

A Sharp Prior-Sensitivity Threshold for Fixed-Partition Support Boundaries

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Abstract

Consider a Poisson support-boundary experiment with intensity scale n , a known partition of $[0, 1]$ into K bins of widths Δ_j , and independent prior densities on the bin heights. For a fixed positive prior whose logarithm is locally Lipschitz, we prove that the posterior is asymptotically equivalent in total variation to the known product-exponential reference whenever $\sum_j (n\Delta_j)^{-2} \rightarrow 0$, uniformly over bounded true heights. On an equal grid this gives $K = o(n^{2/3})$. A finite-sample bound follows by tensorizing squared coordinate errors. A fixed proper Laplace prior proves sharpness for this prior class: if $L^2 K^3 / n^2 \rightarrow \tau$, the limiting total variation distance is $2\Phi(\sqrt{\tau}/2) - 1$, with distance tending to one above the critical scale under the stated localization condition. We also give the resulting integral-posterior approximation and an exact one-bin correction characterization. The exponential reference and the asymptotic K/n bias correction are due to earlier work; our result concerns their range of validity for fixed, known bins.

1 Experiment, prior work and main result

Let $I_1, \dots, I_K$ partition $[0, 1]$ into intervals of lengths $\Delta_j > 0$. Endpoints, being null sets, have no effect. We observe a Poisson point process on $[0, 1] \times \mathbb{R}$ with intensity

$$(x, y) \mapsto n \mathbf{1}\{y \geq f_a(x)\}, \quad f_a = \sum_{j=1}^K a_j \mathbf{1}_{I_j}.$$

Here $n > 0$ is an intensity scale, not the number of observed points. Set $r_j = n\Delta_j$ and let m_j be the minimum observed ordinate in bin j . Each minimum is finite almost surely, and under a true vector a^0 ,

$$m_j - a_j^0 \text{ are independent with laws } \text{Exp}(r_j). \quad (1)$$

The likelihood, up to a data-only factor, is $\prod_j e^{r_j a_j} \mathbf{1}\{a_j \leq m_j\}$. For independent positive prior densities g_j , write Π_m for the posterior. Write Q_m for the reference law under which $m_j - a_j$ are independent $\text{Exp}(r_j)$ variables. Throughout, $d_{\text{TV}}(P, Q) = \sup_B |P(B) - Q(B)|$.

Reiß and Schmidt-Hieber introduced this fixed-partition comparison in their study of compound Poisson priors [1, 2]. Their Supplement, Theorem C.1, proves posterior equivalence under a positive β -Hölder density and

$$\frac{n \min_j \Delta_j}{K^{1/\beta} \log K} \rightarrow \infty.$$

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For $\beta = 1$, its displayed regular-grid sufficient range is $K = o((n/\log n)^{1/2})$. Their proof controls the maximum variation of the product prior over a rectangle. We instead control a squared error after normalizing each coordinate. A positive Lipschitz density has locally Lipschitz logarithm, so the theorem below improves that sufficient range on their $\beta = 1$ class. It also permits densities that have only local logarithmic regularity.

Theorem 1. *Let g be a fixed probability density on $\mathbb{R}$, positive everywhere, with locally Lipschitz logarithm, and use the product prior $\prod_j g(a_j)$. Fix $R < \infty$, $R \geq 0$. If*

$$S_n = \sum_{j=1}^{K_n} (n\Delta_j)^{-2} \longrightarrow 0, \quad (2)$$

then

$$\sup_{\|a^0\|_\infty \leq R} \mathbb{E}_{a^0} d_{\text{TV}}(\Pi_m, Q_m) \longrightarrow 0.$$

In particular, equal bins satisfy (2) if and only if $K_n^3/n^2 \rightarrow 0$.

The number of bins and the partition may vary with n . The prior g and the height bound R are fixed. The rate in the theorem is sharp for this class, in the sense of the proper-prior example in Theorem 5; it is not asserted necessary for each individual prior. Critical dimension phenomena also occur in regular models [3]. Those results assume a smooth regular likelihood and do not directly apply to the support-dependent likelihood here.

The original Oberwolfach discussion [4] already reports conditional positive results and misspecification failure for automatic correction. We do not present that broad discussion as an unsolved universal identity, nor claim a solution for arbitrary random partitions or nonparametric priors. The bounded literature comparison and the exact scope of this note are recorded in the accompanying source package. No absolute priority claim is made.

