

On Meckes's Monotonicity Question for the Expected Volume of a Random Simplex: The Optimal Constant Grows Exponentially

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Abstract

Let V_K be the volume of the simplex spanned by $d + 1$ independent uniform random points in a convex body $K \subset \mathbb{R}^d$. Meckes asked, as recorded by Reitzner in Oberwolfach Report 10/2017, whether there is a constant $c(d)$ with $\lim_d c(d)^{1/d} < \infty$ such that $\mathbb{E}V_K \leq c(d)\mathbb{E}V_L$ whenever $K \subseteq L$, and the report asks for a lower bound for $c(d)$. The displayed definition of $c(d)$ as a minimum is degenerate if read literally; we read the second question as asking for lower bounds for the optimal constant $c^*(d) = \sup_{K \subseteq L} \mathbb{E}V_K / \mathbb{E}V_L$. The answer to the first question is yes, and it is in print: Rademacher proved that it is equivalent to the slicing problem (an equivalence also stated by Reichenwallner and Reitzner), which Klartag and Lehec solved. We add a short self-contained proof, based on Groemer's theorem, of the explicit bound $c^*(d) \leq \theta_d(L_d/L_{B^d})^d$ with $\theta_d \leq ((d+1)\binom{d+8}{8})^{1/2}$, where L_d is the largest isotropic constant in $\mathbb{R}^d$ and B^d is the unit ball; hence $\limsup c^*(d)^{1/d} \leq \sqrt{2\pi e} \sup_n L_n$. For the second question we prove that $c^*(d)$ grows exponentially: $\liminf_d c^*(d)^{1/d} \geq (300657950289351/292457950289351)^{1/2} = 1.01392\dots$, so that $c^*(d) = e^{\Theta(d)}$. Explicitly, $c^*(d) \geq ((d+1)\binom{d+27}{27})^{-1/2} 1.28257669^{\lfloor d/9 \rfloor/2}$ for every d ; this exceeds 1 only for $d \geq 6570$ (a sharper product form only for $d \geq 6525$), so for smaller d it is weaker than the trivial bound $c^*(d) \geq 1$, and the asymptotic statement is the main result. The proof combines Rademacher's identity $\mathbb{E}V_K^2 = \frac{d+1}{d!} \det \text{Cov}(K)$, a Gram-Schmidt and Hölder argument showing that for products of copies of a fixed body $\mathbb{E}V_K$ is within a polynomial factor of $(\mathbb{E}V_K^2)^{1/2}$, and an explicit pair $K_0 \subseteq L_0$ in $\mathbb{R}^9$ with $\det \text{Cov}(K_0) > \det \text{Cov}(L_0)$, certified in exact rational arithmetic: L_0 is the convex hull of a regular simplex and the apexes of pyramids of height $5/4$ (in units of the inradius) on its facets, and K_0 keeps of each pyramid only the bottom $1/11$ of its height. No earlier quantitative lower bound for $c^*(d)$ was found in the sources we searched, and the product idea may be folklore. The exact value of $c^*(d)$ and the limit of $c^*(d)^{1/d}$ remain open. This is an unrefereed note.

1 Introduction

1.1 The question

For a convex body $K \subset \mathbb{R}^d$ (a compact convex set with non-empty interior) let $X_0, \dots, X_d$ be independent random points, uniformly distributed in K , and let

$$V_K := V_d(\text{conv}\{X_0, \dots, X_d\})$$

be the volume of the random simplex they span. In 2006 Meckes [Mec06, Problem 4] conjectured that $\mathbb{E}V_K \leq \mathbb{E}V_L$ whenever $K \subseteq L$, and proposed the weaker conjecture that $\mathbb{E}V_K \leq a^d \mathbb{E}V_L$ for a universal

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constant a ; he explained that, together with a theorem of Klartag [Kla06], the weaker conjecture would give a positive answer to the slicing problem. The monotone version holds for $d = 1, 2$ and fails for $d \geq 4$ [Rad12, Theorem 1], and it fails for $d = 3$ [KRR18].

In his extended abstract in Oberwolfach Report 10/2017, Reitzner recorded the question of Meckes as follows [Rei17, p. 519].

- (Q1) Does there exist $c(d)$ with $\lim_{d \rightarrow \infty} c(d)^{1/d} < \infty$ such that $\mathbb{E}V_K \leq c(d) \mathbb{E}V_L$ for all convex bodies $K \subset L$ in $\mathbb{R}^d$?
- (Q2) Remarking that it would be of interest to compute the lower bound for $c(d)$ given by the counterexamples of Rademacher, he defines $c(d)$ as the minimum over $K \subset L$ of $\mathbb{E}V_K/\mathbb{E}V_L$ and asks: “Compute a lower bound for $c(d)$!”

Together the two questions form record OWR-15215-006 of the UnsolvedMath collection [ula26], where the record is listed as partially solved.

