

A Cubic Heisenberg Centralizer and Non-flat Diamond Argument-Shift Maps

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24 October 2026

Abstract

We compute two small examples separating maximality, maximal transcendence degree and flatness in symmetric Poisson algebras. For the four-dimensional diamond Lie algebra, we classify the regular argument shifts whose natural morphisms are flat. The shift at x^* is not flat, although the algebra satisfies the full codimension-three and invariant-degree hypotheses of the Panyushev–Yakimova maximality theorem. The underlying algebra and this shift subalgebra are due to Ooms. For the five-dimensional Heisenberg algebra we compute the centralizer of a homogeneous cubic: it is a polynomial algebra of transcendence degree two, strictly below the maximal value three. An elementary weighted-degree obstruction for the Airy derivation proves the centralizer formula in all degrees.

1 Setting and attribution

Throughout, $\mathbf{k}$ is an algebraically closed field of characteristic zero. For a finite-dimensional Lie algebra $\mathfrak{q}$, its symmetric algebra $S(\mathfrak{q})$ has the Poisson bracket determined by $\{u, v\} = [u, v]$ for $u, v \in \mathfrak{q}$. Write $Z(\mathfrak{q})$ for its Poisson centre and

$$b(\mathfrak{q}) = \frac{\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q}}{2}.$$

A Poisson-commutative subalgebra has transcendence degree at most $b(\mathfrak{q})$. Maximality means maximality under inclusion, whereas maximal dimension means equality in this bound.

The questions in Panyushev’s Oberwolfach contribution [1, pp. 1201–1203] concern both the converse implication from maximality to maximal dimension and the flatness of argument-shift maps. The latter arises under the hypotheses that the singular coadjoint locus has codimension at least three and that $Z(\mathfrak{q})$ contains $\operatorname{ind} \mathfrak{q}$ algebraically independent homogeneous elements whose degrees sum to $b(\mathfrak{q})$. These hypotheses imply that every regular argument-shift algebra is polynomial and maximal Poisson-commutative [2, Theorem 3.2].

The first question already has a negative answer: Ooms [3, Remark 17(ii)] gives a three-dimensional solvable example. Ooms also supplies the diamond algebra and the two basic shift algebras used below [3, Example 25]. We do not claim these examples as new. Our purpose is to give an explicit flatness classification for all regular diamond shifts and a homogeneous cubic centralizer example in a nilpotent Lie algebra. These computations give negative answers to the two historical universal assertions, with the earlier answer to the first fully credited. No claim of absolute priority for these further computations is made.

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For complex reductive Lie algebras, Moreau proves faithful flatness for every regular shift [4, Theorem 1.2]; see also [5, Section 3.4]. The solvable example below lies outside that positive theorem.

2 All regular shifts of the diamond algebra

Let $\mathfrak{d}$ have basis t, x, y, z , with nonzero basic brackets

$$[t, x] = -x, \quad [t, y] = y, \quad [x, y] = z. \quad (1)$$

Set

$$B = S(\mathfrak{d}) = \mathbf{k}[t, x, y, z], \quad C = xy - tz.$$

For $\xi \in \mathfrak{d}^*$, put

$$(\tau, \alpha, \beta, \gamma) = (\xi(t), \xi(x), \xi(y), \xi(z)), \quad L = \alpha y + \beta x - \gamma t.$$

The argument-shift algebra $\mathcal{F}_\xi(Z(\mathfrak{d}))$ is generated by the nonconstant-in-the-coordinates coefficients in $f(\mu + a\xi)$, for $f \in Z(\mathfrak{d})$. It includes the coefficient at a^0 .

Theorem 2.1. *The Lie algebra $\mathfrak{d}$ is algebraic, has index two, and its singular coadjoint locus has codimension three. Its Poisson centre is $\mathbf{k}[z, C]$, with generator degrees summing to $b(\mathfrak{d}) = 3$. A functional ξ is regular precisely when $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. For every such functional,*

$$\mathcal{F}_\xi(Z(\mathfrak{d})) = \mathbf{k}[z, C, L]$$

is a polynomial maximal Poisson-commutative algebra. Its natural morphism $\pi_\xi : \mathbb{A}^4 \longrightarrow \mathbb{A}^3$ is flat if and only if

$$\gamma \neq 0 \quad \text{or} \quad \alpha\beta \neq 0. \quad (2)$$

In the flat cases it is faithfully flat. In the remaining regular cases its zero fibre is $\mathbb{A}^2$.

