

Proper Riesz Equilibrium Support on Connected Sets of Revolution

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Abstract

For every $0 < s < 1$, we construct a connected compact set of revolution whose Riesz s -equilibrium measure omits a positive-capacity circle of its outer boundary. The generating set can also be chosen connected with nonempty interior. This gives an affirmative answer to the existential support question posed by Brauchart, Hardin and Saff. The construction combines their large-radius negative-distance kernel with a uniform polygonal domination inequality on the unit disk. A strict potential gap gives exact support exclusion at finite radius. The proof does not determine the equilibrium measure and does not require a numerical minimization.

1 The support question

For a compact $A \subset H = \{x + iy : x > 0\}$, define

$$\Gamma(A) = \{(x \cos t, x \sin t, y) : x + iy \in A, 0 \leq t < 2\pi\}.$$

For $0 < s < 1$, the Riesz equilibrium measure $\mu_{s,\Gamma(A)}$ minimizes

$$I_s(\mu) = \iint |X - Y|^{-s} d\mu(X) d\mu(Y)$$

over probability measures on $\Gamma(A)$. Write $\partial_\infty K$ for the boundary of the unbounded component of $\mathbb{R}^3 \setminus K$. It is the outer boundary, not the boundary of the convex hull.

Brauchart, Hardin and Saff gave examples of proper equilibrium support for $0 < s < 1/3$ and asked for examples for every $0 < s < 1$ [1, p. 1036], [2, p. 3], [3, Conjecture 4.2]. The following construction covers the whole interval.

Theorem 1.1. *For each $0 < s < 1$, there is a connected compact $A \subset H$ such that*

$$\text{supp } \mu_{s,\Gamma(A)} \subsetneq \partial_\infty \Gamma(A).$$

The omitted part contains an entire circle of positive s -capacity. The set A may also be chosen to have nonempty interior.

The geometry and its size depend on s . We make no claim about a single set working for all exponents, a circular torus, a convex generating set, or a smooth boundary. The axial reduction, limiting kernel, and strict domination criterion used below are due to Brauchart, Hardin and Saff [2]. The construction uses a polygonal comparison measure to dominate a whole disk uniformly.

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2 A uniform polygonal gap

Put $q = 1 - s \in (0, 1)$. Denote the closed unit disk by $\overline{\mathbb{D}}$.

Lemma 2.1. *For $w \in \overline{\mathbb{D}}$, set*

$$F_q(w) = \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - w|^q dt.$$

Then $F_q(w) - |w|^q \geq q^2/4$.

Proof. By rotation let $w = r \in [0, 1]$. For $r < 1$, the binomial series of $(1 - re^{it})^{q/2}$ converges uniformly in t . Integrating its squared modulus gives

$$F_q(r) = \sum_{n=0}^{\infty} \binom{q/2}{n}^2 r^{2n} \geq 1 + \frac{q^2 r^2}{4}.$$

Continuity extends the inequality to $r = 1$. The tangent-line inequality for the concave function $t^{q/2}$ gives $1 - r^q \geq (q/2)(1 - r^2)$. Hence

$$F_q(r) - r^q \geq \frac{q}{2}(1 - r^2) + \frac{q^2 r^2}{4} \geq \frac{q^2}{4}.$$

□

Choose an integer N with

$$N > 2\pi(8/q^2)^{1/q}, \quad \zeta_j = e^{2\pi i j/N} \quad (0 \leq j < N). \quad (1)$$

This sufficient bound is not intended to be sharp.

Lemma 2.2. *Uniformly for $w \in \overline{\mathbb{D}}$,*

$$\frac{1}{N} \sum_{j=0}^{N-1} |\zeta_j - w|^q - |w|^q > \frac{q^2}{8}. \quad (2)$$

Proof. For $a, b \geq 0$, $|a^q - b^q| \leq |a - b|^q$. Thus $t \mapsto |e^{it} - w|^q$ is q -Hölder with constant one, uniformly in w . On intervals of length $h = 2\pi/N$, the left-endpoint quadrature error is at most

$$\frac{N}{2\pi} \int_0^h u^q du = \frac{h^q}{q+1} < \frac{q^2}{8}.$$

Apply Lemma 2.1. □

3 The angular kernel and a finite-radius bound

The interaction of uniform probability measures on two coaxial circles is

$$K_s(z, w) = \frac{1}{2\pi} \int_0^{2\pi} (|z - w|^2 + 4\Re z \Re w \sin^2(t/2))^{-s/2} dt. \quad (3)$$

It is continuous on compact subsets of $H \times H$, including the diagonal, since $s < 1$. The Riesz equilibrium measure exists and is unique; finite energy is supplied by any circle in $\Gamma(A)$. Uniqueness

and axial symmetry imply that it is the lift of a probability measure $\lambda_{s,A}$ on A , minimizing the continuous K_s -energy

$$J_s(\lambda) = \iint K_s(z, w) d\lambda(z) d\lambda(w).$$

These facts and the reduction are established in [2, Section 2].

