

A Symmetric Counterexample to Brändén's Majorization Conjectures for Mixed Characteristic Polynomials

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October 20, 2026

Abstract

In an extended abstract in Oberwolfach Rep. 14 (2017), no. 1, 771–862 (pp. 825–826), Brändén considered the set $\mathcal{D}_{m,d}$ of stable homogeneous polynomials P of degree d in m variables with $\partial P/\partial x_i(\mathbf{1}) = d/m$ for all i , and the zero vector $\lambda(P)$ of the mixed characteristic polynomial $\chi_P(t) = \prod_i (1 - \partial/\partial x_i)P|_{x=t\mathbf{1}}$. He conjectured that $\lambda(P)$ is majorized by $\lambda(P_\infty)$, where $P_\infty = (x_1 + \cdots + x_m)^d/m^d$ (Conjecture 2), and, for symmetric P , that $\lambda(\Psi_r P)$ is majorized by $\lambda(\Psi_s P)$ for $0 \leq r \leq s$ along a flow Ψ_s tending to P_∞ (Conjecture 5). We show that both conjectures fail for the symmetric polynomial $P = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)/8 \in \mathcal{D}_{3,3}$, by exact proofs that need no numerical computation. Here $\chi_P = (t-1)(4t^2 - 8t + 1)/4$, and $\lambda_1(P) + \lambda_2(P) = 2 + \sqrt{3}/2$ exceeds the same sum for P_∞ because $9\sqrt{3} < 16$. Along the flow, $\Psi_s P$ is again a product of three linear forms, $\chi_{\Psi_s P}$ is an explicit cubic with positive discriminant, and $\lambda_1 + \lambda_2$ of $\Psi_s P$ is smaller than that of P for every $s > 0$; so Conjecture 5 fails with $r = 0$ and every $s > 0$. The flow is printed as $\exp(s(I - T))$, a sign under which the source's Lemma 3 fails; we prove that lemma for $\exp(s(T - I))$, and show that Conjecture 5 fails under either sign. Conjecture 2 itself is not new: it was reportedly refuted in a 2017 KTH degree project (Samuel Zackrisson, DiVA diva2:1106332; full text not read), according to its abstract by way of Wanless's counterexamples to the Holens–Đoković permanent conjecture, and according to a search-engine excerpt also by an explicit example. Our example satisfies the Holens–Đoković inequalities, so it does not arise from Wanless's counterexamples; we claim no priority for it. We found no refutation of Conjecture 5 in the sources searched; since the thesis could not be read, we cannot exclude that it also treats Conjecture 5. This is an unrefereed note.

1 Introduction

1.1 The conjectures

In the solution of the Kadison–Singer problem by Marcus, Spielman and Srivastava, a key step is a bound on the largest zero of mixed characteristic polynomials [MSS15]. In an extended abstract in the report of the Oberwolfach workshop on real algebraic geometry of 2017 [Brä17], Brändén discusses sharp upper bounds of this kind and two conjectures about the whole zero vector. We recall the definitions as printed there.

Let $x = (x_1, \dots, x_m)$, $\mathbf{1} = (1, \dots, 1)$ and $\partial_i = \partial/\partial x_i$. A homogeneous polynomial $P \in \mathbb{R}[x]$ of degree d is *doubly stochastic* if $\partial_i P(\mathbf{1}) = d/m$ for $1 \leq i \leq m$, and P is *stable* if $P(x) \neq 0$ whenever $\operatorname{Im} x_i > 0$ for all i . Let $\mathcal{D}_{m,d}$ be the set of all doubly stochastic stable polynomials of degree d

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in $\mathbb{R}[x_1, \dots, x_m]$; no further condition (such as multiaffinity) is imposed. The *mixed characteristic polynomial* of P is

$$\chi_P(t) = (1 - \partial_1)(1 - \partial_2) \cdots (1 - \partial_m)P \Big|_{x_1 = \dots = x_m = t}.$$

For $P \in \mathcal{D}_{m,d}$ all zeros of χ_P are real, because the operators $1 - \partial_i$ preserve stability [Brä17]; let $\lambda(P) = (\lambda_1(P) \geq \dots \geq \lambda_d(P))$ be these zeros, and put

$$P_\infty := \frac{(x_1 + \dots + x_m)^d}{m^d}.$$

For non-increasing vectors $\alpha, \beta \in \mathbb{R}^d$ one writes $\alpha \preceq \beta$ (β *majorizes* α) if $\alpha_1 + \dots + \alpha_k \leq \beta_1 + \dots + \beta_k$ for $1 \leq k \leq d-1$ and $\alpha_1 + \dots + \alpha_d = \beta_1 + \dots + \beta_d$.

Conjecture 2 (Brändén [Brä17]). *If $P \in \mathcal{D}_{m,d}$, then $\lambda(P) \preceq \lambda(P_\infty)$.*

The inequality for $k=1$, $\lambda_1(P) \leq \lambda_1(P_\infty)$, is [Brä17, Conjecture 1], which, as the source notes, essentially appeared in [MSS15]. The source also notes that the van der Waerden type bound $\partial_1 \cdots \partial_m P \geq m!/m^m$ for $P \in \mathcal{D}_{m,m}$, a version of Gurvits' theorem [Gur08], can be deduced from Conjecture 2.

Let $T(P) = \frac{1}{md}(x_1 + \dots + x_m)(\partial_1 + \dots + \partial_m)P$. The source defines a flow by $\Psi_s = \exp(s(I - T))$, $s \geq 0$, and states ([Brä17, Lemma 3]) that Ψ_s maps $\mathcal{D}_{m,d}$ into itself and that $\Psi_s(P) \rightarrow P_\infty$ as $s \rightarrow \infty$, uniformly on compacts. Let Sym_m be the space of polynomials invariant under all permutations of the variables. By [Brä17, Theorem 4], whose proof is to appear elsewhere, $s \mapsto \lambda_1(\Psi_s(P))$ is monotone for $P \in \mathcal{D}_{m,d} \cap \text{Sym}_m$ (the source says “monotone decreasing”), and in particular $\lambda_1(P) \leq \lambda_1(P_\infty)$.

Conjecture 5 (Brändén [Brä17]). *If $P \in \mathcal{D}_{m,d} \cap \text{Sym}_m$ and $0 \leq r \leq s$, then $\lambda(\Psi_r(P)) \preceq \lambda(\Psi_s(P))$.*

The sign of the flow. Every eigenvalue of T on the forms of degree d lies in $[0, 1]$, and 1 is a simple eigenvalue with eigenvector P_∞ (Proposition 5.1). So $\exp(s(I - T))$ multiplies each spectral component of P for an eigenvalue $\mu < 1$ by $e^{s(1-\mu)} \geq 1$, and $\exp(s(I - T))P$ does not converge to P_∞ unless $P = P(\mathbf{1})P_\infty$; for our example it even leaves the stable polynomials (Theorem 1.2(d)). With the opposite sign, $\Psi_s = \exp(s(T - I))$, Lemma 3 of the source holds (Proposition 5.1); this is presumably the intended flow, and with it “monotone decreasing” in Theorem 4 has to be read as “non-decreasing” for its stated consequence (Remark 4.5). We refute Conjecture 5 under both signs.