Proof. The algebraicity can be seen without classification. Take the algebraic matrix group

$$\left\{ \begin{pmatrix} 1 & u & w \\ 0 & s & v \\ 0 & 0 & 1 \end{pmatrix} : s \in \mathbf{k}^\times, \ u, v, w \in \mathbf{k} \right\}.$$

In its Lie algebra, use $t = \text{diag}(0, 1, 0)$, $x = E_{12}$, $y = E_{23}$ and $z = E_{13}$. Their brackets are (1). The Kirillov matrix is

$$\begin{pmatrix} 0 & -x & y & 0 \\ x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It has rank two away from $x = y = z = 0$, and rank zero on that line. This gives the index, the regular locus and the required codimension.

Directly, z and C are central. Inverting z and eliminating $t = (xy - C)/z$ gives

$$B[z^{-1}] = \mathbf{k}[z, z^{-1}, C, x, y], \quad \{x, y\} = z.$$

Commuting with x, y forces independence of both variables, so the localized centre is $\mathbf{k}[z, z^{-1}, C]$. Its intersection with B is $\mathbf{k}[z, C]$. Indeed, if a minimal positive power of z cleared the denominators of an element of that intersection, its numerator $\sum_i b_i(z)C^i$ would vanish modulo z . This would

give $\sum_i b_i(0)(xy)^i = 0$, forcing every $b_i(z)$ to be divisible by z , contrary to minimality. The elements z, C are algebraically independent, as is seen by setting $t = 0, x = 1$, and their degrees are one and two.

The identities

$$z(\mu + a\xi) = z + a\gamma, \quad (3)$$

$$C(\mu + a\xi) = C + a(L - \tau z) + a^2(\alpha\beta - \tau\gamma) \quad (4)$$

show that the shift algebra is exactly $\mathbf{k}[z, C, L]$. Coefficients of a shift of any polynomial in z, C add no new generators. For $\gamma \neq 0$, algebraic independence follows from the coordinates below. For $\gamma = 0$, regularity gives $(\alpha, \beta) \neq (0, 0)$; after inverting z , solve for y or x from L , and then for t from C . This also proves algebraic independence. The maximality now follows from the Panyushev–Yakimova theorem, whose hypotheses have all been checked.

Write $A = \mathbf{k}[z, C, L]$. We prove the flatness classification by three explicit ring computations.

If $\gamma \neq 0$, let

$$X = x - \frac{\alpha z}{\gamma}, \quad Y = y - \frac{\beta z}{\gamma}.$$

Then X, Y, L, z are polynomial coordinates on B , and

$$C = XY + \frac{zL}{\gamma} - \frac{\alpha\beta z^2}{\gamma^2}.$$

Thus $A = \mathbf{k}[z, L, XY]$. The ring $\mathbf{k}[X, Y]$ is free over $\mathbf{k}[XY]$ with basis

$$\{1\} \cup \{X^n : n \geq 1\} \cup \{Y^n : n \geq 1\}.$$

Factor the largest common power of X, Y from a monomial to obtain existence and uniqueness of this expansion. Hence B is a nonzero free A -module and the morphism is faithfully flat.

If $\gamma = 0$ and $\alpha\beta \neq 0$, eliminate $y = (L - \beta x)/\alpha$. This gives the presentation

$$B \simeq A[t, x]/(\beta x^2 - Lx + \alpha C + \alpha zt).$$

The polynomial is monic in x after division by the nonzero scalar β . Therefore B is free over $A[t]$ with basis $1, x$, and is free over A with basis $\{t^j, xt^j : j \geq 0\}$. Again the morphism is faithfully flat.

Finally, suppose $\gamma = 0$ and $\alpha\beta = 0$. If $\alpha \neq 0$, then $A = \mathbf{k}[z, y, C]$. Base change by $A \rightarrow A/(z, y) = \mathbf{k}[C]$ yields

$$B \otimes_A \mathbf{k}[C] = B/(z, y) = \mathbf{k}[t, x].$$

The nonzero base element C annihilates this nonzero module. Tensoring the injective multiplication-by- C map on $\mathbf{k}[C]$ therefore fails to preserve injectivity. This module is not flat, and base change of a flat module would be flat. Thus B is not flat over A . The case $\beta \neq 0$ is identical with x, y interchanged. Moreover, $(z, y, C) = (z, y)$ in the first case, so the zero fibre is $\text{Spec } \mathbf{k}[t, x] = \mathbb{A}^2$; the other case is symmetric. $\square$

Corollary 2.2. *The regular shift $\xi = x^*$ gives a non-flat morphism*

$$(t, x, y, z) \longmapsto (z, y, xy - tz),$$

although $\mathfrak{d}$ satisfies the full hypotheses of the Panyushev–Yakimova maximality theorem.