Reading of the questions. In (Q1) we read $\lim$ as $\limsup$, as the record does; the answer below works for every reading. In (Q2) the displayed minimum, read literally, is degenerate: replacing K by a homothetic copy $tK \subseteq L$ multiplies $\mathbb{E}V_K$ by t^d , so the infimum is 0 and is not attained. The context, lower bounds coming from counterexamples to monotonicity, shows that the intended quantity is the least admissible constant in (Q1),

$$c^*(d) := \sup \left\{ \frac{\mathbb{E}V_K}{\mathbb{E}V_L} : K \subseteq L \text{ convex bodies in } \mathbb{R}^d \right\}, \quad (1)$$

and we read (Q2) as asking for lower bounds for $c^*(d)$. The record reads it in the same way (“a lower bound for the optimal such $c(d)$ ”). (Under the third conceivable reading, the minimum of $\mathbb{E}V_L/\mathbb{E}V_K$, which equals $1/c^*(d)$, a lower bound is an upper bound for $c^*(d)$, that is, (Q1) again.) Trivially $c^*(d) \geq 1$, and $c^*(1) = c^*(2) = 1$ [Rad12, Theorem 1]. The counterexamples of [Rad12] (a half-ball together with a flat cone whose tip is cut off) and of [KRR18] (a tetrahedron with an infinitesimal variation) show $c^*(d) > 1$ for $d \geq 3$; they are perturbative and give no explicit value. Since $c^*(d) \leq C^d$ by the answer to (Q1), the informative content of a lower bound for $c^*(d)$ is its growth in d .

1.2 Question (Q1) is answered in print

Rademacher [Rad12, Appendix A, Proposition A.4] proved that for every $p \geq 1$ the existence of a universal c_2 with $\mathbb{E}V_K^p \leq c_2^d \mathbb{E}V_M^p$ for all convex bodies $K \subseteq M$ is equivalent to the slicing conjecture, in the form $\sup_n L_n < \infty$ for the isotropic constants (Section 2); in the arXiv version 1008.3944v1 of [Rad12] it is Proposition 20. Reichenwallner and Reitzner also state that the weak conjecture is an equivalent formulation of the slicing problem, citing Rademacher [RR16, Section 1], and so does the source [Rei17, p. 519]. Klartag and Lehec [KL25, Theorem 1.2] proved that $\sup_{n \geq 1} L_n < \infty$. Together these two theorems answer (Q1) positively, with $c(d) = a^d$ for a universal a . This answer is not ours; we found no paper that writes the combination down, but both ingredients are published theorems.

1.3 Results

The isotropic constant L_K , its supremum L_d over convex bodies in $\mathbb{R}^d$ and the unit ball B^d are recalled in Section 2.

Theorem 1.1 (explicit upper bound). *For convex bodies $K \subseteq L$ in $\mathbb{R}^d$,*

$$\frac{\mathbb{E}V_K}{\mathbb{E}V_L} \leq \theta_d \left(\frac{L_K}{L_{B^d}} \right)^d \frac{V_d(K)}{V_d(L)} \leq \theta_d \left(\frac{L_d}{L_{B^d}} \right)^d,$$

where $\theta_d := (\mathbb{E}V_{B^d}^2)^{1/2}/\mathbb{E}V_{B^d}$ satisfies $1 \leq \theta_d \leq ((d+1)\binom{d+8}{8})^{1/2}$. Consequently $\limsup_{d \rightarrow \infty} c^*(d)^{1/d} \leq \sqrt{2\pi e} \sup_n L_n < \infty$.

Theorem 1.2 (exponential lower bound). *For every $d \geq 1$,*

$$c^*(d) \geq \left[(d+1) \prod_{r=1}^d \left(1 + \frac{26.8}{r} \right) \right]^{-1/2} 1.28257669^{\lfloor d/9 \rfloor / 2} \geq \left[(d+1) \binom{d+27}{27} \right]^{-1/2} 1.28257669^{\lfloor d/9 \rfloor / 2}. \quad (2)$$

The middle term of (2) is larger than 1 if and only if $d \geq 6525$, and the right-hand side if and only if $d \geq 6570$; for $d \leq 6524$ the bound (2) is weaker than the trivial bound $c^*(d) \geq 1$. Moreover

$$\liminf_{d \rightarrow \infty} c^*(d)^{1/d} \geq \left(\frac{300657950289351}{292457950289351} \right)^{1/2} = 1.0139221956 \dots \quad (3)$$

Corollary 1.3. $1.01392 \leq \liminf_d c^*(d)^{1/d} \leq \limsup_d c^*(d)^{1/d} \leq \sqrt{2\pi e} \sup_n L_n < \infty$. In particular $\log c^*(d)$ is of exact order d , that is, $c^*(d) = e^{\Theta(d)}$.

Thus the exponential allowance in Meckes's weak conjecture cannot be replaced by a subexponential function of d . The main result is the asymptotic statement (3). The explicit bound (2) is non-trivial only in very high dimension: for $3 \leq d \leq 6524$ we obtain nothing beyond the known $c^*(d) > 1$. (A Monte Carlo experiment, which is evidence and not proof, suggests $c^*(9) \geq 1.1$; see Remark 6.5.)

