}{100}\right)^6 = \frac{2368521}{1.28 \cdot 10^{12}},$$

so $Q_1(d') > 582 Q_2(d')$, but $d' \notin \mathcal{G}(\mathcal{B})$.

Proof. The values follow from (3) with $a = \frac{9}{100}$, $b = \frac{24}{100}$, and $d'_4 = \frac{24}{100} \leq \frac{27}{100}$ gives the convex-hull inequalities. Since $1 - 4 \cdot \frac{9}{100} = (\frac{4}{5})^2$ and $1 - 4 \cdot \frac{6}{25} = (\frac{1}{5})^2$, we have $(f(d'_j))_j = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{5})$ and $\rho_4(d') = 3 \cdot \frac{4}{5} - \frac{1}{5} = \frac{11}{5} > 2$. By Theorem 1.1, $d' \notin \mathcal{G}(\mathcal{B})$. $\square$

The boundary point $(\frac{9}{100}, \frac{9}{100}, \frac{9}{100}, \frac{21}{100}) = h(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{10})$ is attained by the tensor $\sqrt{7/10} e_{0000} + \sqrt{1/10} (e_{1001} + e_{0101} + e_{0011})$ of Lemma 2.2; as d_4 grows to $\frac{1}{4}$ the point leaves $\mathcal{G}(\mathcal{B})$ at once, while $Q_1 > 5 \cdot 10^{-4} > 4 \cdot 10^{-6} > Q_2$ by (3).

Lemma 3.2. *Let $2 \leq m \leq n$, let $d \in [0, \frac{1}{4}]^m$, $s = \sum_j d_j$, and let $\hat{d} = (d, 0, \dots, 0) \in [0, \frac{1}{4}]^n$. Then $Q_1(\hat{d}) = Q_1(d) s^{n-m}$ and $Q_2(\hat{d}) = \frac{1}{2}(2Q_2(d))^{2^{n-m}}$, where on the right Q_1, Q_2 are the polynomials for m variables. Moreover $f(\hat{d}) = (f(d), 0, \dots, 0)$.*

Proof. For $i \leq m$ the i -th factor of Q_1 is unchanged, and for $i > m$ it equals s . Each of the 2^n linear forms in $Q_2(\hat{d})$ coincides with the form for its first m signs, so every form of the m -variable product occurs 2^{n-m} times. Finally $f(0) = 0$. $\square$

Proposition 3.3. *For $0 < x \leq \frac{1}{12}$ the point $d(x) = (x, x, x, 3x - x^2)$ lies in $(0, \frac{1}{4})^4$, satisfies $Q_1 > Q_2$, and is not in $\mathcal{G}(\mathcal{B})$.*

Proof. With $a = x$, $b = 3x - x^2$, (3) gives $Q_1 = x^5(4 - x)^3$ and $Q_2 = \frac{1}{2}x^8(6 + x)^2(2 - x)^6$, so

$$Q_1 - Q_2 = x^5 \left[(4 - x)^3 - \frac{1}{2}x^3(6 + x)^2(2 - x)^6 \right]$$

with $(4 - x)^3 \geq (\frac{47}{12})^3 > 60$ and $\frac{1}{2}x^3(6 + x)^2(2 - x)^6 \leq \frac{1}{2}(\frac{1}{12})^3(\frac{73}{12})^2 2^6 < 0.69$ on $(0, \frac{1}{12}]$. Also $3x - x^2 < \frac{1}{4}$ there. Let $r_1 = \sqrt{1 - 4x}$ and $r_4 = \sqrt{1 - 4b} = \sqrt{1 - 12x + 4x^2}$. The point is outside $\mathcal{G}(\mathcal{B})$ if $\rho_4 = 3r_1 - r_4 > 2$, that is, $3r_1 - 2 > r_4$. For $0 < x < \frac{5}{36}$ we have $r_1 > \frac{2}{3}$, so both sides are nonnegative, and since $(3r_1 - 2)^2 - r_4^2 = 4(3 - 6x - x^2 - 3r_1)$ and $3 - 6x - x^2 > 0$,