Set

$$c_s = \frac{1}{2\pi} \int_0^{2\pi} (4 \sin^2(t/2))^{-s/2} dt,$$

$$\kappa_s = \frac{1}{\pi} \int_0^\infty [u^{-s} - (1+u^2)^{-s/2}] du > 0.$$

The second integral converges at both ends. Define

$$B_s = \frac{s}{8\pi(1-s)} + \frac{s}{3} + \frac{s}{2\pi(s+1)}$$

and

$$E_s(R) = c_s s 2^{s+1} R^{-s} + \kappa_s 2^{q+1} R^{-1} + 2^{s+4} B_s R^{-s-1}. \quad (4)$$

Lemma 3.1. *For $R \geq 2$ and $z, w \in \overline{\mathbb{D}}$,*

$$|R\{K_s(R+z, R+w) - c_s R^{-s}\} + \kappa_s |z-w|^q| \leq E_s(R). \quad (5)$$

In particular, the left expression tends to zero uniformly on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ as $R \rightarrow \infty$.

This is the R -normalization of the $2R$ -normalized limit in [2, Lemma 4.1 and equations (4.3)–(4.6)]. We include an estimate to make the transfer to finite radius explicit.

Proof. Let $d = |z-w|$ and $T = ((R + \Re z)(R + \Re w))^{1/2}$, so $R-1 \leq T \leq R+1$. Write $K(T, d)$ for the angular integral in (3). Then $K(T, 0) = c_s T^{-s}$. Put $g_d(u) = u^{-s} - (u^2 + d^2)^{-s/2}$. The substitution $u = 2T \sin(t/2)$, after using symmetry, gives

$$T\{K(T, 0) - K(T, d)\} = \frac{1}{\pi} \int_0^{2T} g_d(u) \left(1 - \frac{u^2}{4T^2}\right)^{-1/2} du.$$

Scaling yields $\pi^{-1} \int_0^\infty g_d(u) du = \kappa_s d^q$. For $d = 0$ both sides vanish. For $d > 0$, the mean value theorem gives $0 \leq g_d(u) \leq (s/2)d^2 u^{-s-2}$. On $0 \leq u \leq T$, the weight minus one is at most $u^2/(4T^2)$; hence the error on that interval is at most $s d^2 T^{-s-1}/(8\pi(1-s))$. For the two tails use

$$\int_T^{2T} \left(1 - \frac{u^2}{4T^2}\right)^{-1/2} du = \frac{2\pi T}{3}, \quad \int_T^\infty u^{-s-2} du = \frac{T^{-s-1}}{s+1}.$$

Their sum gives

$$|T\{K(T, 0) - K(T, d)\} - \kappa_s d^q| \leq B_s d^2 T^{-s-1}.$$

Finally, $d \leq 2$, $T \geq R/2$, and

$$R|T^{-s} - R^{-s}| \leq s 2^{s+1} R^{-s}, \quad |R/T - 1| \leq 2/R.$$

Substitution gives precisely (4)–(5). □

4 Strict domination and the connected star

We recall the elementary strict domination criterion [2, Lemma 3.6]. If λ minimizes a continuous symmetric kernel energy on a compact A , with energy V and potential U , then first variations toward point masses give $U \geq V$ on A . Since $\int U d\lambda = V$, continuity gives $U = V$ on $\text{supp } \lambda$. Consequently, if a probability measure ν on A satisfies

$$K_s(a, w) > \int_A K_s(v, w) d\nu(v) \quad (w \in A), \quad (6)$$

then $a \notin \text{supp } \lambda$: integrate against λ , interchange the integrals, and obtain $U(a) > \int U d\nu \geq V$.

Let

$$T_N = \bigcup_{j=0}^{N-1} [0, \zeta_j], \quad A = R + T_N, \quad \nu = \frac{1}{N} \sum_j \delta_{R+\zeta_j}.$$

Choose $R > 3$ so large that

$$E_s(R) < \kappa_s q^2 / 32. \quad (7)$$

Such a finite radius exists by Lemma 3.1. Subtracting the average vertex row from the center row in (5), and applying (2), gives

$$K_s(R, R+w) - \frac{1}{N} \sum_j K_s(R + \zeta_j, R+w) > \frac{\kappa_s q^2}{16R} \quad (w \in \mathbb{D}). \quad (8)$$

Criterion (6) excludes R from $\text{supp } \lambda_{s,A}$. By continuity, it excludes a neighborhood of R , not merely the point itself. Notice that ν is a comparison measure; no claim is made that the actual equilibrium measure has uniform vertex masses.