2 A squared-tilt estimate

Let $q(y) = e^{-y}\mathbf{1}\{y \geq 0\}$. In coordinates $Y_j = r_j(m_j - a_j)$, the posterior density relative to q is h_j/Z_j , where

$$h_j(y) = \frac{g_j(m_j - y/r_j)}{g_j(m_j)}, \quad Z_j = \mathbb{E}_q h_j. \quad (3)$$

Under Q_m , the Y_j are independent $\text{Exp}(1)$ variables. The following estimate is deterministic and does not use a typical-data event.

Lemma 2. *Suppose $h > 0$, $h(0) = 1$ and $|\log h(y)| \leq \varepsilon y$ for $y \geq 0$, where $0 \leq \varepsilon \leq 1/4$. If $p = qh/\mathbb{E}_q h$, then*

$$\chi^2(p, q) \leq \frac{2\varepsilon^2(1+\varepsilon)^2}{(1-2\varepsilon)(1-\varepsilon)} \leq 9\varepsilon^2. \quad (4)$$

Consequently, for independent coordinates satisfying these hypotheses,

$$d_{\text{TV}} \left(\bigotimes_j p_j, \bigotimes_j q \right) \leq \min \left\{ 1, 3 \left(\sum_j \varepsilon_j^2 \right)^{1/2} \right\}. \quad (5)$$

Proof. The envelope gives $Z = \mathbb{E}_q h \geq (1 + \varepsilon)^{-1}$. Moreover $|h - 1| \leq e^{\varepsilon y} - 1$, whence

$$\text{Var}_q(h) \leq \mathbb{E}_q(h - 1)^2 \leq \frac{1}{1 - 2\varepsilon} - \frac{2}{1 - \varepsilon} + 1 = \frac{2\varepsilon^2}{(1 - 2\varepsilon)(1 - \varepsilon)}.$$

Divide by Z^2 to obtain the first inequality. The coefficient $2(1 + \varepsilon)^2/((1 - 2\varepsilon)(1 - \varepsilon))$ is at most $25/3 < 9$ on $[0, 1/4]$. For $H^2(P, Q) = \int (\sqrt{p} - \sqrt{q})^2$, Cauchy–Schwarz gives $d_{\text{TV}}(P, Q) \leq H(P, Q)$, and $H^2(P, Q) \leq \chi^2(P, Q)$. Hellinger affinities multiply under products, so $H^2(\otimes_j P_j, \otimes_j Q_j) \leq \sum_j H^2(P_j, Q_j)$. This proves (5), including the case $\varepsilon_j = 0$. $\square$

Proposition 3. *If $\log g_j$ is globally L_j -Lipschitz and $r_j \geq 4L_j$, then for every finite vector m ,*

$$d_{\text{TV}}(\Pi_m, Q_m) \leq \min \left\{ 1, 3 \left(\sum_j \frac{L_j^2}{r_j^2} \right)^{1/2} \right\}.$$

Proof. Equation (3) gives $|\log h_j(y)| \leq L_j y / r_j$. Apply Lemma 2. The bijective coordinate transformation between a and Y preserves total variation. $\square$

Proper Laplace, logistic and Cauchy densities satisfy a global condition of this kind. A Gaussian density does not have a globally Lipschitz logarithm, but is covered by the localization argument that follows.

3 Localization and unequal bins

Proof of Theorem 1. Put $I = [-R - 1, R + 1]$, $c = \min_I g > 0$, and choose $L \geq 1$ at least the Lipschitz constant of $\log g$ on I . Define $u = g$ on I and extend $\log u$ linearly with slope $+L$ to the left and $-L$ to the right. Then $U = \int_{\mathbb{R}} u < \infty$, $\log u$ is globally L -Lipschitz, and the normalized prior u/U has the same posterior as u .

Consider the event $E = \{-R \leq m_j \leq R + 1/2 \text{ for every } j\}$. By (1),

$$\sup_{\|a^0\|_{\infty} \leq R} \mathbb{P}_{a^0}(E^c) \leq \sum_j e^{-r_j/2}. \quad (6)$$

On E , the g -posterior and the u -posterior in coordinate j agree conditional on $B_j = \{a_j \geq -R - 1\}$. Their likelihood normalizers, with the common factor $e^{r_j m_j}$ removed, are each at least

$$\int_{m_j - 1/2}^{m_j} e^{r_j(a - m_j)} g(a) da \geq \frac{c}{r_j} (1 - e^{-r_j/2}).$$

The integration interval lies in I . Outside B_j , $a - m_j \leq -1$, so the corresponding numerator is at most e^{-r_j} for g and Ue^{-r_j} for u . For $r_j \geq 2$, comparison of the tail masses gives

$$d_{\text{TV}}(\Pi_{g,j}, \Pi_{u,j}) \leq C r_j e^{-r_j}, \quad C = \frac{1 + U}{c(1 - e^{-1})}.$$