$$3r_1 - 2 > r_4 \iff 3 - 6x - x^2 > 3r_1 \iff (3 - 6x - x^2)^2 - 9(1 - 4x) = 30x^2 + 12x^3 + x^4 > 0,$$

which holds for $x > 0$. $\square$

Proof of Corollary 1.2. (a) is Proposition 3.1. (b) Let $n \geq 5$ and $\hat{d} = (d', 0, \dots, 0)$. By Lemma 3.2, $f(\hat{d})_4 = \frac{2}{5} > \frac{3}{10} = \sum_{j \neq 4} f(\hat{d})_j$, so $\hat{d} \notin \mathcal{G}(\mathcal{B})$. Put $q = 2Q_2(d') = (\frac{57}{100})^2 (\frac{15}{100})^6 < 3.8 \cdot 10^{-6}$. Since $2^{n-4} \geq n-3$ and $0 < q < 1$, Lemma 3.2 gives

$$Q_1(\hat{d}) = Q_1(d') \left(\frac{51}{100} \right)^{n-4} > \frac{q}{2} q^{n-4} = \frac{1}{2} q^{n-3} \geq \frac{1}{2} q^{2^{n-4}} = Q_2(\hat{d}),$$

because $Q_1(d') > q/2 = Q_2(d')$ and $\frac{51}{100} > q$. (c) is Proposition 3.3; the claim about cQ_2 follows from $Q_1 = x^5(4-x)^3$ and $Q_2 = O(x^8)$. For the interior: at $\hat{d}$ (with $\hat{d} = d'$ if $n = 4$) the inequalities $Q_1 > Q_2$ and $\rho_4 > n-2$ are strict ($\rho_4(\hat{d}) - (n-2) = \frac{1}{5}$), and both sides are continuous on the cube. So all points of $(0, \frac{1}{4})^n$ close to $\hat{d}$ are counterexamples. $\square$

Remark 3.4. A second example, where the numbers $1 - 4d_j$ are not squares, is $d^* = (\frac{3}{50}, \frac{3}{50}, \frac{3}{50}, \frac{171}{1000})$. Here $Q_1(d^*) = (\frac{231}{1000})^3 \frac{9}{1000} = 110937519 \cdot 10^{-12}$ and $q^* = 2Q_2(d^*) = (\frac{369}{1000})^2 (\frac{111}{1000})^6 < 2.6 \cdot 10^{-7}$, and $\rho_4(d^*) = 3\sqrt{19/25} - \sqrt{79/250} > 2$, as one checks by squaring twice: $\frac{1}{4}(9 \cdot \frac{19}{25} - 4 - \frac{79}{250}) = \frac{631}{1000} > 0$ and $(\frac{631}{1000})^2 > \frac{79}{250}$. For the padded points $(d^*, 0, \dots, 0)$, Lemma 3.2 gives $Q_1 = Q_1(d^*) 0.351^{n-4} > \frac{1}{2}(q^*)^{n-3} \geq Q_2$, because $Q_1(d^*) > q^*/2$ and $0.351 > q^*$.

Remark 3.5. The failure is not confined to small regions. A Monte Carlo computation ($4 \cdot 10^5$ uniform points per dimension, floating point) puts the fraction of the cube occupied by $\{Q_1 \geq Q_2\} \setminus \mathcal{G}(\mathcal{B})$ at about 13.6%, 7.6% and 3.0% for $n = 4, 5, 6$, while it found no point of $\mathcal{G}(\mathcal{B}) \setminus \{Q_1 \geq Q_2\}$. For $n = 3$ the roles are reversed: $\{Q_1 \geq Q_2\} \subseteq \mathcal{G}(\mathcal{B})$ by Theorem 1.3, and $\mathcal{G}(\mathcal{B}) \setminus \{Q_1 \geq Q_2\}$ fills about 1.1% of the cube.