Method. Three ingredients are combined. (a) For independent identically distributed points, $\mathbb{E}V^2 = \frac{d+1}{d!} \det \text{Cov}$ (Lemma 2.1, due to Rademacher), so second moments of V become covariance determinants, which are multiplicative over products. (b) A Gram–Schmidt and Hölder argument (Lemma 3.2) gives $\mathbb{E}V \geq [(d+1) \prod_{r \leq d} (1 + C_0/r)]^{-1/2} (\mathbb{E}V^2)^{1/2}$ for laws with scalar covariance satisfying a fourth-moment condition, which holds for products of a fixed body with a constant C_0 independent of the number of factors (Lemma 3.4); so for such products $\mathbb{E}V$ is within a polynomial factor of $(\mathbb{E}V^2)^{1/2}$. (c) An explicit pair $K_0 \subseteq L_0$ of polytopes in $\mathbb{R}^9$ with $\det \text{Cov}(K_0)/\det \text{Cov}(L_0) = 1.28257 \dots > 1$ (Proposition 5.3), certified in exact rational arithmetic: L_0 is the convex hull of a regular simplex and the apexes of pyramids of height $5/4$ (in units of the inradius) erected on its facets, and K_0 keeps of each pyramid only the slab between its base and the parallel hyperplane at $1/11$ of its height. For $d = 9m + j$ the bodies $K = K_0^m \times [-1, 1]^j$ and $L = L_0^m \times [-1, 1]^j$ (after normalization) then satisfy $\mathbb{E}V_K/\mathbb{E}V_L \geq \text{poly}(d)^{-1} 1.28257 \dots^{m/2}$. As by-products we note that the constant in Vempala's covariance version of the question grows exponentially (Remark 6.4), and that for the cube $\log \mathbb{E}V_{[0,1]^d} = -\frac{1}{2} \log d! - \frac{d}{2} \log 12 + O(\log d)$ (Remark 3.5).

Credits and limits. The question is Meckes's [Mec06], as recorded by Reitzner [Rei17]. The equivalence of (Q1) with the slicing problem is Rademacher's [Rad12] (it is also stated in [RR16]), and the slicing problem was solved by Klartag and Lehec [KL25]; Groemer's theorem, Miles's formula and Kingman's formula are classical and are cited, not proved. Theorem 1.1 is a self-contained variant of the known answer to (Q1). New, to our knowledge, are the quantitative exponential lower bound of Theorem 1.2 and Corollary 1.3, and the product bound of Theorem 6.1 on which they rest. The product idea is simple and may be folklore, but we found no earlier statement of it (see “Scope and priority” in Section 8). The exact value of $c^*(d)$ for $d \geq 3$ and the limit of $c^*(d)^{1/d}$, if it exists, remain open.

2 Preliminaries

We write $\langle \cdot, \cdot \rangle$ and $|\cdot|$ for the Euclidean inner product and norm, B^d for the unit ball, $\kappa_d = V_d(B^d) = \pi^{d/2}/\Gamma(1 + \frac{d}{2})$ and $\omega_d = d\kappa_d$. For a random vector X with finite second moments, $\text{Cov } X$ is its

covariance matrix. For a convex body $K \subset \mathbb{R}^d$ and X uniformly distributed in K we write $\mu(K) = \mathbb{E}X$ for the centroid and $A(K) = \text{Cov } X$ for the covariance matrix of K , and

$$L_K := \frac{(\det A(K))^{1/(2d)}}{V_d(K)^{1/d}}, \quad L_d := \sup\{L_K : K \subset \mathbb{R}^d \text{ a convex body}\}$$

for the isotropic constant, normalized as in [KL25, (2)] and [Rad12, (A1)]. L_K is invariant under invertible affine maps, L_d is finite for each d (for instance by John's theorem), and $\sup_d L_d < \infty$ by [KL25, Theorem 1.2]. Since $A(B^d) = \frac{1}{d+2}I_d$,

$$L_{B^d} = (d+2)^{-1/2} \kappa_d^{-1/d}. \quad (4)$$

If T is an invertible affine map with linear part T' , then V_{TK} has the law of $|\det T'|V_K$ and $A(TK) = T'A(K)T'^T$; hence the ratios $\mathbb{E}V_K/\mathbb{E}V_L$ and $\det A(K)/\det A(L)$ do not change when the same T is applied to K and L .

Lemma 2.1 (second moment, [Rad12, Lemma 4]). *Let $X_0, \dots, X_d$ be independent identically distributed random vectors in $\mathbb{R}^d$ with $\mathbb{E}|X_0|^2 < \infty$ and covariance matrix A . Then*

$$\mathbb{E}[V_d(\text{conv}\{X_0, \dots, X_d\})^2] = \frac{d+1}{d!} \det A.$$

In particular $\mathbb{E}V_K^2 = \frac{d+1}{d!} \det A(K) = \frac{d+1}{d!} L_K^2 V_d(K)^2$ for every convex body $K \subset \mathbb{R}^d$.