The zero fibre has dimension two, whereas the fibre $(z, y, C) = (1, 0, 0)$ consists of $z = 1, y = t = 0$ and has dimension one. The free generators z, y, C do not form a regular sequence: modulo z, y , the third generator is zero in the nonzero ring $\mathbf{k}[t, x]$. Thus the unrestricted regular-sequence question stated in [4, Question 1] has the same obstruction. The invariant ring and the shift algebra in this corollary are those already recorded by Ooms; the obstruction is their displayed base-change computation.

3 An elementary cubic Heisenberg centralizer

Let $\mathfrak{h}_5$ have basis x, t, q, p, z , with

$$[x, t] = [q, p] = z$$

and all other basic brackets zero. It is the algebraic nilpotent Lie algebra spanned, in that order, by

$$E_{12}, \quad E_{24}, \quad E_{13}, \quad E_{34}, \quad E_{14}.$$

Put

$$H = t + \frac{p^2 - xq^2}{2}, \quad F = 2z^2t + zp^2 - xq^2.$$

Theorem 3.1. *In $S(\mathfrak{h}_5)$ one has*

$$\text{Cent}(H) = \mathbf{k}[z, H], \quad \text{Cent}(F) = \mathbf{k}[z, F].$$

In particular, $\mathbf{k}[z, F]$ is a graded maximal Poisson-commutative polynomial subalgebra of transcendence degree two, whereas $b(\mathfrak{h}_5) = 3$.

The central step is the following all-degree polynomial calculation. It does not use a nonintegrability theorem or a finite search as a substitute for proof.

Lemma 3.2. *Over any characteristic-zero field,*

$$\ker(\partial_x + p\partial_q + xq\partial_p : \mathbf{k}[x, q, p] \rightarrow \mathbf{k}[x, q, p]) = \mathbf{k}.$$

Proof. Write $D = \partial_x + L_0$, where $L_0 = p\partial_q + xq\partial_p$. Both preserve total degree in q, p , so it suffices to consider a D -invariant polynomial R homogeneous of degree n in these two variables. If $n = 0$, then $R \in \mathbf{k}[x]$ and $DR = R'$, giving only constants.

Suppose $n > 0$. Give x, q, p weights 2, 0, 1, respectively. The operator L_0 raises weight by one, whereas ∂_x lowers it by two. Let R_N be the highest-weight part of a nonzero R . The highest-weight equation is $L_0 R_N = 0$.

Over the extension $\mathbf{k}(x)(s)$ with $s^2 = x$, set $u = p + sq$, $v = p - sq$. Then

$$L_0 = s(u\partial_u - v\partial_v).$$

Its kernel on homogeneous degree n polynomials is zero for odd n , and is spanned by $(uv)^m$ if $n = 2m$. Consequently $n = 2m$, $m \geq 1$, and

$$R_N = c x^a K^m, \quad K = p^2 - xq^2, \quad c \in \mathbf{k}^\times, \quad a \geq 0.$$

To justify the last form, its coefficient of p^{2m} is a polynomial in x , and weight homogeneity forces it to be a single monomial.

The equation at weight $N - 2$ is

$$L_0 R_{N-3} = -\partial_x R_N, \tag{5}$$

with a missing component interpreted as zero. Projection onto the zero eigenspace of L_0 , namely the coefficient of $u^m v^m$, annihilates its image. But

$$\partial_x(x^a K^m) = ax^{a-1} K^m - mx^a q^2 K^{m-1}, \quad q^2 = \frac{(u-v)^2}{4x}.$$

The zero-eigenvalue component of $q^2 K^{m-1}$ is $-K^m/(2x)$. Hence the projection of the right side of (5) is

$$-c \left(a + \frac{m}{2} \right) x^{a-1} K^m \neq 0.$$

This is a contradiction in characteristic zero. The weight-zero variable q causes no completion issue: R is a polynomial of fixed q, p -degree and has finitely many weight components. $\square$

Proof of Theorem 3.1. Since the Heisenberg bracket is z times the canonical bracket, the polynomial derivation $\delta = \{\cdot, H\}/z$ is defined on the whole polynomial ring. The polynomial coordinate change $h = H$, with inverse $t = h - (p^2 - xq^2)/2$, gives

$$\delta(x) = 1, \quad \delta(q) = p, \quad \delta(p) = xq, \quad \delta(h) = \delta(z) = 0.$$

Applying Lemma 3.2 coefficient by coefficient in h, z shows that its kernel is $\mathbf{k}[h, z]$. The ring is a domain, so $\{G, H\} = z\delta(G) = 0$ is equivalent to $\delta(G) = 0$. This proves the first centralizer identity, including at the algebraic level before any specialization of z .