Remark 3.6. Seigal explains (2) by noting that $\{Q_1 \geq Q_2\}$ is the connected component of the complement of $\{Q_1 = Q_2\}$ in the cube that contains the points $(\frac{1}{4} - \epsilon, \dots, \frac{1}{4} - \epsilon)$ [Sei18, Section 3]. This reading does not rescue the conjecture, since d' lies in that component. On the path $d' \rightarrow (\frac{9}{100}, \frac{9}{100}, \frac{9}{100}, \frac{21}{100}) \rightarrow (\frac{1}{5}, \dots, \frac{1}{5}) \rightarrow (\frac{1}{4}, \dots, \frac{1}{4})$ all points have the form (a, a, a, b) , and the bounds $a + b \geq \frac{3}{10}$, $3a - b \geq \frac{3}{100}$, $0 < 9a - b \leq 2$ and $|a - b| \leq \frac{15}{100}$ hold (the four quantities are affine on each segment). So (3) gives $Q_1 \geq (\frac{3}{10})^3 \frac{3}{100} > 8 \cdot 10^{-4}$ and $Q_2 \leq \frac{1}{2} \cdot 2^2 (\frac{15}{100})^6 < 2.3 \cdot 10^{-5}$ along the path.

Remark 3.7. Seigal's test checks that Gram tuples of random tensors satisfy $Q_1 \geq Q_2$ [Sei19, Section 4.2], that is, the inclusion $\mathcal{G}(\mathcal{B}) \subseteq \{Q_1 \geq Q_2\}$. It fails for $n = 3$, at the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ of Remark 4.2. For $n = 4, 5, 6$, none of about $5 \cdot 10^5$ random points $h(\lambda)$ of $\mathcal{G}(\mathcal{B})$ (λ uniform in P_n) has $Q_1 < Q_2$, and for $n = 4$ the largest value of Q_2/Q_1 we found on $\mathcal{G}(\mathcal{B})$ is $\frac{27}{512}$, at $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0)$.

Problem 3.8. Is $\mathcal{G}(\mathcal{B}) \subseteq \{d : Q_1(d) \geq Q_2(d)\}$ for every $n \geq 4$?

4 The correct analogue of Theorem 1.4

For $\varepsilon \in \{\pm 1\}^n$ and $r \in [0, 1]^n$ let $L_\varepsilon(r) = n - 2 - \sum_j \varepsilon_j r_j$, so that $N_n = \prod_\varepsilon L_\varepsilon$. Write $L_+ = L_{(1, \dots, 1)}$ and let $\ell_i = n - 2 + r_i - \sum_{j \neq i} r_j = L_+ + 2r_i$ be the form L_ε for the vector ε with a single -1 , in position i . By Theorem 1.1, $d \in \mathcal{G}(\mathcal{B})$ if and only if $\ell_i \geq 0$ for all i .

Proof of Theorem 1.3. If ε has $k \geq 2$ entries -1 , then $L_\varepsilon \geq (n-2) - (n-k) + \sum_{j: \varepsilon_j = -1} r_j \geq 0$, with equality only if $k = 2$, $r_j = 0$ for the two indices with $\varepsilon_j = -1$ and $r_j = 1$ for the others. Hence $N_n = L_+ \cdot \prod_i \ell_i \cdot R$ with $R \geq 0$ on $[0, 1]^n$, and $R = 0$ only at points d with two coordinates $\frac{1}{4}$ and the others 0; these lie in $\mathcal{G}(\mathcal{B})$. Similarly $M_i = K_i S_i$, where $K_i = n - 2 - \sum_{j \neq i} r_j = L_+ + r_i$ and S_i is the product of the factors of M_i with at least one minus sign. A factor with $k \geq 1$ minus signs is at least $k-1 + \sum_{j: \varepsilon_j = -1} r_j \geq 0$, so $S_i \geq 0$, and $S_i = 0$ forces $r_a = 0$ for some $a \neq i$ and $r_j = 1$ for all $j \notin \{i, a\}$.