The problem is record OWR-15219-013 of the UnsolvedMath collection [ula26], where it is listed as open.

1.2 Results

Throughout,

$$P := \frac{1}{8}(x_1 + x_2)(x_1 + x_3)(x_2 + x_3), \quad A(q) := \frac{1-q}{3}I + \frac{2+q}{6}(J - I) \quad (q \in \mathbb{R}),$$

where I and J are the 3×3 identity and all-ones matrices, and for an $n \times n$ matrix M we write $P_M(x) = \prod_{i=1}^n \sum_j m_{ij}x_j$. Then $P = P_{A(1)}$, $A(1) = \frac{1}{2}(J - I)$, and $P_\infty = P_{A(0)}$, $A(0) = \frac{1}{3}J$, for $m = d = 3$.

Theorem 1.1 (Conjecture 2 fails). $P \in \mathcal{D}_{3,3} \cap \text{Sym}_3$,

$$\chi_P(t) = \frac{1}{4}(t-1)(4t^2 - 8t + 1), \quad \lambda(P) = \left(1 + \frac{\sqrt{3}}{2}, 1, 1 - \frac{\sqrt{3}}{2}\right), \quad \chi_{P_\infty}(t) = t^3 - 3t^2 + 2t - \frac{2}{9},$$

and $\lambda_3(P) < \lambda_3(P_\infty)$. Hence $\lambda_1(P) + \lambda_2(P) > \lambda_1(P_\infty) + \lambda_2(P_\infty)$, so $\lambda(P) \not\preceq \lambda(P_\infty)$.

Numerically $\lambda(P_\infty) \approx (2.0966484, 0.7647601, 0.1385915)$, so the sums for $k = 2$ are $2.8660254 > 2.8614085$, while $1.8660 \leq 2.0966$ for $k = 1$, in accordance with [Brä17, Theorem 4]. For $d = 2$, Conjecture 2 is Conjecture 1 of the source, since the sum of the zeros is fixed (Lemma 2.1); so the failure at $k = 2 = d - 1$ occurs in the smallest degree in which Conjecture 2 says more.

Theorem 1.2 (Conjecture 5 fails).

- (a) For every real s , $\exp(s(T - I))P = P_{A(q)}$ with $q = e^{-s/3}$. For $0 \leq q \leq 1$ the matrix $A(q)$ is nonnegative and doubly stochastic, so $\exp(s(T - I))P \in \mathcal{D}_{3,3} \cap \text{Sym}_3$ for $s \geq 0$.
- (b) $\chi_{P_{A(q)}}(t) = \chi_q(t) := t^3 - 3t^2 + (2 + \frac{q^2}{4})t - \frac{2}{9} - \frac{q^2}{12} + \frac{q^3}{18}$, and the discriminant of χ_q is $\Delta(q) = \frac{1}{48}(2 + q)^3(16 - 24q + 18q^2 - 7q^3)$, which is positive for $-2 < q \leq 1$.
- (c) Let $\Psi_s = \exp(s(T - I))$. For every $s > 0$, $\lambda_3(\Psi_s P) > \lambda_3(P)$, hence $\lambda_1(\Psi_s P) + \lambda_2(\Psi_s P) < \lambda_1(P) + \lambda_2(P)$ and $\lambda(\Psi_0 P) \not\leq \lambda(\Psi_s P)$. So Conjecture 5 fails for P with $r = 0$ and every $s > 0$.
- (d) Let $\Psi_s = \exp(s(I - T))$ as printed, and $s > 0$. Then $\Psi_s P = P_{A(u)}$ with $u = e^{s/3} > 1$, a product of linear forms with a negative coefficient, and $\chi_{\Psi_s P} = \chi_u$. The polynomial $\Psi_s P$ is not stable, so Lemma 3 of the source fails for this sign and every $s > 0$. Let $u_0 \approx 1.2873544$ be the unique real zero of $7u^3 - 18u^2 + 24u - 16$ and $s_0 = 3 \ln u_0 \approx 0.7578$. For $s > s_0$ the polynomial $\chi_{\Psi_s P}$ even has non-real zeros. For $0 < s \leq s_0$ all zeros are real and $\lambda_1(\Psi_s P) < \lambda_1(P)$, so $\lambda(\Psi_0 P) \not\leq \lambda(\Psi_s P)$.

In Theorem 1.2 the zeros of $\chi_{\Psi_s P}$ are shown to be real by the discriminant in (b) alone. So the refutation of Conjecture 5 uses neither Lemma 3 of the source nor the stability of $\Psi_s P$, and it holds whichever sign of the flow the source intended (for the printed sign see (d) and Section 4.4). For instance, $s = 3 \ln 2$ gives the zeros $\approx (2.050152, 0.807166, 0.142681)$, and $s = 3 \ln \frac{6}{5}$ under the printed sign gives $\approx (1.68913, 1.18819, 0.12268)$. Section 3 also explains how Wanless's counterexamples to the Holens–Đoković conjecture give counterexamples of another kind (Proposition 3.1), and Section 4 locates the failures on the orbit of P (Remarks 4.3 to 4.5).

1.3 What is new, what is not

The question, the definitions, Lemma 3, Theorem 4 and both conjectures are Brändén's [Brä17]; mixed characteristic polynomials and the bound on their largest zero are due to Marcus, Spielman and Srivastava [MSS15], the van der Waerden type bound to Gurvits [Gur08].

Conjecture 2 is not claimed as new: it was reportedly refuted in a 2017 KTH degree project (Samuel Zackrisson [Zac17], DiVA diva2:1106332; full text not read). According to its abstract, which we saw only as quoted in search-engine results, the thesis shows by way of Wanless's counterexample to the Holens–Đoković conjecture [Wan99] that the zeros of associated Laguerre polynomials need not majorize those of other mixed characteristic polynomials; by Lemma 2.2 this is Conjecture 2 in the square case $m = d$. A second excerpt of the full text, again seen only in search-engine results, attributes this conjecture to Marcus, in personal communication with Brändén, and says that it is disproved by an explicit choice of three weights w_1, w_2, w_3 , with a computation in an appendix. So the thesis apparently contains an explicit counterexample of its own, which we have not seen and which may be as small as ours. We could not read the thesis (the DiVA site refused connections from our machine). A reader with access should check the theorem or example number, the example with the weights w_1, w_2, w_3 and whether it is symmetric or 3×3 , whether the thesis names Brändén's conjectures, and whether it contains anything on Conjecture 5 or on the flow Ψ_s . Proposition 3.1 is our reconstruction of the route described in the abstract, credited to Wanless [Wan99] and to Marcus and Lopes [ML57]. Our example does not arise in this way: it satisfies all Holens–Đoković inequalities, and the Tverberg–Friedland and Gurvits coefficient bounds (Remark 3.2).

