

Log-Free Supersaturation of Collinear Triples in Three-Dimensional Grids

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Abstract

Let $F(n, m)$ be the minimum number of unordered collinear triples in an m -point subset of $[n]^3$. We prove that $F(n, m) = \Theta(m^4/n^6)$ whenever $12n^2 \leq m \leq n^3$, with absolute constants. The lower bound removes the logarithmic loss in the Balogh–Solymosi supersaturation estimate. Its proof counts all short primitive positive directions and uses the exact partition of the grid into maximal lattice chains. For the upper bound, we adapt Roche-Newton’s periodic finite-field cap construction to arbitrary side lengths and exact cardinalities, using an anisotropic quadratic form and translation averaging. In particular, the minimum has order n^{6-4s} for every fixed $0 \leq s < 1$ and $m = \lceil n^{3-s} \rceil$. The result determines the dense-regime order, not an exact minimum or optimal leading constant; the threshold $m \asymp n^2$ is not resolved.

1 Introduction

Write $[n] = \{1, \dots, n\}$, and let $t(S)$ be the number of three-element subsets of a finite set $S \subset \mathbb{R}^3$ whose points lie on one real line. Define

$$F(n, m) = \min\{t(S) : S \subset [n]^3, |S| = m\}.$$

Solymosi asked for the minimum number of collinear triples in subsets of the cubic grid in the 2017 Oberwolfach Discrete Geometry problem session [2, Question 11.1]. The question records a lower bound of order $n^{6-4s}/\log n$ for sets of size n^{3-s} and asks whether it is sharp. Balogh and Solymosi prove this estimate for $0 \leq s < 1$ and sufficiently large n [1, Lemma 4.2].

Roche-Newton [3, Theorem 4 and Corollary 2] gives matching-exponent constructions: for every fixed $0 \leq s \leq 1$ and arbitrarily large n , there are sets of at least n^{3-s} points with $O(n^{6-4s})$ collinear triples. We remove the logarithmic denominator from the lower bound. We also explain a uniform form of his upper-bound method.

Theorem 1.1. *For all integers n, m with $12n^2 \leq m \leq n^3$,*

$$\frac{m^4}{62208n^6} \leq F(n, m) \leq 134784 \frac{m^4}{n^6}. \quad (1)$$

The upper bound holds for every $1 \leq m \leq n^3$.

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Corollary 1.2. *For every fixed $0 \leq s < 1$, as $n \rightarrow \infty$,*

$$F(n, \lceil n^{3-s} \rceil) = \Theta(n^{6-4s}).$$

Proof. The ratio $\lceil n^{3-s} \rceil / n^2$ tends to infinity, while $\lceil n^{3-s} \rceil \asymp n^{3-s}$. Apply Theorem 1.1. $\square$

The constants in (1) are not optimized. The theorem gives an order of growth, not a formula for the exact minimum or its leading constant. The endpoint $s = 1$ is different: $F(p, p^2) = 0$ for every odd prime p , by the familiar quadratic-cap construction recalled below. The separate bipartite rigidity question in the same source is already answered by known constructions, including Solymosi and White's knight framework [4, Theorem 2]; it is not a new claim here.

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2 Primitive directions and chain populations

A nonzero vector $v \in \mathbb{Z}^3$ is *primitive* if $\gcd(v_1, v_2, v_3) = 1$. The integer points on a line of primitive direction v have the form $x + tv$, $t \in \mathbb{Z}$, for any integer point x on the line. Indeed, if $\lambda v \in \mathbb{Z}^3$, Bezout's identity gives integers c_i with $\sum_i c_i v_i = 1$, so $\lambda = \sum_i c_i (\lambda v_i) \in \mathbb{Z}$.