Let $d \in \mathcal{G}(\mathcal{B})$, so all $\ell_i \geq 0$. If $L_+ \leq 0$, then $N_n \leq 0$. If $L_+ > 0$, then $N_n \geq 0$ and $K_i > 0$, hence $M_i \geq 0$, for all i .

Let $d \in [0, \frac{1}{4}]^n \setminus \mathcal{G}(\mathcal{B})$, so $\ell_m < 0$ for some m . For $i \neq k$ we have $\ell_i + \ell_k = 2(n-2) - 2 \sum_{j \neq i, k} r_j \geq 0$; hence $\ell_k > 0$ for all $k \neq m$. Moreover $L_+ = \ell_m - 2r_m < 0$ and $R > 0$, so $N_n > 0$. Also $K_m = \ell_m - r_m < 0$. If $S_m = 0$, then $r_a = 0$ and $r_j = 1$ for $j \notin \{m, a\}$, which gives $\ell_m = r_m \geq 0$, a contradiction. So $S_m > 0$ and $M_m < 0$. Thus d lies in neither of the two sets. $\square$

Proposition 4.1. *For $n = 3$, $N_3 = -512(Q_1 - Q_2)$ and $M_k = 16((d_i - d_j)^2 + \frac{1}{2}(d_i + d_j) - \frac{3}{16})$ for $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. For any x , the product of $x - \varepsilon_2 r_2 - \varepsilon_3 r_3$ over $(\varepsilon_2, \varepsilon_3) \in \{\pm 1\}^2$ is $((x - r_2)^2 - r_3^2)((x + r_2)^2 - r_3^2) = x^4 - 2\sigma x^2 + \delta^2$ with $\sigma = r_2^2 + r_3^2$ and $\delta = r_2^2 - r_3^2$. For $x = 1 \mp r_1$ we have $x^2 = \alpha \mp \beta$ with $\alpha = 1 + r_1^2$ and $\beta = 2r_1$, and the product becomes $U \mp V$ with $U = \alpha^2 + \beta^2 - 2\sigma\alpha + \delta^2$ and $V = 2\beta(\alpha - \sigma)$. Hence $N_3 = U^2 - V^2$. Put $c_i = \sum_{j \neq i} d_j - d_i$ and $E = \sum_i d_i^2 - 2 \sum_{i < j} d_i d_j$. Substituting $r_j^2 = 1 - 4d_j$ gives $\alpha - \sigma = 4c_1$, $U = 16(c_1 + E)$ and $V^2 = 256(1 - 4d_1)c_1^2$, so

$$N_3 = 256(E^2 + 2c_1 E + 4d_1 c_1^2) = 256(E^2 + 2c_1(E + 2d_1 c_1)) = 256(E^2 - 2c_1 c_2 c_3),$$

since $E + 2d_1 c_1 = (d_2 - d_3)^2 - d_1^2 = -c_2 c_3$. Now $Q_1 = c_1 c_2 c_3$, and $Q_2 = \frac{1}{2}E^2$: the same computation with $x = \sqrt{d_1}$ and r_j replaced by $\sqrt{d_j}$ shows that the four forms with $\varepsilon_1 = 1$ in Q_2 have product $d_1^2 - 2(d_2 + d_3)d_1 + (d_2 - d_3)^2 = E$, and the four with $\varepsilon_1 = -1$ give E again. So $N_3 = -512(Q_1 - Q_2)$. Finally

$$M_k = (1 - (r_i + r_j)^2)(1 - (r_i - r_j)^2) = (1 - r_i^2 - r_j^2)^2 - 4r_i^2 r_j^2 = 16(d_i - d_j)^2 + 8(d_i + d_j) - 3. \quad \square$$

With Proposition 4.1, the case $n = 3$ of Theorem 1.3 is [Sei18, Theorem 1.4] with the inequality in (1) reversed; this gives an independent proof of the corrected statement, based on [DSL17].