Here one may bound the distance by the sum of the two tail probabilities, because the conditional distributions on B_j coincide. Telescoping products and applying Proposition 3 yield, once $r_{\min} \geq \max(2, 4L)$,

$$\sup_{\|a^0\|_{\infty} \leq R} \mathbb{E}_{a^0} d_{\text{TV}}(\Pi_m, Q_m) \leq 3L\sqrt{S_n} + C \sum_j r_j e^{-r_j} + \sum_j e^{-r_j/2}. \quad (7)$$

Condition (2) forces $r_{\min} \rightarrow \infty$. The two sums on the right are at most, respectively,

$$S_n \sup_{r \geq r_{\min}} r^3 e^{-r}, \quad S_n \sup_{r \geq r_{\min}} r^2 e^{-r/2}.$$

Both vanish. This also shows why no additional minimum-width logarithmic condition is needed. For equal bins $r_j = n/K$, $S_n = K^3/n^2$. $\square$

4 The integral posterior and credible intervals

Let $\theta = \sum_j \Delta_j a_j$ and $\hat{\theta} = \sum_j \Delta_j m_j$. The reference law and the sampling law give the exact identities

$$\theta \mid Q_m \stackrel{d}{=} \hat{\theta} - G_K/n, \quad n(\hat{\theta} - \theta_0) \stackrel{d}{=} G_K, \quad G_K \sim \text{Gamma}(K, 1). \quad (8)$$

Indeed, $\Delta_j(m_j - a_j) = Y_j/n$, and the same calculation applies to the independent sampling excesses.

Corollary 4. *Under Theorem 1, the marginal posterior of θ approaches the first law in (8) in expected total variation, uniformly over bounded truths. If also $K \rightarrow \infty$, then*

$$\sup_{\|a^0\|_\infty \leq R} \mathbb{E}_{a^0} d_{\text{TV}} \left(\Pi_m^\theta, N \left(\hat{\theta} - \frac{K}{n}, \frac{K}{n^2} \right) \right) \rightarrow 0.$$

For each fixed $\alpha \in (0, 1)$, the equal-tailed posterior interval of mass $1 - \alpha$ has frequentist coverage tending uniformly to $1 - \alpha$.

Proof. Total variation contracts under a measurable map, proving the first assertion. By Stirling's formula the density of $(G_K - K)/\sqrt{K}$ converges pointwise to the standard normal density. Scheffé's lemma then gives convergence in total variation, proving the normal approximation.

For coverage, let $b_u(m)$ be the u -quantile of the reference law $\hat{\theta} - G_K/n$. Its density is positive on $(-\infty, \theta)$; the actual integral posterior is also continuous and strictly increasing there. If $d_{\text{TV}}(\Pi_m^\theta, Q_m^\theta) \leq \eta < \min(\alpha, 1 - \alpha)/2$, each actual u -quantile lies between $b_{u-\eta}(m)$ and $b_{u+\eta}(m)$. Thus the posterior equal-tailed interval lies between reference intervals with nominal masses $1 - \alpha - 2\eta$ and $1 - \alpha + 2\eta$. The second identity in (8) gives these reference intervals exactly their nominal sampling coverages. The bad event has probability tending to zero uniformly, by Theorem 1 and Markov's inequality. First take the limit in n , then let $\eta \downarrow 0$. $\square$

The K/n correction and the limiting normal law are established in [2, Corollary C.2]. The contribution here is their enlarged range under the stated regularity assumptions. No convergence of unbounded posterior moments is inferred from total variation convergence.

5 A proper prior attaining the critical threshold

Theorem 5. *Take equal bins, put $r = n/K$, and use the fixed Laplace prior $g(a) = (L/2)e^{-L|a|}$ with $L > 0$. Let $a_j^0 = -b$ for a fixed $b > 0$. Suppose $\varepsilon = L/r \rightarrow 0$ and $Ke^{-br} \rightarrow 0$. If $K\varepsilon^2 \rightarrow \tau \in [0, \infty)$, then, in sampling probability and in expectation,*

$$d_{\text{TV}}(\Pi_m, Q_m) \longrightarrow 2\Phi(\sqrt{\tau}/2) - 1. \quad (9)$$

If $K\varepsilon^2 \rightarrow \infty$, this distance tends to one. The same statements hold for the marginal laws of θ .