What we claim is the refutation of Conjecture 5 (Theorem 1.2), for which no printed answer was found in the sources searched (Section 6), together with the explicit symmetric counterexample to Conjecture 2 on which it rests, for which we claim no priority; and a proof of Lemma 3 of the source for $\exp(s(T - I))$ (Proposition 5.1), which is not needed for Theorems 1.1 and 1.2. The unread thesis might contain our example or treat Conjecture 5. We make no claim about Conjecture 5 for all pairs $0 < r < s$ (some pairs on the orbit of P satisfy it, Remark 4.4), nor about which (m, d) admit counterexamples.

2 Coefficients of mixed characteristic polynomials

For $S \subseteq \{1, \dots, m\}$ let $\partial^S = \prod_{i \in S} \partial_i$.

Lemma 2.1. *Let $Q \in \mathbb{R}[x_1, \dots, x_m]$ be homogeneous of degree d . Then*

$$\chi_Q(t) = \sum_{k=0}^d (-1)^k E_k(Q) t^{d-k}, \quad E_k(Q) := \sum_{|S|=k} \partial^S Q(\mathbf{1}).$$

If Q is doubly stochastic, then $E_0(Q) = Q(\mathbf{1}) = 1$ and $E_1(Q) = d$; so χ_Q is monic and the sum of its zeros is d .

Proof. Expanding the product gives $\prod_i (1 - \partial_i) = \sum_S (-1)^{|S|} \partial^S$. The polynomial $\partial^S Q$ is homogeneous of degree $d - |S|$ (or zero), so $\partial^S Q(t\mathbf{1}) = t^{d-|S|} \partial^S Q(\mathbf{1})$. If Q is doubly stochastic, $E_1(Q) = \sum_i \partial_i Q(\mathbf{1}) = d$, and Euler's identity $\sum_i x_i \partial_i Q = dQ$ at $x = \mathbf{1}$ gives $dQ(\mathbf{1}) = d$. $\square$

So, for doubly stochastic Q, Q' of degree d with real-rooted χ , the equality in $\lambda(Q) \preceq \lambda(Q')$ holds automatically; for $d = 3$ the condition $\lambda(Q) \preceq \lambda(Q')$ consists of $\lambda_1(Q) \leq \lambda_1(Q')$ and $\lambda_3(Q) \geq \lambda_3(Q')$, the latter being equivalent to $\lambda_1(Q) + \lambda_2(Q) \leq \lambda_1(Q') + \lambda_2(Q')$.

For an $n \times n$ matrix M let $\sigma_k(M)$ be the sum of the permanents of all $k \times k$ submatrices of M ($\sigma_0 = 1$).

Lemma 2.2. *Let $M = (m_{ij})$ be a nonnegative doubly stochastic $n \times n$ matrix. Then $P_M \in \mathcal{D}_{n,n}$ and $E_k(P_M) = \sigma_k(M)$, so $\chi_{P_M}(t) = \sum_k (-1)^k \sigma_k(M) t^{n-k}$. Moreover $P_{J_n/n} = P_\infty$ for $m = d = n$, $\sigma_k(J_n/n) = \binom{n}{k}^2 k! n^{-k}$, and*

$$\chi_{P_\infty}(t) = \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \frac{k!}{n^k} t^{n-k} = (-1)^n \frac{n!}{n^n} L_n(nt), \quad L_n(y) = \sum_{j=0}^n \binom{n}{j} \frac{(-y)^j}{j!},$$

where L_n is the Laguerre polynomial.

Proof. Let $L_i = \sum_j m_{ij} x_j$. If $\text{Im } x_j > 0$ for all j , then $\text{Im } L_i > 0$, since row i is nonnegative and nonzero; so $P_M = \prod_i L_i \neq 0$, and P_M is stable. By the Leibniz rule, $\partial^S \prod_i L_i$ is the sum, over all injective maps $\varphi : S \rightarrow \{1, \dots, n\}$, of $\prod_{j \in S} m_{\varphi(j)j} \prod_{i \notin \varphi(S)} L_i$. Since $L_i(\mathbf{1}) = 1$ (row sums), $\partial^S P_M(\mathbf{1}) = \sum_{|R|=|S|} \text{per } M[R, S]$, and summing over $|S| = k$ gives $E_k(P_M) = \sigma_k(M)$. In particular $\partial_j P_M(\mathbf{1}) = \sum_i m_{ij} = 1$ (column sums), so P_M is doubly stochastic. For $M = J_n/n$ every $k \times k$ submatrix has permanent $k!/n^k$, and there are $\binom{n}{k}^2$ of them. Finally, putting $j = n - k$, $(-1)^n n! n^{-n} \binom{n}{j} (-nt)^j / j! = (-1)^k \binom{n}{k}^2 k! n^{-k} t^{n-k}$, because $n!/(n-k)! = \binom{n}{k} k!$. $\square$

The last formula is the reason why Laguerre polynomials appear in [Zac17]. For $n = 3$ it gives $\chi_{P_\infty} = t^3 - 3t^2 + 2t - \frac{2}{9}$.

Remark 2.3 (multiaffine polynomials). If Q is multiaffine and homogeneous of degree d with $Q(\mathbf{1}) = 1$, then $\chi_Q = (t-1)^d$, since $\prod_i (1 - \partial_i)$ maps each monomial $\prod_{i \in S} x_i$, $|S| = d$, to $\prod_{i \in S} (x_i - 1)$. Then $\lambda(Q) = (1, \dots, 1)$ is majorized by every non-increasing vector with sum d . So under a multiaffine reading of $\mathcal{D}_{m,d}$ Conjecture 2 would be trivial, and P_∞ would not belong to the class; the van der Waerden application of the source also needs the products $P_M \in \mathcal{D}_{m,m}$, which are not multiaffine. This supports the literal reading, which we follow.

3 Conjecture 2

Proof of Theorem 1.1. Since $P = P_{A(1)}$ with the nonnegative doubly stochastic matrix $A(1) = \frac{1}{2}(J - I)$, Lemma 2.2 gives $P \in \mathcal{D}_{3,3}$. P is symmetric, because a permutation of the variables permutes the three factors. The 2×2 submatrices of $J - I$ with equal row and column sets are $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the others are of the form $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ up to the order of rows and columns; all nine have permanent 1, and $\text{per}(J - I) = 2$. So $\sigma_1(A(1)) = 3$, $\sigma_2(A(1)) = \frac{9}{4}$, $\sigma_3(A(1)) = \frac{1}{4}$, and by Lemma 2.2

$$\chi_P(t) = t^3 - 3t^2 + \frac{9}{4}t - \frac{1}{4} = \frac{1}{4}(t-1)(4t^2 - 8t + 1),$$

whose zeros are 1 and $1 \pm \frac{\sqrt{3}}{2}$. The formula for χ_{P_∞} is Lemma 2.2 with $n = 3$. Put $y = t - 1$ and $g(y) := \chi_{P_\infty}(1 + y) = y^3 - y - \frac{2}{9}$. Then $g(-1) = -\frac{2}{9} < 0$, $g(-\frac{1}{2}) = \frac{11}{72} > 0$, $g(\frac{1}{2}) < 0$ and $g(2) > 0$, so χ_{P_∞} has three real zeros $\mu_1 > \mu_2 > \mu_3$, and $\mu_3 - 1 < -\frac{1}{2}$. Since $g'(y) = 3y^2 - 1 > 0$ for $y^2 > \frac{1}{3}$, g is increasing on $(-\infty, -\frac{1}{\sqrt{3}}]$, which contains -1 and $-\frac{\sqrt{3}}{2}$, and