Remark 4.2 (the misprint in [Sei18, Theorem 1.4]). The inequality " $\leq \frac{3}{16}$ " in (1) appears in the arXiv preprint, in the accepted manuscript, in the published version, in the zbMATH review and in [Sei19, Theorem 4.8]. As printed, the statement is false in both directions. The tensor $(e_{000} + e_{111})/\sqrt{2}$ has $G_i = \frac{1}{2}I$, so $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathcal{G}(\mathcal{B})$; there $Q_1 - Q_2 = \frac{1}{64} - \frac{9}{512} = -\frac{1}{512} < 0$, and the left side of (1) equals $\frac{1}{4} > \frac{3}{16}$, so the printed statement excludes this point. Conversely, $(\frac{1}{5}, \frac{1}{100}, \frac{1}{100})$ satisfies $Q_1 < Q_2$ and (1) for all pairs, so the printed statement includes it, although it violates $d_1 \leq d_2 + d_3$ and is not even in the convex hull of $\mathcal{G}(\mathcal{B})$. The proof in [Sei18] obtains the second region by requiring the three polynomials $(d_i - d_j)^2 + \frac{1}{2}(d_i + d_j) - \frac{3}{16}$ to be positive, which is the corrected inequality. On the grid $\{0, \frac{1}{12}, \dots, 1\}^3$ of values of r (2197 points, exact arithmetic), the printed version disagrees with $\mathcal{G}(\mathcal{B})$ at 772 points and the corrected version at none.

5 No single polynomial

Proof of Theorem 1.4. For $n = 2$ and any real 2×2 matrix M we have $\det(MM^T) = \det(M)^2 = \det(M^T M)$, so $\mathcal{G}(\mathcal{B})$ lies on the diagonal $d_1 = d_2$; by Theorem 1.1 it is the whole segment, which is $\{d \in [0, \frac{1}{4}]^2 : -(d_1 - d_2)^2 \geq 0\}$.

Let $n \geq 3$ and suppose F exists. The map $\varphi(r) = ((1 - r_j^2)/4)_j$ is a homeomorphism from $(0, 1)^n$ onto $(0, \frac{1}{4})^n$; let $V = \varphi^{-1}(U)$, an open set. Put $\tilde{F} = F \circ \varphi$, a polynomial in r that is even in every r_j . Let $\Pi = \{r : \ell_i(r) \geq 0 \text{ for all } i\}$. By Theorem 1.1, for $r \in V$ we have $\tilde{F}(r) \geq 0$ if and only if $r \in \Pi$. The proof uses two kinds of points.

Facet points. For $0 < \delta < \frac{1}{n-1}$ let $r^\delta = (1 - \delta, \dots, 1 - \delta, 1 - (n-1)\delta)$. Then $\ell_n(r^\delta) = 0$ and $\ell_k(r^\delta) = (2n-4)\delta > 0$ for $k < n$. With $d = \varphi(r^\delta)$ one finds $\sum_{j < n} d_j - d_n = (n-1)(n-2)\delta^2/4 > 0$, and

the other inequalities defining U hold as well, so $r^\delta \in V$. Hence, for each i (permute the coordinates), the set $Z_i = \{r \in V : \ell_i(r) = 0, \ell_k(r) > 0 \text{ for } k \neq i\}$ is a nonempty open subset of the hyperplane $H_i = \{\ell_i = 0\}$.

Points on $L_+ = 0$. On $H_+ = \{L_+ = 0\}$ we have $\ell_i = 2r_i$, so $H_+ \cap (0, 1)^n$ lies in the interior of Π . The point $r^* = (\frac{n-2}{n}, \dots, \frac{n-2}{n})$ lies on H_+ , and $\varphi(r^*)$ has equal positive coordinates, so it lies in U . Hence $H_+ \cap V$ is a nonempty open subset of H_+ contained in the interior of Π .