Proof. We include the short proof ([Rad12, Lemma 4] is stated for uniform points in a convex body). Let M be the $(d+1) \times (d+1)$ matrix with rows $Y_i = (1, X_i)$, $i = 0, \dots, d$. Subtracting the row Y_0 from the others shows $d! V_d(\text{conv}\{X_0, \dots, X_d\}) = |\det M|$. Let $G = \mathbb{E}[Y_0 Y_0^T]$. Expanding both determinants and using that the rows are independent and identically distributed,

$$\mathbb{E}(\det M)^2 = \sum_{s,t} \text{sgn}(s) \text{sgn}(t) \prod_i \mathbb{E}[M_{i,s(i)} M_{i,t(i)}] = \sum_{s,t} \text{sgn}(s) \text{sgn}(t) \prod_i G_{s(i),t(i)},$$

the sums running over permutations s, t of $\{0, \dots, d\}$. For fixed s substitute $u = t \circ s^{-1}$; then $\text{sgn}(s) \text{sgn}(t) = \text{sgn}(u)$ and $\prod_i G_{s(i),t(i)} = \prod_j G_{j,u(j)}$, so the inner sum is $\det G$ and $\mathbb{E}(\det M)^2 = (d+1)! \det G$. Finally $G = \begin{pmatrix} 1 & \mu^T \\ \mu & \mathbb{E}X_0 X_0^T \end{pmatrix}$ with $\mu = \mathbb{E}X_0$, and the Schur complement of the entry 1 gives $\det G = \det(\mathbb{E}X_0 X_0^T - \mu \mu^T) = \det A$. Hence $\mathbb{E}V_d(\text{conv}\{X_0, \dots, X_d\})^2 = (d+1)! \det A / (d!)^2$. $\square$

Theorem 2.2 (Groemer [Gro74]; see [SW08, Theorem 8.6.3] and [Rad12, Theorem 6]). *For every convex body $K \subset \mathbb{R}^d$, $\mathbb{E}V_K/V_d(K) \geq \mathbb{E}V_{B^d}/\kappa_d$, with equality if and only if K is an ellipsoid.*

The ball. By Miles's formula [Mil71], in the form of [SW08, Theorem 8.2.3] and [Rad12, Theorem 7], for every integer $k \geq 1$

$$\mathbb{E}V_{B^d}^k = \frac{1}{(d!)^k} \left(\frac{\kappa_{d+k}}{\kappa_d} \right)^{d+1} \frac{\kappa_{d(d+k+1)}}{\kappa_{(d+1)(d+k)}} \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}. \quad (5)$$

For $k = 1$ this has the closed form, going back to Kingman [Kin69],

$$\frac{\mathbb{E}V_{B^d}}{\kappa_d} = \frac{\binom{d+1}{(d+1)/2}}{2^d \binom{(d+1)^2/2}}^{d+1}, \quad \binom{x}{x/2} := \frac{\Gamma(x+1)}{\Gamma(\frac{x}{2}+1)^2}, \quad (6)$$

which gives the classical values $\frac{1}{3}$, $\frac{35}{48\pi^2}$ and $\frac{9}{715}$ for $d = 1, 2, 3$. We use (5) and (6) only for the numerical values in Remark 4.1; our program checks numerically that (5) with $k = 1$ agrees with (6) and that (5) with $k = 2$ agrees with Lemma 2.1 (Section 8). We put

$$\theta_d := \frac{(\mathbb{E}V_{B^d}^2)^{1/2}}{\mathbb{E}V_{B^d}} \geq 1,$$

the inequality being the Cauchy–Schwarz inequality.

3 The first moment of the volume compared with the second

By the Cauchy–Schwarz inequality $\mathbb{E}V \leq (\mathbb{E}V^2)^{1/2}$. In this section we show that a reverse inequality holds up to a polynomial factor in d for laws with scalar covariance satisfying the following fourth-moment condition.

Definition 3.1. Let X be a random vector in $\mathbb{R}^d$ with $\mathbb{E}|X|^4 < \infty$, mean μ and covariance matrix $\sigma^2 I_d$, $\sigma > 0$. We say that X satisfies condition (H) with constant $C_0 \geq 0$ if

$$(H) \quad \text{Var}(|P(X - \mu) + c|^2) \leq C_0 \sigma^2 (\sigma^2 r + |c|^2)$$

for every $r \in \{1, \dots, d\}$, every orthogonal projection P of rank r and every c in the range of P .

Lemma 3.2 (Gram–Schmidt and Hölder). *Let X satisfy (H) with constant C_0 , let $X_0, \dots, X_d$ be independent copies of X and $V = V_d(\text{conv}\{X_0, \dots, X_d\})$. Then*

$$\mathbb{E}V \geq \sigma^d \prod_{r=1}^d (r + C_0)^{-1/2} = \left[(d+1) \prod_{r=1}^d \left(1 + \frac{C_0}{r}\right) \right]^{-1/2} (\mathbb{E}V^2)^{1/2}.$$

If $N \geq C_0$ is an integer, then $\prod_{r=1}^d (1 + C_0/r) \leq \binom{d+N}{N}$.