Lemma 2.1. *If $v \in \mathbb{Z}^3$ is primitive and $0 < \|v\|_\infty < n$, the maximal chains of step v in $[n]^3$ partition the grid. Their number is*

$$L_v = n^3 - \prod_{i=1}^3 (n - |v_i|) \leq n^2 \sum_{i=1}^3 |v_i| \leq 3n^2 \|v\|_\infty. \quad (2)$$

Proof. Every chain has a unique initial point $x \in [n]^3$ with $x - v \notin [n]^3$. There are $\prod_i (n - |v_i|)$ grid points whose predecessor remains in the grid, proving the equality. The initial points lie in the union of three boundary strips, the i th of which contains $n^2 |v_i|$ points. This proves the inequalities, including mixed-sign and zero coordinates. $\square$

Lemma 2.2. *For every positive integer r , the number of primitive vectors in $\{1, \dots, r\}^3$ is at least $3r^3/4$.*

Proof. A nonprimitive vector has all coordinates divisible by some integer $d \geq 2$. The union bound therefore gives

$$\#\{\text{nonprimitive vectors}\} \leq \sum_{d=2}^r \left\lfloor \frac{r}{d} \right\rfloor^3 \leq r^3 \sum_{d=2}^\infty d^{-3} < \frac{r^3}{4}.$$

For the final inequality, separate the term $d = 2$ and bound the tail by $\int_2^\infty x^{-3} dx = 1/8$. The assertion also includes $r = 1$. $\square$

Lemma 2.3. *Let $k_1, \dots, k_L$ be nonnegative integers summing to m , where $L \geq 1$ and $m \geq 4L$. Then*

$$\sum_{j=1}^L \binom{k_j}{3} \geq \frac{m^3}{216L^2}.$$

Proof. The indices with $k_j \leq 2$ account for at most $2L$ of the total population. The remaining indices therefore have population at least $m/2$ and form a nonempty set J . For every integer $k \geq 3$, $k^3 \leq 27\binom{k}{3}$: the equivalent inequality $2k^2 \leq 9(k-1)(k-2)$ has difference $(k-3)(7k-6) \geq 0$. Hölder's inequality now yields

$$\sum_{j=1}^L \binom{k_j}{3} \geq \frac{1}{27} \sum_{j \in J} k_j^3 \geq \frac{1}{27} \frac{(\sum_{j \in J} k_j)^3}{|J|^2} \geq \frac{m^3}{216L^2}.$$

□

3 The log-free lower bound

Let $S \subset [n]^3$ have $m \geq 12n^2$ points, and set

$$r = \left\lfloor \frac{m}{12n^2} \right\rfloor.$$

Then $1 \leq r \leq n/12$ and $r \geq m/(24n^2)$. Let D be the primitive vectors in $\{1, \dots, r\}^3$. For a fixed $v \in D$, Lemma 2.1 partitions the grid into $L_v \leq 3n^2r \leq m/4$ chains. Their populations in S sum to m . By Lemma 2.3, the number of selected triples in this direction is at least

$$\frac{m^3}{216L_v^2} \geq \frac{m^3}{1944n^4r^2}.$$

Every such triple is collinear. Moreover, a three-element collinear set determines a unique line direction. Distinct primitive vectors in the positive octant are not proportional, so no triple is counted twice when the last bound is summed over D . Lemma 2.2 therefore gives

$$t(S) \geq \frac{3r^3}{4} \frac{m^3}{1944n^4r^2} = \frac{m^3r}{2592n^4} \geq \frac{m^4}{62208n^6}.$$

This proves the lower bound in Theorem 1.1. Unlike the prime-coordinate family in [1, Lemma 4.2], the family D has a positive constant proportion of all vectors in its box. The chain count supplies the needed uniform population estimate without a prime-distribution estimate or a hypothesis on the shape of S .

4 A uniform version of the cap upper bound

The finite-field cap construction and its use of short primitive directions are due to the method in [3, Theorem 4, Section 6]. We give the details for arbitrary n, m . The changes are to use an anisotropic form over any odd prime field, average over translations, and then take a subset of the desired size.

Lemma 4.1. *Let p be an odd prime and let $d \in \mathbb{F}_p$ be a nonsquare. The set*

$$A = \{(x, y, x^2 - dy^2) : x, y \in \mathbb{F}_p\} \subset \mathbb{F}_p^3$$

has p^2 elements and meets every affine line in at most two points.