Step 1: ℓ_i divides $\tilde{F}$. Let $p \in Z_i$. Then $p \in \Pi$, so $\tilde{F}(p) \geq 0$. For small $s > 0$ the point $p - se_i$ lies in V and has $\ell_i = -s < 0$, so $\tilde{F}(p - se_i) < 0$. By continuity $\tilde{F}(p) = 0$. Thus $\tilde{F}$ vanishes on a nonempty open subset of H_i , hence on H_i , and the affine form ℓ_i divides $\tilde{F}$. (In particular $\tilde{F} \neq 0$.)

Step 2: equal multiplicities. The sign changes $r_j \mapsto -r_j$ leave $\tilde{F}$ invariant and permute the 2^n affine forms L_ε transitively. Since $n - 2 \neq 0$, no two of these forms are proportional. As $\mathbb{R}[r]$ is a unique factorization domain, all L_ε divide $\tilde{F}$ with the same multiplicity $k \geq 1$, and $\tilde{F} = \prod_\varepsilon L_\varepsilon^k \cdot \tilde{G}$, where no L_ε divides $\tilde{G}$.

Step 3: k is odd. Neither $\tilde{G}$ nor a form $L_\varepsilon \neq \ell_n$ vanishes identically on H_n , as none of them is divisible by ℓ_n . So their zero sets in H_n are nowhere dense, and there is $p \in Z_n$ where $u = \tilde{G} \prod_{L_\varepsilon \neq \ell_n} L_\varepsilon^k$ does not vanish. Near p , $\tilde{F} = \ell_n^k u$ and u has the sign of $u(p)$. Points near p with $\ell_n > 0$ lie in $V \cap \Pi$, where $\tilde{F} \geq 0$, so $u(p) > 0$. Points near p with $\ell_n < 0$ lie in $V \setminus \Pi$, where $\tilde{F} < 0$, so $(-1)^k u(p) < 0$. Hence k is odd.

Step 4: contradiction. As in Step 3, choose $p \in H_+ \cap V$ where $u_+ = \tilde{G} \prod_{L_\varepsilon \neq L_+} L_\varepsilon^k$ does not vanish. Near p , $\tilde{F} = L_+^k u_+$ with k odd, so $\tilde{F}$ takes negative values arbitrarily close to p . But a neighbourhood of p lies in V and in the interior of Π , where $\tilde{F} \geq 0$. This contradiction proves the first statement. If $\mathcal{G}(\mathcal{B})$ had one of the two other forms, then intersecting with U would give a description that we have just excluded. $\square$

The obstruction is the hyperplane $\sum_j r_j = n - 2$, a component of the zero set of N_n crossing the interior of the Gram locus. Similarly, $\mathcal{G}(\mathcal{B}) \cap U$ is not the closure in U of a set $\{F > 0\} \cap U$.

Verification. The claims were checked as follows. (1) A self-contained program by the author, `verify_paper.py` (standard library only, a few seconds), recomputes every number and identity of the note from the definitions in exact arithmetic. Products over sign vectors of sums of square roots (Q_2 , N_n , M_i) are evaluated by an exact recursion, without closed forms; with polynomial coefficients the same recursion yields (3), Proposition 4.1 and the expansion of N_4 . It checks Theorem 1.3 at 4,750 exact points ($n = 3, \dots, 6$, including points on facets and on $L_+ = 0$), Lemma 2.2 for 230 random rational $\lambda \in P_n$, $n \leq 8$, all bounds of Section 3 and the grid of Remark 4.2, and it tests Lemmas 2.1 and 2.3 in floating point. A second program gives the Monte Carlo estimates. (2) Earlier programs, written with a separate exact polynomial library, produced the same values; their expansion of N_4 is byte-identical. (3) A first *independent verification run*, itself AI-assisted, re-derived the proofs line by line and recomputed the claims with separately written code: Q_2 in multiquadratic number fields, Proposition 4.1 by exact evaluation on a grid, Theorem 1.3 at 7,000 exact points, Lemma 2.2 by linear programming and by numerical inversion of the Gram map, and the distance from d' to $\mathcal{G}(\mathcal{B})$ by optimization over the unit ball (it equals the distance to $h(P_4)$, about 0.0233). It found that an earlier justification of the polygon failure in Proposition 3.3 was invalid, although the conclusion was correct; the argument given here is the corrected one. (4) A second independent verification run, also AI-assisted, read the sources again, re-derived the proofs line by line and recomputed the claims with its own code; it expanded Q_2 , N_n , M_i fully (so (3) and Propositions 3.3 and 4.1 become polynomial identities), found a box of radius 10^{-3} around d' of counterexamples (interval arithmetic) and reran all earlier programs, with identical outputs. All programs and outputs are in the source files.