Proof. Step 1: one distance. Let $E \subset \mathbb{R}^d$ be an affine subspace of dimension $k-1$, where $1 \leq k \leq d$, so that its codimension is $r = d - k + 1 \geq 1$. Choose $a \in E$ and let P be the orthogonal projection onto the orthogonal complement of the linear subspace $E - a$; it has rank r , and $\text{dist}(x, E) = |P(x - a)|$. Thus $Y := \text{dist}(X, E) = |PZ + c|$ with $Z = X - \mu$ and $c = P(\mu - a)$ in the range of P . Since $\mathbb{E}Z = 0$ and $\text{Cov } Z = \sigma^2 I_d$,

$$T := \mathbb{E}Y^2 = \mathbb{E}|PZ|^2 + |c|^2 = \sigma^2 \text{tr } P + |c|^2 = \sigma^2 r + |c|^2.$$

Hölder’s inequality with exponents $\frac{3}{2}$ and 3, applied to $Y^2 = Y^{2/3} Y^{4/3}$, gives $\mathbb{E}Y^2 \leq (\mathbb{E}Y)^{2/3} (\mathbb{E}Y^4)^{1/3}$, that is $\mathbb{E}Y \geq T^{3/2} (\mathbb{E}Y^4)^{-1/2}$. By (H), $\mathbb{E}Y^4 = T^2 + \text{Var}(Y^2) \leq T^2 + C_0 \sigma^2 T$, hence $\mathbb{E}Y \geq T/(T + C_0 \sigma^2)^{1/2}$. The function $t \mapsto t/(t + b)^{1/2}$ is increasing on $(0, \infty)$ for $b \geq 0$, and $T \geq \sigma^2 r$; therefore

$$\mathbb{E} \text{dist}(X, E) \geq m_r := \frac{\sigma r}{(r + C_0)^{1/2}}. \quad (7)$$

If E is an affine subspace of dimension less than $k-1$, choose an affine subspace $E' \supseteq E$ of dimension $k-1$; since $\text{dist}(\cdot, E) \geq \text{dist}(\cdot, E')$, (7) for E' gives $\mathbb{E} \text{dist}(X, E) \geq m_{d-k+1}$ again.

Step 2: Gram–Schmidt. For $x_0, \dots, x_d \in \mathbb{R}^d$,

$$d! V_d(\text{conv}\{x_0, \dots, x_d\}) = |\det(x_1 - x_0, \dots, x_d - x_0)| = \prod_{k=1}^d \text{dist}(x_k, \text{aff}\{x_0, \dots, x_{k-1}\}).$$

Indeed, the absolute value of the determinant is the product of the norms of the vectors obtained from $x_1 - x_0, \dots, x_d - x_0$ by the Gram–Schmidt process, and the k -th of these norms is the distance of $x_k - x_0$ from the span of $x_1 - x_0, \dots, x_{k-1} - x_0$, which is the distance of x_k from $\text{aff}\{x_0, \dots, x_{k-1}\}$. (If the points are affinely dependent, both sides vanish.)

Step 3: conditioning. Put $D_k = \text{dist}(X_k, \text{aff}\{X_0, \dots, X_{k-1}\})$. Since X_k is independent of $(X_0, \dots, X_{k-1})$, Fubini's theorem and Step 1, applied for fixed values of $X_0, \dots, X_{k-1}$ to the affine hull of these k points (of dimension at most $k-1$), give

$$\mathbb{E}[D_1 \cdots D_k] = \mathbb{E}[D_1 \cdots D_{k-1} \mathbb{E}[D_k \mid X_0, \dots, X_{k-1}]] \geq m_{d-k+1} \mathbb{E}[D_1 \cdots D_{k-1}].$$

By induction, $\mathbb{E}[d! V] = \mathbb{E}[D_1 \cdots D_d] \geq \prod_{k=1}^d m_{d-k+1} = \prod_{r=1}^d \sigma r(r + C_0)^{-1/2}$, which is the first inequality.

Step 4. By Lemma 2.1, $\mathbb{E}V^2 = \frac{d+1}{d!} \sigma^{2d}$, and $\sigma^d \prod_{r=1}^d (r + C_0)^{-1/2} = (\frac{d!}{d+1} \mathbb{E}V^2)^{1/2} \prod_{r=1}^d (r + C_0)^{-1/2} = [(d+1) \prod_{r=1}^d (1 + C_0/r)]^{-1/2} (\mathbb{E}V^2)^{1/2}$. Finally, for an integer $N \geq C_0$, $\prod_{r=1}^d (1 + C_0/r) \leq \prod_{r=1}^d \frac{r+N}{r} = \binom{d+N}{N}$. $\square$

Lemma 3.3 (the ball). *If X is uniformly distributed in B^d , then $\mathbb{E}X = 0$, $\text{Cov } X = \frac{1}{d+2} I_d$, and X satisfies (H) with $C_0 = 8$.*