$$g(-\frac{\sqrt{3}}{2}) = -\frac{3\sqrt{3}}{8} + \frac{\sqrt{3}}{2} - \frac{2}{9} = \frac{\sqrt{3}}{8} - \frac{2}{9} < 0,$$

because $(9\sqrt{3})^2 = 243 < 256 = 16^2$. Hence $g < 0$ on $(-\infty, -\frac{\sqrt{3}}{2}]$, and $\mu_3 - 1 > -\frac{\sqrt{3}}{2}$, that is, $\lambda_3(P_\infty) = \mu_3 > 1 - \frac{\sqrt{3}}{2} = \lambda_3(P)$. Both zero vectors have the sum 3 (Lemma 2.1), so $\lambda_1(P) + \lambda_2(P) = 3 - \lambda_3(P) > 3 - \mu_3 = \mu_1 + \mu_2$. $\square$

The next proposition explains how counterexamples of another kind arise. We take the Holens–Doković conjecture in the form stated in the zbMATH review of [Wan99] (Zbl 0941.15003, where a factor n is missing from the displayed inequality): for every doubly stochastic $n \times n$ matrix M and $1 \leq k \leq n$,

$$k n \sigma_k(M) \geq (n - k + 1)^2 \sigma_{k-1}(M). \quad (1)$$

Equality holds for $M = J_n/n$ by Lemma 2.2. Wanless showed that the conjecture fails [Wan99].

Proposition 3.1. *Let M be a nonnegative doubly stochastic $n \times n$ matrix with $\lambda(P_M) \preceq \lambda(P_\infty)$, where $m = d = n$. Then M satisfies (1) for $1 \leq k \leq n$. Consequently every counterexample of Wanless [Wan99] to the Holens–Doković conjecture gives a polynomial $P_M \in \mathcal{D}_{n,n}$ for which Conjecture 2 fails.*

Proof. The zeros of χ_{P_M} are real (Lemma 2.2 and [Brä17]), the coefficients $(-1)^k \sigma_k(M)$ alternate in sign, and the constant term is $\pm \text{per } M \neq 0$, since the permanent of a doubly stochastic matrix is at least $n!/n^n$ [Gur08]; so the zeros are positive, and $e_k(\lambda(P_M)) = \sigma_k(M) > 0$ for the elementary symmetric functions e_k . The function e_k/e_{k-1} is concave on the positive orthant (Marcus and Lopes [ML57]) and symmetric, hence Schur-concave [MOA11, Chapter 3]: $\alpha \preceq \beta$ implies $(e_k/e_{k-1})(\alpha) \geq (e_k/e_{k-1})(\beta)$. With $\alpha = \lambda(P_M)$ and $\beta = \lambda(P_\infty)$, Lemma 2.2 gives

$$\frac{\sigma_k(M)}{\sigma_{k-1}(M)} \geq \frac{\sigma_k(J_n/n)}{\sigma_{k-1}(J_n/n)} = \left(\frac{\binom{n}{k}}{\binom{n}{k-1}} \right)^2 \frac{k}{n} = \frac{(n - k + 1)^2}{kn}. \quad \square$$

We have not read [Zac17]; its abstract suggests that it argues in this way, but Proposition 3.1 is our reconstruction.

Remark 3.2 (the example is not of this kind). For $M = A(1)$ we have $(\sigma_0, \dots, \sigma_3) = (1, 3, \frac{9}{4}, \frac{1}{4})$, and the three inequalities (1) read $9 \geq 9$, $\frac{27}{2} \geq 12$ and $\frac{9}{4} \geq \frac{9}{4}$; they hold. So Theorem 1.1 does not come from a counterexample to the Holens–Đoković conjecture. The example also satisfies other coefficient inequalities that Conjecture 2 would imply through Schur-concave functions: $\sigma_k(A(1)) \geq \sigma_k(J/3)$, that is $3 \geq 3$, $\frac{9}{4} \geq 2$, $\frac{1}{4} \geq \frac{2}{9}$ (Tverberg’s conjecture, proved by Friedland [Fri82]), and in particular per $A(1) = \frac{1}{4} \geq \frac{3!}{3^3}$, the van der Waerden type bound of Gurvits [Gur08] quoted in [Brä17]. So majorization of the zeros fails here although all these coefficient inequalities hold.

Remark 3.3 (a family; computation only). For $n \geq 3$ let C_n be the permutation matrix of the cyclic shift, $A_n = \frac{1}{2}(I + C_n)$ and $P_n = P_{A_n} = \prod_{i=1}^n \frac{1}{2}(x_i + x_{i+1})$ (indices modulo n); then $P_3 = P$. The support of A_n is the bipartite graph of a cycle of length $2n$, so $2^k \sigma_k(A_n)$ is the number $\frac{2n}{2n-k} \binom{2n-k}{k}$ of k -matchings of that cycle. An exact computation (Sturm sequences and rational isolating intervals, Section 6) shows that χ_{P_n} and χ_{P_∞} have n distinct real zeros and that $\lambda_n(P_n) < \lambda_n(P_\infty)$ for every $3 \leq n \leq 12$. So Conjecture 2 fails for P_n , at $k = n - 1$, for these n . Numerically $\lambda_n(P_n)$ agrees with $1 - \cos \frac{\pi}{2n}$; we have proved neither this nor the failure for $n > 12$.

4 Conjecture 5

4.1 The operator T on symmetric cubics

Let $m = d = 3$, so $T = \frac{1}{9}e_1D$ with $D = \partial_1 + \partial_2 + \partial_3$, and let e_1, e_2, e_3 be the elementary symmetric polynomials in x_1, x_2, x_3 . D is a derivation with $De_1 = 3$, $De_2 = 2e_1$ and $De_3 = e_2$. Put

$$B_0 := 3x_1x_2x_3 - (x_1^3 + x_2^3 + x_3^3) = 3e_1e_2 - e_1^3, \quad C_0 := \prod_{i=1}^3 (3x_i - e_1) = 2e_1^3 - 9e_1e_2 + 27e_3.$$

Lemma 4.1. $TP_\infty = P_\infty$, $TB_0 = \frac{1}{3}B_0$, $TC_0 = 0$, and $P = P_\infty + \frac{1}{36}B_0 - \frac{1}{216}C_0$. Moreover $\chi_{B_0} = 9t - 3$ and $\chi_{C_0} = -12$.