The finite star T_N has connected complement. A point within the unit disk but off the spokes can move outward along its argument without meeting a spoke. Outside the disk, the points can be joined around it. Since $R > 3$, the translated paths can remain in the right half-plane and reach its boundary axis. Lifting them in meridian planes shows that $\mathbb{R}^3 \setminus \Gamma(A)$ is connected. Moreover, $\Gamma(A)$ is a finite union of two-dimensional surfaces, so has empty interior. Thus

$$\partial_\infty \Gamma(A) = \Gamma(A).$$

In particular, the excluded circle $\Gamma(\{R\})$ is part of the outer boundary. Its uniform probability measure has energy $c_s R^{-s} < \infty$, so it has positive s -capacity. This proves the first assertion of Theorem 1.1. It also avoids the ambiguity of adding a polar isolated point.

5 Generating sets with nonempty interior

Fix N, R above. The strict uniform gap (8) and continuity give $\varepsilon \in (0, 1/2)$ and $\delta_0 > 0$ such that

$$K_s(R + z, R + w) > \frac{1}{N} \sum_j K_s(R + \zeta_j, R + w) \quad (9)$$

whenever $|z| < \varepsilon$ and $\text{dist}(w, T_N) \leq \delta_0$. Choose

$$0 < \delta < \min\{\delta_0, \tfrac{1}{2}\varepsilon \sin(\pi/N), 1/4\}, \quad C_\delta = T_N + \overline{B(0, \delta)}, \quad A_\delta = R + C_\delta.$$

The set C_δ is a finite union of closed capsules, each convex and containing the origin. It is therefore compact, connected and star-shaped, with nonempty interior.

Consider

$$z_\delta = \frac{\delta}{\sin(\pi/N)} e^{i\pi/N}.$$

Its distance to each adjacent spoke is δ . Their orthogonal projections are within the spokes because $|z_\delta| < 1/2$; no other spoke is nearer. Small inward and outward radial moves show that $z_\delta \in \partial C_\delta$. Also $|z_\delta| < \varepsilon$, so (9) and strict domination exclude $R + z_\delta$ from the equilibrium support on A_δ .

For completeness, a point outside a compact star-shaped planar set can escape along its radial ray, and points sufficiently far away can be joined around it. Thus its complement is connected. Here $C_\delta \subset \overline{B(0, 1 + \delta)}$, and $R > 3$ leaves room for these paths in the meridian half-plane. Lifting as before shows that $\Gamma(A_\delta)$ has connected complement. Since the cylindrical-coordinate map is locally a product away from the rotation axis, $\Gamma(\partial A_\delta) = \partial \Gamma(A_\delta) = \partial_\infty \Gamma(A_\delta)$. The omitted circle $\Gamma(\{R + z_\delta\})$ is therefore on the outer boundary and has positive s -capacity. This proves the remaining assertion of Theorem 1.1.

6 Exact parameter certificates

The proof is analytic and covers real s , without numerical optimization. Its sufficient bounds also admit exact rational certificates when $s = a/b \in (0, 1)$ is rational. Elementary estimates of the defining integrals give

$$c_s \leq \frac{2^{-s}}{1-s}, \quad \frac{7s}{22(1-s)} < \kappa_s < \frac{1}{3} \left(\frac{1}{1-s} + \frac{s}{2(s+1)} \right),$$

where $3 < \pi < 22/7$ was used. For the lower bound on κ_s , integrate $u^{-s} - 1$ over $(0, 1)$; for the upper bound, use u^{-s} there and $(s/2)u^{-s-2}$ on $(1, \infty)$. The bound for c_s follows from $\sin t \geq 2t/\pi$ on $[0, \pi/2]$. Also

$$B_s < \frac{s}{24(1-s)} + \frac{s}{3} + \frac{s}{6(s+1)}.$$

Take

$$N = 8 \left\lceil 8/q^2 \right\rceil^{\lceil 1/q \rceil}, \quad R = L^b, \quad L = 2^m.$$

The first choice implies (1). With $R^{-s} = L^{-a}$, the preceding rational bounds and $2^{q+1} \leq 4$, $2^{s+4} \leq 32$ make (7) a strict rational inequality; increasing m eventually satisfies it. The accompanying standard-library program checks 129 rational exponents and 2,020 auxiliary concavity inequalities, identically in ordinary and optimized execution. These checks replay sufficient bounds; they are not a formal verification or a substitute for the universal proof.

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References

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