Proof. The radial part $|X|$ has density dt^{d-1} on $[0, 1]$, so $\mathbb{E}|X|^2 = \frac{d}{d+2}$ and $\mathbb{E}|X|^4 = \frac{d}{d+4}$; by rotation invariance $\text{Cov } X = \sigma^2 I_d$ with $\sigma^2 = \frac{1}{d+2}$, and $\mathbb{E}X_i X_j X_k X_l = c_4(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where $c_4 d(d+2) = \mathbb{E}|X|^4$ gives $c_4 = \frac{1}{(d+2)(d+4)}$. Let P have rank r and $c = Pc$. Then $|PX+c|^2 = U+2W+|c|^2$ with $U = X^\top P X$ and $W = \langle c, X \rangle$. We have $\mathbb{E}U^2 = c_4((\text{tr } P)^2 + 2 \text{tr } P^2) = c_4(r^2 + 2r)$ and $\mathbb{E}U = \sigma^2 r$, so

$$\text{Var } U = \frac{r^2 + 2r}{(d+2)(d+4)} - \frac{r^2}{(d+2)^2} = \frac{2r(d+2-r)}{(d+2)^2(d+4)} \leq 2r\sigma^4,$$

and $\text{Var } W = \sigma^2 |c|^2$. Since $\text{Var}(A+B) \leq 2 \text{Var } A + 2 \text{Var } B$, $\text{Var}(U+2W) \leq 2 \text{Var } U + 8 \text{Var } W \leq 4r\sigma^4 + 8\sigma^2 |c|^2 \leq 8\sigma^2(\sigma^2 r + |c|^2)$. $\square$

Lemma 3.4 (products). *Let $\mathbb{R}^d = \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$ and let $X = (X^1, \dots, X^m)$ have independent blocks $X^a \in \mathbb{R}^{d_a}$ with $\mathbb{E}|X^a|^4 < \infty$, $\mathbb{E}X^a = \mu^a$ and $\text{Cov } X^a = \sigma^2 I_{d_a}$, with the same $\sigma > 0$ for all a . Put $Z^a = X^a - \mu^a$ and*

$$\Lambda_a := \lambda_{\max}(\mathbb{E}[|Z^a|^2 Z^a (Z^a)^\top]) / \sigma^4.$$

Then X satisfies (H) with $C_0 = \max(8, 4 + 2 \max_a \Lambda_a)$. If $|Z^a| \leq \rho$ almost surely, then $\Lambda_a \leq \rho^2 / \sigma^2$.

Proof. Let P be an orthogonal projection of rank r , $c = Pc$, $Z = X - \mu$. As in the previous proof, $|PZ+c|^2 = U+2W+|c|^2$ with $U = Z^\top P Z$ and $W = \langle c, Z \rangle$, $\text{Var } W = \sigma^2 |c|^2$ and $\text{Var}(U+2W) \leq 2 \text{Var } U + 8 \text{Var } W$. Write P in blocks P_{ab} , so that $U = \sum_{a,b} (Z^a)^\top P_{ab} Z^b$ and $\mathbb{E}U^2 = \sum_{a,b,e,f} \mathbb{E}[(Z^a)^\top P_{ab} Z^b (Z^e)^\top P_{ef} Z^f]$. By independence and $\mathbb{E}Z^a = 0$, a term vanishes if some block index occurs exactly once in (a,b,e,f) . The remaining terms have $a=b=e=f$, or $a=b \neq e=f$, or $a=e \neq b=f$, or $a=f \neq b=e$. The terms with $a=b \neq e=f$ equal $\sigma^4 \text{tr } P_{aa} \text{tr } P_{ee}$ and cancel against the corresponding terms of $(\mathbb{E}U)^2 = \sigma^4 (\sum_a \text{tr } P_{aa})^2$; by $\mathbb{E}Z^a (Z^a)^\top = \sigma^2 I$ and $P_{ba} = P_{ab}^\top$, the terms with $a=e \neq b=f$ and those with $a=f \neq b=e$ both equal $\sigma^4 \|P_{ab}\|_F^2$. Hence

$$\text{Var } U = \sum_a \text{Var}((Z^a)^\top P_{aa} Z^a) + 2\sigma^4 \sum_{a \neq b} \|P_{ab}\|_F^2, \quad \sum_{a \neq b} \|P_{ab}\|_F^2 \leq \|P\|_F^2 = \text{tr } P = r.$$

For $Q = P_{aa}$ we have $0 \leq Q \leq I$, so $0 \leq Z^\top Q Z \leq |Z|^2$ for $Z = Z^a$ and

$$\text{Var}(Z^\top Q Z) \leq \mathbb{E}[|Z|^2 Z^\top Q Z] = \text{tr}(Q \mathbb{E}[|Z|^2 Z Z^\top]) \leq \Lambda_a \sigma^4 \text{tr } Q,$$

and $\sum_a \text{tr } P_{aa} = r$. Altogether, with $\Lambda = \max_a \Lambda_a$, $\text{Var}(|PZ + c|^2) \leq 2(\Lambda + 2)\sigma^4 r + 8\sigma^2 |c|^2 \leq \max(8, 4 + 2\Lambda) \sigma^2 (\sigma^2 r + |c|^2)$. If $|Z^a| \leq \rho$, then $|Z^a|^2 Z^a (Z^a)^\top \preceq \rho^2 Z^a (Z^a)^\top$, whose expectation is $\rho^2 \sigma^2 I$. $\square$