For the cubic, work first over $\mathbf{k}(z)$ and use $\bar{x} = x/z, \bar{t} = t/z, \bar{q} = q/z, \bar{p} = p/z$. Then

$$F = 2z^3 \left(\bar{t} + \frac{\bar{p}^2 - \bar{x}\bar{q}^2}{2} \right).$$

The rescaled Poisson bracket is a nonzero scalar multiple of the canonical one. The same lemma over $\mathbf{k}(z)$ therefore gives $\text{Cent}_{\mathbf{k}(z)[x,t,q,p]}(F) = \mathbf{k}(z)[F]$. To take the intersection with $S(\mathfrak{h}_5)$, consider $\sum_{i=0}^N a_i(z)F^i$, where $a_i(z) \in \mathbf{k}(z)$. Its coefficient of x^N is $(-q^2)^N a_N(z)$. Polynomiality forces $a_N(z) \in \mathbf{k}[z]$; subtract that multiple of F^N and proceed by induction. Thus the intersection is $\mathbf{k}[z, F]$.

The generators z, F are algebraically independent, for example after localizing at z and observing the nonzero coefficient $2z^2$ of t in F . Their algebra is Poisson-commutative. Any commutative extension consists of elements commuting with F , and hence is contained in its already computed centralizer. This proves maximality. Its generators have degrees one and three, so it is graded. Finally, the Kirillov matrix has rank four for $z \neq 0$, and rank zero for $z = 0$. Thus $\text{ind } \mathfrak{h}_5 = 1$ and $b(\mathfrak{h}_5) = 3$. $\square$

4 Scope and reproducibility

For completeness, Ooms's earlier answer to the dimension question uses $\mathfrak{r} = \langle a, b, c \rangle$ with $[a, b] = b$, $[a, c] = c$. The derivation $\{a, -\}$ acts on $a^i b^j c^k$ by the scalar $j + k$, so its polynomial kernel is $\mathbf{k}[a]$. Hence $\mathbf{k}[a]$ is maximal but has transcendence degree one, below $b(\mathfrak{r}) = 2$. The Heisenberg example above adds nilpotence and an explicit homogeneous cubic generator; it does not change the attribution of the unrestricted negative answer.

The diamond counterexample concerns the full codimension-three hypotheses. The Heisenberg example is used only for the general dimension question: its singular coadjoint locus has codimension one, so it is not offered as a counterexample under the argument-shift maximality hypotheses. No quantization assertion is made.

The accompanying Python script checks polynomial bracket and shift identities exactly, and computes bounded centralizer dimensions by modular elimination over two explicitly checked prime fields. It includes all outgoing monomials, without discarding boundary terms. The homogeneous cubic centralizer is checked through degree ten. These finite computations are diagnostic evidence;

the unrestricted statements follow from the written proofs. The work was developed with OpenAI Codex assistance in derivation, literature search, exact computation and exposition. The author is responsible for the manuscript. This is an unrefereed preprint; no independent review or proof-assistant verification is claimed.

References

- [1] D. Panyushev, joint work with O. Yakimova, On maximal Poisson-commutative subalgebras in the symmetric algebra of a Lie algebra, in *Algebraic Groups*, Oberwolfach Reports **4** (2007), report 22, pp. 1201–1203. doi:10.4171/OWR/2007/22.
- [2] D. I. Panyushev and O. S. Yakimova, The argument shift method and maximal commutative subalgebras of Poisson algebras, *Math. Res. Lett.* **15** (2008), 239–249.
- [3] A. I. Ooms, The Poisson center and polynomial, maximal Poisson commutative subalgebras, especially for nilpotent Lie algebras of dimension at most seven, *J. Algebra* **365** (2012), 83–113. doi:10.1016/j.jalgebra.2012.04.029. Author manuscript: arXiv:1110.0363, Remark 17(ii) and Example 25.
- [4] A. Moreau, A remark on Mishchenko–Fomenko algebras and regular sequences, *Selecta Math. (N.S.)* **24** (2018), 2651–2657. doi:10.1007/s00029-017-0357-z.
- [5] P. Crooks and M. Röser, On the fibres of Mishchenko–Fomenko systems, *Doc. Math.* **25** (2020), 1195–1239. doi:10.4171/DM/774.