Proof. $T(e_1^3) = \frac{1}{9}e_1 \cdot 9e_1^2 = e_1^3$. $DB_0 = 3(3e_2 + 2e_1^2) - 9e_1^2 = 9e_2 - 3e_1^2$, so $TB_0 = e_1e_2 - \frac{1}{3}e_1^3 = \frac{1}{3}B_0$. Each factor $3x_i - e_1$ of C_0 is killed by D , so $DC_0 = 0$. Since $P = \frac{1}{8}(e_1e_2 - e_3)$, the decomposition follows from $\frac{1}{27}e_1^3 + \frac{1}{36}(3e_1e_2 - e_1^3) - \frac{1}{216}(2e_1^3 - 9e_1e_2 + 27e_3) = \frac{1}{8}e_1e_2 - \frac{1}{8}e_3$. For χ we use Lemma 2.1. $B_0(\mathbf{1}) = 0$ and $DB_0(\mathbf{1}) = 0$; the mixed second derivatives of $x_1^3 + x_2^3 + x_3^3$ vanish and those of $3x_1x_2x_3$ are $3x_k$, so $E_2(B_0) = 9$, and $E_3(B_0) = \partial_1\partial_2\partial_3B_0 = 3$. For C_0 , every term of the Leibniz expansion of $\partial^S C_0$ with $|S| \leq 2$ contains an undifferentiated factor $3x_i - e_1$, which vanishes at $\mathbf{1}$; and $E_3(C_0) = \text{per}(3I - J) = 8 + 3 \cdot 2 - 2 = 12$. $\square$

Lemma 4.2. For every $s \in \mathbb{R}$, with $q = e^{-s/3}$,

$$\exp(s(T - I))P = P_\infty + \frac{q^2}{36}B_0 - \frac{q^3}{216}C_0 = P_{A(q)}, \quad \chi_{P_{A(q)}} = \chi_q.$$

Proof. By Lemma 4.1, P_∞, B_0, C_0 are eigenvectors of $T - I$ for the eigenvalues $0, -\frac{2}{3}, -1$, which gives the first equality. For a matrix $\alpha I + \beta(J - I)$ we have $(\alpha I + \beta(J - I))x = \beta e_1 \mathbf{1} + \gamma x$ with $\gamma = \alpha - \beta$, so its product polynomial is $\prod_i (\beta e_1 + \gamma x_i) = \beta^2 \alpha e_1^3 + \beta \gamma^2 e_1 e_2 + \gamma^3 e_3$. For $A(q)$, $\alpha = \frac{1-q}{3}$, $\beta = \frac{2+q}{6}$ and $\gamma = -\frac{q}{2}$, and the coefficients are $\frac{(1-q)(2+q)^2}{108} = \frac{1}{27} - \frac{q^2}{36} - \frac{q^3}{108}$, $\frac{q^2(2+q)}{24}$ and $-\frac{q^3}{8}$. By Lemma 4.1 these are also the coefficients of $P_\infty + \frac{q^2}{36}B_0 - \frac{q^3}{216}C_0$ in the basis e_1^3, e_1e_2, e_3 . Finally $Q \mapsto \chi_Q$ is linear, so $\chi_{P_{A(q)}} = \chi_{P_\infty} + \frac{q^2}{36}(9t - 3) + \frac{q^3}{216} \cdot 12 = \chi_q$. $\square$

Equivalently, by Lemma 2.2, $\sigma_2(A(q)) = 2 + \frac{q^2}{4}$ and $\text{per } A(q) = \frac{2}{9} + \frac{q^2}{12} - \frac{q^3}{18}$. In the variable $y = t - 1$,

$$\chi_q(1+y) = y^3 + \left(\frac{q^2}{4} - 1\right)y + \frac{(q-1)(q+2)^2}{18}, \quad \frac{d}{dy}\chi_q(1+y) = 3y^2 + \frac{q^2}{4} - 1. \quad (2)$$

4.2 Proof of Theorem 1.2(a)–(c)

(a) is Lemma 4.2 together with Lemma 2.2: for $0 \leq q \leq 1$ the entries $\frac{1-q}{3}$ and $\frac{2+q}{6}$ of $A(q)$ are nonnegative and its line sums are $\frac{1-q}{3} + \frac{2+q}{3} = 1$; $P_{A(q)} = \prod_i(\beta e_1 + \gamma x_i)$ is symmetric.

(b) The formula for χ_q is Lemma 4.2. For a cubic $y^3 + py + r$ the discriminant is $-4p^3 - 27r^2$. With (2),

$$\Delta(q) = \frac{(4-q^2)^3}{16} - \frac{(q-1)^2(q+2)^4}{12} = \frac{(2+q)^3}{48}(3(2-q)^3 - 4(q-1)^2(q+2)) = \frac{(2+q)^3}{48}h(q),$$

where $h(q) = 16 - 24q + 18q^2 - 7q^3$. Since $h'(q) = -3(7q^2 - 12q + 8) < 0$ (the discriminant of $7q^2 - 12q + 8$ is $144 - 224 < 0$), h is decreasing, and $h(1) = 3$; so $h > 0$ on $(-\infty, 1]$ and $\Delta > 0$ on $(-2, 1]$. (Expanded, $\Delta(q) = \frac{8}{3} - q^2 + \frac{2}{3}q^3 - \frac{1}{2}q^5 - \frac{7}{48}q^6$.) Thus χ_q has three distinct real zeros for $-2 < q \leq 1$.

(c) Let $s > 0$, so that $0 < q < 1$, and let $a = 1 - \frac{\sqrt{3}}{2} = \lambda_3(P)$. By (2) with $y = -\frac{\sqrt{3}}{2}$, where $y^3 + (\frac{q^2}{4} - 1)y = -\frac{\sqrt{3}}{8}(q^2 - 1)$,

$$\chi_q(a) = (q-1) \left[\frac{(q+2)^2}{18} - \frac{\sqrt{3}}{8}(q+1) \right] = -\frac{1-q}{72} [4q^2 + (16 - 9\sqrt{3})(1+q)] < 0,$$

since $16 > 9\sqrt{3}$. For $t \leq a$ we have $(t-1)^2 \geq \frac{3}{4} > \frac{1}{3}$, so χ_q is increasing on $(-\infty, a]$ by (2). Hence $\chi_q < 0$ on $(-\infty, a]$, all zeros of χ_q exceed a , and $\lambda_3(\Psi_s P) > \lambda_3(P)$. Both zero vectors have the sum 3, so $\lambda_1(\Psi_s P) + \lambda_2(\Psi_s P) < \lambda_1(P) + \lambda_2(P)$, which is the failure of the inequality for $k = 2$ in $\lambda(\Psi_0 P) \leq \lambda(\Psi_s P)$. $\square$

For $q = 0$ ($s \rightarrow \infty$) this computation is the proof of Theorem 1.1. In general, for the convergent flow, Conjecture 5 with $r = 0$ implies Conjecture 2 for symmetric P (zeros depend continuously on the coefficients, and majorization is a closed condition), so every symmetric counterexample to Conjecture 2 refutes Conjecture 5 for large s ; Theorem 1.2 gives every $s > 0$.