The quantity Λ_a does not change if Z^a is replaced by tUZ^a with $t > 0$ and U orthogonal. For the uniform distribution on an interval $[-t, t]$, $\sigma^2 = t^2/3$ and $\mathbb{E}Z^4 = t^4/5$, so $\Lambda = \frac{9}{5}$ and $4 + 2\Lambda = 7.6$: products of intervals satisfy (H) with $C_0 = 8$.

Remark 3.5 (the cube). For the unit cube $C^d = [0, 1]^d$, a product of d intervals with $\sigma^2 = \frac{1}{12}$, Lemma 3.4 gives (H) with $C_0 = 8$, and Lemma 2.1, Lemma 3.2 and the Cauchy–Schwarz inequality give

$$\left[(d+1) \binom{d+8}{8} \right]^{-1/2} \leq \frac{\mathbb{E}V_{C^d}}{\left(\frac{d+1}{d!}\right)^{1/2} 12^{-d/2}} \leq 1,$$

hence $\log \mathbb{E}V_{C^d} = -\frac{1}{2} \log d! - \frac{d}{2} \log 12 + O(\log d)$. The source asks for the asymptotic behaviour of $\mathbb{E}V_{C^d}$ in another open problem [Rei17, p. 519] (record OWR-15215-007 of [ula26]); the remark determines it on the logarithmic scale only, and we do not claim novelty for it.

4 The first question: an explicit upper bound

Proof of Theorem 1.1. Let $K \subseteq L$ be convex bodies in $\mathbb{R}^d$. By the Cauchy–Schwarz inequality and Lemma 2.1,

$$\mathbb{E}V_K \leq (\mathbb{E}V_K^2)^{1/2} = \left(\frac{d+1}{d!}\right)^{1/2} L_K^d V_d(K).$$

By Theorem 2.2, $\mathbb{E}V_L \geq V_d(L) \mathbb{E}V_{B^d}/\kappa_d$, and by the definition of θ_d and Lemma 2.1, $\mathbb{E}V_{B^d} = (\mathbb{E}V_{B^d}^2)^{1/2}/\theta_d = \left(\frac{d+1}{d!}\right)^{1/2} L_{B^d}^d \kappa_d/\theta_d$. Dividing,

$$\frac{\mathbb{E}V_K}{\mathbb{E}V_L} \leq \theta_d \left(\frac{L_K}{L_{B^d}}\right)^d \frac{V_d(K)}{V_d(L)} \leq \theta_d \left(\frac{L_d}{L_{B^d}}\right)^d,$$

because $V_d(K) \leq V_d(L)$ and $L_K \leq L_d$. We know $\theta_d \geq 1$. By Lemma 3.3 the uniform distribution on B^d satisfies (H) with $C_0 = 8$, so Lemma 3.2 with $N = 8$ gives $\theta_d \leq ((d+1)\binom{d+8}{8})^{1/2}$.

For the asymptotics, (4) gives $L_{B^d}^{-2} = (d+2)\kappa_d^{2/d} = \pi(d+2)\Gamma(\frac{d}{2}+1)^{-2/d}$, and Stirling’s formula $\log \Gamma(\frac{d}{2}+1) = \frac{d}{2} \log \frac{d}{2} - \frac{d}{2} + O(\log d)$ gives $\Gamma(\frac{d}{2}+1)^{2/d} = \frac{d}{2e} e^{O(\log d/d)}$, so $L_{B^d} \rightarrow (2\pi e)^{-1/2}$. Moreover $\theta_d^{1/d} \rightarrow 1$, since $1 \leq \theta_d \leq (d+1)^{1/2}(d+8)^4$. Hence $c^*(d) \leq \theta_d (L_d/L_{B^d})^d$ for every d and

$$\limsup_{d \rightarrow \infty} c^*(d)^{1/d} \leq \sqrt{2\pi e} \limsup_{d \rightarrow \infty} L_d \leq \sqrt{2\pi e} \sup_n L_n,$$

which is finite by [KL25, Theorem 1.2]. $\square$

Remark 4.1 (numerical values). From (5) we obtain $\theta_1 = 1.2247$, $\theta_2 = 1.3192$, $\theta_3 = 1.3851$, $\theta_{10} = 1.6712$, $\theta_{100} = 2.7810$ and $\theta_{1000} = 4.9072$; the bound of Lemma 3.2 is crude ($\theta_{10} \leq 693.8$). The quantity $(\theta_d/L_{B^d}^d)^{1/d}$ lies between 3.99 and 4.25 for $1 \leq d \leq 1000$ and tends to $\sqrt{2\pi e} = 4.1327\dots$; so Theorem 1.1 gives $c^*(d) \leq (4.25 L_d)^d$ for $d \leq 1000$. These values are floating-point evaluations (Section 8) and are not used in any proof.