Scope and priority. Theorem 1.1 is due to [DSL17] and, for complex tensors, to [HSS03]; Krämer [Krä19] relates the feasibility problem for tensor singular values to the quantum marginal problem and cites [HSS03] for qubits. We only translate these results into Seigal’s coordinates; Lemma 2.3 is the one step we did not find stated. The arXiv preprint (its only version), the accepted manuscript (PubMed Central, PMC6051437), the version of record (read in the publisher-typeset copy on the author’s web page) and the thesis [Sei19] all state the conjecture without qualification. The zbMATH review restates it without comment. OpenAlex and Semantic Scholar list, besides [DSL17, Krä19, Sei19], three works that cite [Sei18] for other purposes. Searches of Crossref, OpenAlex, Semantic Scholar, zbMATH, the arXiv API and the web found no statement that the conjecture is false, and no occurrence of Theorems 1.3 and 1.4 or of the correction in Remark 4.2. We did not search the quantum-information literature systematically; since the polygon inequalities are well known there, the observation may have been made informally. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [Bra04] S. Bravyi. Requirements for compatibility between local and multipartite quantum states. *Quantum Information and Computation*, 4(1):12–26, 2004. arXiv:quant-ph/0301014.
- [DSL17] I. Domanov, A. Stegeman, and L. De Lathauwer. On the largest multilinear singular values of higher-order tensors. *SIAM Journal on Matrix Analysis and Applications*, 38(4):1434–1453, 2017. arXiv:1612.03751. doi:10.1137/16M110770X.
- [HSS03] A. Higuchi, A. Sudbery, and J. Szulc. One-qubit reduced states of a pure many-qubit state: polygon inequalities. *Physical Review Letters*, 90(10):107902, 2003. arXiv:quant-ph/0209085. doi:10.1103/PhysRevLett.90.107902.
- [HU17] W. Hackbusch and A. Uschmajew. On the interconnection between the higher-order singular values of real tensors. *Numerische Mathematik*, 135(3):875–894, 2017. doi:10.1007/s00211-016-0819-9.
- [Krä19] S. Krämer. A geometric description of feasible singular values in the tensor train format. *SIAM Journal on Matrix Analysis and Applications*, 40(3):1153–1178, 2019. arXiv:1701.08437. doi:10.1137/18M1192408.
- [Sei17] A. Seigal. Higher order singular values: Gram determinants of real binary tensors. *Oberwolfach Reports*, 14(2):1226–1227, 2017. Report No. 20/2017, *Algebraic Statistics*; issue published 2018. doi:10.4171/OWR/2017/20.
- [Sei18] A. Seigal. Gram determinants of real binary tensors. *Linear Algebra and its Applications*, 544:350–369, 2018. arXiv:1612.04420 (2016, only version); accepted manuscript PMC6051437. doi:10.1016/j.laa.2018.01.019.
- [Sei19] A. Seigal. *Structured Tensors and the Geometry of Data*. PhD thesis, University of California, Berkeley, 2019. <https://escholarship.org/uc/item/9jv5j0f4>.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-15428-003.