4.3 Further properties of the orbit

Remark 4.3 (where Conjecture 2 fails on the orbit). The matrices $A(q)$, $0 \leq q \leq 1$, form the family $\alpha I + \frac{1-\alpha}{2}(J - I)$, $0 \leq \alpha = \frac{1-q}{3} \leq \frac{1}{3}$. Let $\mu_1 > \mu_2 > \mu_3$ be the zeros of $\chi_{P_\infty} = \chi_0$ and $q^* = \frac{3-9\mu_3}{2} \approx 0.876338$. Then $\lambda(P_{A(q)}) \leq \lambda(P_\infty)$ fails exactly for $q^* < q \leq 1$, that is, for $0 \leq \alpha < \alpha^* = \frac{9\mu_3-1}{6} \approx 0.0412206$, or $0 \leq s < 3\ln(1/q^*) \approx 0.3960$. Indeed $\chi_q - \chi_0 = q^2(\frac{t}{4} - \frac{1}{12}) + \frac{q^3}{18}$, so $\chi_q(\mu_j) = \frac{q^2}{18}(q - \frac{3-9\mu_j}{2})$. By the proof of Theorem 1.1 and $g(-\frac{1}{\sqrt{3}}) = \frac{2\sqrt{3}-2}{9} > 0$, $g(\frac{1}{\sqrt{3}}) < 0$, we have $\mu_3 < 1 - \frac{1}{\sqrt{3}}$ and $\mu_1 > 1 + \frac{1}{\sqrt{3}}$. By (2), χ_q is non-decreasing on $(-\infty, 1 - \frac{1}{\sqrt{3}}]$ and on $[1 + \frac{1}{\sqrt{3}}, \infty)$, and $\chi_q(1 - \frac{1}{\sqrt{3}}) = \frac{8-3q^2}{12\sqrt{3}} + \frac{(q-1)(q+2)^2}{18} \geq \frac{5}{12\sqrt{3}} - \frac{2}{9} > 0$ for $0 \leq q \leq 1$ (as $45^2 > 3 \cdot 24^2$). So $\lambda_3(P_{A(q)})$ is the only zero below $1 - \frac{1}{\sqrt{3}}$, and it is smaller than μ_3 exactly when $\chi_q(\mu_3) > 0$, that is, $q > q^*$. Moreover $\chi_q(\mu_1) > 0$ for $q > 0$, so $\lambda_1(P_{A(q)}) < \mu_1$, and the inequality for $k = 1$ always holds on the orbit.

Remark 4.4 (pairs $0 < r < s$). For $0 < r < s$ the inequality for $k = 2$ in $\lambda(\Psi_r P) \leq \lambda(\Psi_s P)$ fails exactly when $\lambda_3(\Psi_r P) < \lambda_3(\Psi_s P)$. Rational isolating intervals give $\lambda_3(q = \frac{99}{100}) \leq 0.1344152245 < 0.1426813715 \leq \lambda_3(q = \frac{1}{2})$, so Conjecture 5 also fails for $r = 3\ln \frac{100}{99}$, $s = 3\ln 2$. Numerically, λ_3

increases from $\lambda_3(P) \approx 0.13397$ as q decreases from 1 to $\frac{4}{7}$, where $\lambda_3 = \frac{1}{7}$, and then decreases to $\mu_3 \approx 0.13859$. (At a critical point of λ_3 we have $3\lambda_3 = 1 - q$ by the formula for $\frac{\partial}{\partial q}\chi_q$ in Remark 4.5, and $\chi_q(\frac{1-q}{3}) = -\frac{(q+2)^2(7q-4)}{108}$ vanishes for $0 < q \leq 1$ only at $q = \frac{4}{7}$; indeed $\chi_{4/7}(t) = \frac{1}{343}(7t-1)(49t^2 - 140t + 82)$.) So some pairs satisfy the inequality: $\lambda_3(q = \frac{1}{2}) > 0.1426 > 0.1390 > \lambda_3(q = \frac{1}{10})$ (isolating intervals), and with Remark 4.5 $\lambda(\Psi_r P) \preceq \lambda(\Psi_s P)$ for $r = 3 \ln 2$, $s = 3 \ln 10$. We make no claim about the set of all failing pairs.

Remark 4.5 (Theorem 4 of the source). Along the convergent flow $s \mapsto \lambda_1(\Psi_s P)$ is strictly increasing, from $\lambda_1(P) = 1 + \frac{\sqrt{3}}{2} \approx 1.86603$ to $\mu_1 \approx 2.09665$ (a grid computation shows the same). Indeed $\frac{\partial}{\partial q}\chi_q(t) = \frac{q}{6}(3t - 1 + q)$, which is positive at $t = \lambda_1(q)$ for $0 < q \leq 1$, because $\chi_q(1) = \frac{(q-1)(q+2)^2}{18} \leq 0$ forces $\lambda_1(q) \geq 1$; and $\frac{\partial}{\partial t}\chi_q > 0$ at the simple largest zero. By the implicit function theorem $\lambda_1(q)$ is decreasing in $q = e^{-s/3}$. So, for the convergent flow, “monotone decreasing” in [Brä17, Theorem 4] has to mean “non-decreasing”, in accordance with its stated consequence $\lambda_1(P) \leq \lambda_1(P_\infty)$.

4.4 The printed sign

Proof of Theorem 1.2(d). Since $\exp(s(I - T)) = \exp((-s)(T - I))$, Lemma 4.2 gives $\Psi_s P = P_{A(u)}$ and $\chi_{\Psi_s P} = \chi_u$ with $u = e^{s/3} > 1$; the diagonal entry $\frac{1-u}{3}$ of $A(u)$ is negative. A stable form Q of degree d has no zero x with $x_1, \dots, x_m > 0$, since ix has positive imaginary parts and $Q(ix) = i^d Q(x)$. The first factor $\frac{1-u}{3}x_1 + \frac{2+u}{6}(x_2 + x_3)$ of $P_{A(u)}$ vanishes at $x = (\frac{2+u}{u-1}, 1, 1)$; so $\Psi_s P$ is not stable for any $s > 0$, contrary to Lemma 3 as printed. (Its coefficients are also unbounded as $s \rightarrow \infty$.) By (b), $\Delta(u) = \frac{1}{48}(2+u)^3 h(u)$ with h strictly decreasing, and $h(\frac{6}{5}) = \frac{128}{125} > 0 > -\frac{25}{8} = h(\frac{3}{2})$; so h has a unique real zero $u_0 \in (\frac{6}{5}, \frac{3}{2})$, $u_0 \approx 1.2873544$. For $u > u_0$, $\Delta(u) < 0$, and χ_u has two non-real zeros. (This gives a second proof of instability for $s > s_0 = 3 \ln u_0$: the polynomial $\Psi_s P$ is doubly stochastic for every s , since the line sums of $A(u)$ are 1 and the computation of $\partial_j P_M(\mathbf{1})$ in Lemma 2.2 does not use nonnegativity; if it were stable, it would lie in $\mathcal{D}_{3,3}$ and $\chi_{\Psi_s P}$ would be real-rooted [Brä17].) For $1 < u \leq u_0$ all zeros of χ_u are real. Let $b = 1 + \frac{\sqrt{3}}{2} = \lambda_1(P)$. By (2) with $y = \frac{\sqrt{3}}{2}$,

$$\chi_u(b) = (u-1) \left[\frac{(u+2)^2}{18} + \frac{\sqrt{3}}{8}(u+1) \right] = (u^2-1) \left(\frac{1}{6} + \frac{\sqrt{3}}{8} \right) + \frac{u^3-1}{18} > 0,$$

and χ_u is increasing on $[b, \infty)$, since $(t-1)^2 \geq \frac{3}{4}$ there. So every real zero of χ_u is smaller than b , and $\lambda_1(\Psi_s P) < \lambda_1(P) = \lambda_1(\Psi_0 P)$: the inequality for $k = 1$ in $\lambda(\Psi_0 P) \preceq \lambda(\Psi_s P)$ fails. $\square$

5 Lemma 3 of the source for the convergent flow

This section is not needed for Theorems 1.1 and 1.2. It justifies the choice of sign made in the introduction. Let V_k be the space of real forms of degree k in $x_1, \dots, x_m$, $D = \sum_i \partial_i$, $e_1 = \sum_i x_i$, and $T = \frac{1}{md} e_1 D$ on V_d .