Proof. Let $E_- = \{m_j < 0 \text{ for every } j\}$. Equation (1) gives $\mathbb{P}(E_-^c) \leq Ke^{-br}$. On E_- , the admissible heights satisfy $a_j \leq m_j < 0$. The posterior deficits $D_j = m_j - a_j$ are therefore independent $\text{Exp}(r+L)$ variables, whereas the reference deficits are independent $\text{Exp}(r)$ variables. Their product Hellinger affinity is

$$A_K = \left(\frac{2\sqrt{1+\varepsilon}}{2+\varepsilon} \right)^K, \quad A_K^2 = \left(1 - \frac{\varepsilon^2}{(2+\varepsilon)^2} \right)^K. \quad (10)$$

If $K\varepsilon^2 \rightarrow 0$, then $A_K \rightarrow 1$ and $d_{\text{TV}} \leq \sqrt{2(1-A_K)} \rightarrow 0$. If $K\varepsilon^2 \rightarrow \infty$, then $A_K \rightarrow 0$ and $d_{\text{TV}} \geq 1 - A_K \rightarrow 1$.

For $0 < \tau < \infty$, let $S = \sum_j D_j$. The likelihood ratio of posterior to reference is $(1+\varepsilon)^K e^{-r\varepsilon S}$, whose positive region is

$$S \leq s_* = \frac{K \log(1+\varepsilon)}{r\varepsilon}.$$

With F_K the distribution function of $\text{Gamma}(K, 1)$, the exact distance on E_- is consequently

$$F_K((r+L)s_*) - F_K(rs_*). \quad (11)$$

Here $K \rightarrow \infty$. Taylor expansion gives

$$\frac{rs_* - K}{\sqrt{K}} \rightarrow -\frac{\sqrt{\tau}}{2}, \quad \frac{(r+L)s_* - K}{\sqrt{K}} \rightarrow \frac{\sqrt{\tau}}{2}.$$

The gamma central limit theorem proves (9). On E_- , the affine identity $\theta = \hat{\theta} - S/K$ preserves the likelihood-ratio test, so its marginal distance is exactly the product distance. The complement of E_- has vanishing probability, and all distances are at most one, proving both modes of convergence. $\square$

Since $K\varepsilon^2 = L^2 K^3/n^2$, this example proves sharpness at K of order $n^{2/3}$ within the class in Theorem 1. The localization condition holds at that scale and, for example, for $K = \lfloor n^{3/4} \rfloor$. We do not assert failure for every sequence above the threshold or for every prior.

On E_- the posterior mean correction can be computed directly:

$$\hat{\theta} - \mathbb{E}(\theta \mid m) = \frac{K}{n + LK}, \quad \frac{K/n - K/(n + LK)}{\sqrt{K}/n} = \frac{LK^{3/2}}{n + LK}.$$

Thus prior sensitivity reaches the fluctuation scale at exactly the same threshold. These moment identities follow from the explicit exponential laws, not from a total variation approximation.

6 An exact one-bin identity

Proposition 6. Fix $n > 0$. Let π be positive and locally integrable on $\mathbb{R}$, and suppose $Z(m) = \int_{-\infty}^m e^{na} \pi(a) da$ is finite and positive, with $\int_{-\infty}^m |a| e^{na} \pi(a) da < \infty$, for every $m \in \mathbb{R}$. The posterior mean for likelihood $e^{na} \mathbf{1}\{a \leq m\}$ equals $m - 1/n$ for every m if and only if π is a positive constant almost everywhere. In particular, no proper full-line prior has this identity for every observation.

Proof. Put $M(m) = \int_{-\infty}^m a e^{na} \pi(a) da$. The assumed identity is $M = (m - 1/n)Z$. These functions are locally absolutely continuous. Differentiation almost everywhere yields $mZ' = Z + (m - 1/n)Z'$, hence $Z' = nZ$. It follows that $Z(m) = Ce^{nm}$ for some $C > 0$ and $\pi = nC$ almost everywhere. Conversely, integration with a positive constant prior gives exactly $m - 1/n$. $\square$

The full-line and all-observation hypotheses are essential. This proposition does not rule out asymptotic correction, correction under a bounded parameter space, or a different exact correction. Nor does it resolve the broader source question for random partitions, shared hyperparameters or misspecified true boundaries, where our factorization or sampling pivot need not apply.

The accompanying exact-arithmetic tests check the tilt bound, finite mixtures of exponential tilts and Laplace formulas; they are regression checks, not a replacement for the proofs. This unrefereed note was developed with AI assistance in mathematical exploration, literature comparison, drafting and computation. It has not undergone independent review or formal proof-assistant verification.

References

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