Proof. Insert $a + tv$, with $v \neq 0$, into the defining equation. If $(v_1, v_2) \neq (0, 0)$, the quadratic coefficient $v_1^2 - dv_2^2$ is nonzero: its vanishing would make d a square or force $v_1 = v_2 = 0$. The resulting polynomial has at most two roots. If $v_1 = v_2 = 0$, then $v_3 \neq 0$ and the equation is nonconstant linear. The size assertion follows from the graph parametrization. □

For $u \in \mathbb{F}_p^3$, set

$$S_u = \{x \in [n]^3 : x \bmod p \in A + u\}.$$

Each grid point belongs to exactly p^2 of the p^3 translates, and hence

$$\frac{1}{p^3} \sum_{u \in \mathbb{F}_p^3} |S_u| = \frac{n^3}{p}. \quad (3)$$

Proposition 4.2. *Every translate satisfies $t(S_u) \leq 832n^6/p^4$.*

Proof. On an integer line $x + tv$ of primitive direction v , reduction modulo p is a nonconstant affine field line. By Lemma 4.1, selected integer parameters occupy at most two residue classes modulo p . If the line contains three selected points, two parameters have the same residue. Their nonzero difference is a multiple of p , so, with $h = \|v\|_\infty$, the coordinate of largest magnitude gives $ph \leq n - 1$. Only $1 \leq h \leq \lfloor (n - 1)/p \rfloor$ can contribute.

Fix such a direction. It has at most $3n^2h$ grid chains by Lemma 2.1. The parameter interval on a chain has span at most $(n - 1)/h$. Its selected population is therefore at most

$$2 \left(\frac{n - 1}{ph} + 1 \right) \leq \frac{4n}{ph}.$$

Using $\binom{k}{3} \leq k^3/6$, the triples in this direction number at most

$$3n^2h \frac{1}{6} \left(\frac{4n}{ph} \right)^3 = \frac{32n^5}{p^3h^2}.$$

There are at most $(2h + 1)^3 - (2h - 1)^3 = 24h^2 + 2 \leq 26h^2$ integer vectors of norm h . Including nonprimitive vectors and both orientations only increases this upper estimate. Thus

$$t(S_u) \leq \sum_{h=1}^{\lfloor (n-1)/p \rfloor} \frac{832n^5}{p^3} \leq \frac{832n^6}{p^4}.$$

If $p \geq n$, the sum is empty and the claim remains true. □

We also record the elementary full-grid bound

$$t([n]^3) \leq 104n^6. \quad (4)$$

Indeed, a direction of norm h , $1 \leq h \leq n - 1$, has at most $3n^2h$ chains, each with at most $(n - 1)/h + 1 \leq 2n/h$ points. Its contribution is at most $4n^5/h^2$. Multiplying by at most $26h^2$ directions per shell and summing at most n shells proves (4). For $n = 1$ there are no triples.

Proof of the upper bound in Theorem 1.1. Put $a = n^3/m$. If $a \geq 6$, set $k = \lfloor a/2 \rfloor$. Then $k \geq 3$ and $k \geq a/3$. Bertrand's postulate supplies a prime $k < p < 2k \leq a$, which is odd and satisfies $p > a/3$. Equation (3) gives a translate containing at least $n^3/p \geq m$ grid points. Any m of them form a set with at most

$$\frac{832n^6}{p^4} \leq 67392 \frac{m^4}{n^6}$$

triples. If $1 \leq a < 6$, take any m -point subset of the full grid; (4) gives at most $104n^6 \leq 104 \cdot 6^4 m^4/n^6 = 134784 m^4/n^6$ triples. □

5 The threshold and supplementary checks

For $n = p$ an odd prime, take the representatives of A from Lemma 4.1 in $\{0, \dots, p-1\}^3$ and shift by $(1, 1, 1)$. This gives p^2 grid points and no real collinear triple. To see the latter directly, a primitive integer line has parameter span less than p inside this cube. Three selected parameters would therefore give three distinct points on a field line, contradicting the cap property. Consequently $F(p, p^2) = 0$. This known endpoint phenomenon explains why the positive lower estimate is restricted to a genuinely denser regime. The behavior for densities near this threshold, exact minimum values and optimal constants remain outside the present result.

The accompanying standard-library Python program checks primitive counts, signed grid-chain partitions, population inequalities, floor bounds, explicit dense examples, finite-field caps and their translated lifts. Direct cross-product enumeration is compared with an independent line-key count on small grids. It also reproduces the known knight-framework rank as a separately attributed source check. Both ordinary and optimized execution perform 68,705 exact checks and return identical output. These are finite regressions, not a replacement for the universally quantified proofs above.

References

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