Proposition 5.1. *Let $m, d \geq 1$.*

- (a) *Every eigenvalue of T on V_d lies in $\{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\}$, and 1 is an algebraically simple eigenvalue with eigenvector P_∞ .*
- (b) *For $Q \in V_d$, $(TQ)(\mathbf{1}) = Q(\mathbf{1})$, and the vector $\nabla Q(\mathbf{1}) = (\partial_i Q(\mathbf{1}))_i$ satisfies $\nabla(TQ)(\mathbf{1}) = K \nabla Q(\mathbf{1})$ with $K = \frac{1}{md}(J + m(d-1)I)$. Hence $\exp(s(T - I))$ preserves the value at $\mathbf{1}$ and maps doubly stochastic forms to doubly stochastic forms.*
- (c) *For $s \geq 0$, $\exp(s(T - I))$ maps every stable form Q with $Q(\mathbf{1}) > 0$ to a stable form.*

(d) $\exp(s(T - I))Q \rightarrow Q(\mathbf{1})P_\infty$ as $s \rightarrow \infty$, coefficientwise and hence uniformly on compact sets.

In particular Lemma 3 of [Brä17] holds for $\Psi_s = \exp(s(T - I))$.

Proof. (a) Write $T = \frac{1}{md}MD$, where $D : V_k \rightarrow V_{k-1}$ and $M : V_{k-1} \rightarrow V_k$ is multiplication by e_1 . Since D is a derivation and $De_1 = m$, we have $DM = mI + MD$ on V_{k-1} . For linear maps $X : U \rightarrow W$ and $Y : W \rightarrow U$, the maps XY and YX have the same nonzero eigenvalues, with the same algebraic multiplicities. We show by induction on k that the eigenvalues of MD on V_k lie in $\{0, m, \dots, km\}$ and that km is simple. For $k = 0$, $MD = 0$ on the one-dimensional space V_0 . If the claim holds for $k - 1$, then $DM = mI + MD$ has its eigenvalues on V_{k-1} in $\{m, \dots, km\}$, with km simple, and so do the nonzero eigenvalues of MD on V_k . Moreover $MDe_1^k = km e_1^k$. For $k = d$ divide by md .

(b) By Euler's identity $DQ(\mathbf{1}) = dQ(\mathbf{1})$, so $(TQ)(\mathbf{1}) = \frac{m}{md}DQ(\mathbf{1}) = Q(\mathbf{1})$. Next $\partial_i(e_1 DQ) = DQ + e_1 \partial_i DQ$, and Euler's identity for the form $\partial_i Q$ of degree $d - 1$ gives $D\partial_i Q(\mathbf{1}) = (d - 1)\partial_i Q(\mathbf{1})$; so $\partial_i(TQ)(\mathbf{1}) = \frac{1}{md}(\sum_j \partial_j Q(\mathbf{1}) + m(d - 1)\partial_i Q(\mathbf{1}))$. Hence $\nabla \exp(s(T - I))Q(\mathbf{1}) = \exp(s(K - I))\nabla Q(\mathbf{1})$, and $K \frac{d}{m} \mathbf{1} = \frac{d}{m} \mathbf{1}$. Likewise the value at $\mathbf{1}$ is preserved.

(c) First let $c > 0$ and let Q be a stable form with $Q(\mathbf{1}) > 0$; we show that $Q + ce_1 DQ$ is stable, by a standard argument going back to Lieb and Sokal [LS81]. Let $H = \{z : \text{Im } z > 0\}$ and $F(x, y) = Q(x + y\mathbf{1})$. Fix $x \in H^m$; then $f(y) := F(x, y)$ is a polynomial in y that does not vanish on H , and $f'(y) = (DQ)(x + y\mathbf{1})$. Let $y, w \in H$. If $f' \equiv 0$ then $f(y) + wf'(y) = f(y) \neq 0$. Otherwise $\text{Im} \frac{f'(y)}{f(y)} = \sum_\zeta \text{Im} \frac{1}{y - \zeta} < 0$, the sum running over the zeros ζ of f , which satisfy $\text{Im } \zeta \leq 0$; while $\text{Im}(-1/w) > 0$. So $f(y) + wf'(y) \neq 0$ in all cases. Taking $w = ce_1(x) \in H$ we get $G(x, y) := Q(x + y\mathbf{1}) + ce_1(x)(DQ)(x + y\mathbf{1}) \neq 0$ on $H^m \times H$. As $\varepsilon \downarrow 0$, the polynomials $G(\cdot, i\varepsilon)$, which do not vanish on H^m , converge coefficientwise to $Q + ce_1 DQ$; by Hurwitz's theorem the limit does not vanish on H^m or is identically zero, and it is not zero, because its value at $\mathbf{1}$ is $Q(\mathbf{1})(1 + cmd) > 0$. So $Q + ce_1 DQ$ is stable.

Now, for $0 < h < 1$, $(I + h(T - I))Q = (1 - h)(Q + \frac{h}{(1-h)md}e_1 DQ)$ is stable, with the same value $Q(\mathbf{1})$ at $\mathbf{1}$ by (b). Iterating, the forms $(I + \frac{s}{n}(T - I))^n Q$ are stable for $n > s$, with value $Q(\mathbf{1})$ at $\mathbf{1}$. On the finite-dimensional space V_d they converge to $\exp(s(T - I))Q$ as $n \rightarrow \infty$; by Hurwitz's theorem again, the limit is stable, since its value at $\mathbf{1}$ is $Q(\mathbf{1}) > 0$.

(d) By (a), every generalized eigenspace of $T - I$ other than the line $\mathbb{R}P_\infty$ belongs to an eigenvalue $\mu - 1 < 0$, on which $\exp(s(T - I))$ is $e^{s(\mu - 1)}$ times a polynomial in s . So $\exp(s(T - I))Q$ converges to the component $c(Q)P_\infty$ of Q in $\mathbb{R}P_\infty$. By (b) the value at $\mathbf{1}$ is constant along the flow, and $P_\infty(\mathbf{1}) = 1$, so $c(Q) = Q(\mathbf{1})$.

For $Q \in \mathcal{D}_{m,d}$ we have $Q(\mathbf{1}) = 1$ (Lemma 2.1); by (b), (c) and (d), $\exp(s(T - I))Q \in \mathcal{D}_{m,d}$ for $s \geq 0$ and $\exp(s(T - I))Q \rightarrow P_\infty$. $\square$

6 Computations, verification and scope

All theorems, propositions and lemmas of this note are proved without computation; only the numerical values quoted, the interval certificates in Remark 4.4 and Remark 3.3 rest on computations. The reproducibility package contains the exact program of this note, `check_note.py` (Python standard library, arithmetic in $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{3})$, about one second, 95 checks, all passed). It checks the finite statements of Sections 2–4: χ_P and χ_{P_∞} from the operator definition and from permanent minors, the sign certificates and isolating intervals, Lemmas 4.1 and 4.2 (identities of degree at most 3 in q , checked at ten values), the spectrum of T on all ten cubic forms, Proposition 5.1(a) for eleven small pairs (m, d) and (b), (d) for three of them, the discriminant and u_0 , the values $\chi_q(1 \pm \frac{\sqrt{3}}{2})$, the instances quoted, Remarks 2.3, 3.2 and 4.3, the Laguerre identity for $n \leq 8$, and Remark 3.3 for $3 \leq n \leq 12$. The package also contains the programs with which the results were first obtained

(`original/`, `sympy` and `mpmath`) and those of the three verification runs (`verification_run_A/`, `verification_run_B/` and `independent_run_2/`), with their recorded outputs; all were run again from the package with identical outputs (the recorded output of `original/stress.py` has one extra final line, `exit 0`, written by the shell).

Verification. The results were first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first run read pages 825–826 of the report (text extracted from the publisher’s PDF), re-derived all proofs line by line, and recomputed χ_P and χ_{P_∞} in two ways (from the definition and from the determinantal formula of [MSS15]), the spectrum of T on all cubic forms, the flow at $s = 3 \ln 2$ and the root isolations. The second run recomputed everything with its own exact programs, including the exponential of T on all cubic forms for both signs and the sign certificates in $\mathbb{Q}(\sqrt{3})$; it found the product form of the orbit (Lemma 4.2), the pair of Remark 4.4, the family of Remark 3.3, the Laguerre identity and Remark 2.3, and it checked the literature. Neither run found a mathematical error. Their required corrections concerned the literature and the wording and were made: the status of Conjecture 2 is stated as reported in [Zac17], the description of [GL21] was corrected, and the citation of the report was made consistent. The factorised discriminant, the complete proofs in Remarks 4.3 and 4.5 and the proof of the Laguerre identity for all n were added when the note was written.

The third run examined the final text. It read pages 825–826 of the report again in the publisher’s PDF and compared every definition and statement quoted in Section 1 with them; re-derived all proofs line by line, including those added when the note was written; and wrote its own exact program (`sympy`), which computes χ from the operator definition, the matrix of T on all ten cubic forms and $\exp(\pm s(T - I))P$ symbolically in s , the discriminant, the $\mathbb{Q}(\sqrt{3})$ identities, u_0 , q^* , α^* , the isolating intervals of Remark 4.4, the cycle family of Remark 3.3 for $3 \leq n \leq 12$ and the Laguerre identity for $n \leq 8$ (101 checks, none failed), with numerical tests of Proposition 3.1 and Proposition 5.1(c). It re-ran the whole package from the archive, re-read Theorem 7.1 of [GL21], checked all DOIs with Crossref and repeated the literature search. It found no mathematical error. Its required corrections were made: the position of the maximum of λ_3 in Remark 4.4 (previously given numerically as about 0.55) is now $q = \frac{4}{7}$; Theorem 1.2(d) now states that $\Psi_s P$ is not stable for any $s > 0$ under the printed sign; the novelty claim for the counterexample to Conjecture 2 was withdrawn after a new excerpt of [Zac17] (Section 1.3); and the description of the search, of `check_note.py` and of one reference were corrected.

Scope and priority. The note refutes Conjectures 2 and 5 of [Brä17] by one symmetric example in $\mathcal{D}_{3,3}$; it does not decide which (m, d) admit counterexamples, nor which pairs $0 < r < s$ violate Conjecture 5 on the orbit of P . The question, the definitions, Lemma 3, Theorem 4 and both conjectures are due to Brändén [Brä17], mixed characteristic polynomials and the bound on their largest zero to Marcus, Spielman and Srivastava [MSS15], the van der Waerden type bound to Gurvits [Gur08]; Proposition 3.1 rests on Wanless [Wan99] and Marcus–Lopes [ML57], Proposition 5.1(c) on an argument of Lieb and Sokal [LS81] and on Hurwitz’s theorem. Conjecture 2 was reportedly refuted in the KTH degree project of Samuel Zackrisson [Zac17] (DiVA diva2:1106332; full text not read; Section 1.3), apparently also with an explicit example of its own; we claim neither that refutation nor any priority for a counterexample to Conjecture 2, and the thesis might contain our example or treat Conjecture 5.

No printed answer to Conjecture 5 was found in the sources searched on 20 October 2026, and no example like ours outside the unread thesis: the arXiv API (mixed characteristic polynomials; majorization with characteristic, stable, hyperbolic or real-rooted polynomials and with permanents; doubly stochastic stable polynomials; the Holens–Đoković conjecture; the papers of Brändén), zb-

MATH (Brändén’s papers since 2017, mixed characteristic polynomials, the review of [Wan99], the title of [Brä17], Zackrisson), Crossref, Semantic Scholar and web search. Semantic Scholar lists six works citing the whole report (2016–2021); by their titles none of them concerns mixed characteristic polynomials, and their texts were not read. OpenAlex refused the list of citing works (rate limit), and the DiVA site refused connections. A 2026 collection of counterexamples found with AI assistance (arXiv:2608.29595), returned by the web search, treats other problems of Borcea and Brändén and not these conjectures (full text searched). Brändén’s later paper [Brä18] contains no majorization statement (full text searched); [RL20] and the other works found bound only the largest zero or concern other majorization statements. In particular, Theorem 7.1 of [GL21], credited there to Brändén through personal correspondence, is a majorization statement of Schur–Horn type for hyperbolic polynomials: for p in their class $\text{HStab}_n(\mathbf{1})$, the roots of $t \mapsto p(t\mathbf{1} - x)$ are majorized by x . It is not a bound on a largest zero, does not compare χ_P with χ_{P_∞} , and settles neither conjecture. A search that finds nothing is not a proof of novelty. What was read: [Brä17], pages 825–826 in the publisher’s PDF; [GL21], arXiv v4, around Theorem 7.1 and the definition of HStab_n ; [Brä18], arXiv version, searched by keywords; of [Wan99], the bibliographic data and the zbMATH review; of [Zac17], the title page, the abstract and one further excerpt, all only as quoted in search-engine results. The theorems of [MSS15, Gur08, ML57, Fri82, LS81, MOA11] are quoted as standard and were not re-read for this note.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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