Remark 4.2. (a) Rademacher’s proof of the implication from the slicing conjecture to the weak conjecture for the first moment [Rad12, Proposition A.4] bounds the second moments by means of the isotropic constant and passes from the second moment to the first by a Khinchine-type inequality [Rad12, Lemma A.3], which costs a factor c^{d+1} . In Theorem 1.1, Groemer’s theorem and Lemma 3.2

for the ball reduce this loss to the polynomial factor θ_d . (b) If $\sup_n L_n = 1/e$, the value expected under the strong form of the slicing problem (simplices maximize the isotropic constant; see [KL25, p. 3]), then Theorem 1.1 gives $\limsup_d c^*(d)^{1/d} \leq \sqrt{2\pi e}/e = \sqrt{2\pi}/e = 1.5203\dots$. This is conditional and is not used.

5 A pair of polytopes in $\mathbb{R}^9$

Construction. Let $n \geq 2$ and let $\Delta \subset \mathbb{R}^n$ be an n -simplex with barycentric coordinates $\lambda_0, \dots, \lambda_n$ (affine functions with $\sum_i \lambda_i = 1$ and $\Delta = \{\lambda_0, \dots, \lambda_n \geq 0\}$), centroid g , facets $F_i = \Delta \cap \{\lambda_i = 0\}$ and facet centroids c_i . For $h > 0$ and $i = 0, \dots, n$ let

$$a_i = g + (1+h)(c_i - g), \quad \text{Pyr}_i = \text{conv}(F_i \cup \{a_i\}), \quad \beta = \frac{n+1+h}{nh},$$

so that Pyr_i is a pyramid over F_i with apex a_i beyond F_i . For $0 < s_0 < 1$ put

$$L_0 := \Delta \cup \text{Pyr}_0 \cup \dots \cup \text{Pyr}_n, \quad K_0 := L_0 \cap \left\{ x : \lambda_i(x) \geq -\frac{s_0 h}{n+1} \text{ for } i = 0, \dots, n \right\}.$$

Since $\lambda(g) = \frac{1}{n+1}(1, \dots, 1)$ and c_i has $\lambda_i = 0$ and $\lambda_j = \frac{1}{n}$ for $j \neq i$,

$$\lambda_i(a_i) = -\frac{h}{n+1}, \quad \lambda_j(a_i) = \frac{n+1+h}{n(n+1)} = \beta \frac{h}{n+1} \quad (j \neq i). \quad (8)$$

The condition $\beta \geq 1$ is equivalent to $h \leq \frac{n+1}{n-1}$.

Lemma 5.1. *Let $0 < h \leq \frac{n+1}{n-1}$ and $0 < s_0 < 1$.*

- (a) $L_0 = Q := \{x : \lambda_j(x) + \beta\lambda_i(x) \geq 0 \text{ for all } i \neq j\}$. Hence L_0 is a convex polytope, and $L_0 = \text{conv}(\Delta \cup \{a_0, \dots, a_n\})$.
- (b) K_0 is a convex polytope with $\Delta \subseteq K_0 \subseteq L_0$, and $K_0 = \Delta \cup T_0 \cup \dots \cup T_n$ with $T_i = \{(1-s)y + sa_i : y \in F_i, 0 \leq s \leq s_0\}$.

Proof. The pyramids. We claim that $\text{Pyr}_i = P_i := \{x : \lambda_i(x) \leq 0, \lambda_j(x) + \beta\lambda_i(x) \geq 0 \ (j \neq i)\}$. Let $x = (1-s)y + sa_i$ with $y \in F_i$ and $s \in [0, 1]$. By (8), $\lambda_i(x) = -\frac{sh}{n+1} \leq 0$ and $\lambda_j(x) + \beta\lambda_i(x) = (1-s)\lambda_j(y) \geq 0$ for $j \neq i$; so $\text{Pyr}_i \subseteq P_i$. Conversely let $x \in P_i$. Summing the n inequalities over $j \neq i$ gives $1 - \lambda_i + n\beta\lambda_i \geq 0$, and $n\beta - 1 = \frac{n+1}{h}$, so $\lambda_i(x) \geq -\frac{h}{n+1}$. Put $s = -\frac{(n+1)\lambda_i(x)}{h} \in [0, 1]$. If $s = 1$, the sum of the n inequalities is an equality, so each of them is, and $\lambda(x) = \lambda(a_i)$ by (8), i.e. $x = a_i$. If $s < 1$, put $y = (x - sa_i)/(1-s)$; then $\lambda_i(y) = 0$ and $\lambda_j(y) = (\lambda_j(x) + \beta\lambda_i(x))/(1-s) \geq 0$, so $y \in F_i$ and $x = (1-s)y + sa_i \in \text{Pyr}_i$.

(a) $\Delta \subseteq Q$ because $\beta > 0$. Let $x \in \text{Pyr}_i$. The inequalities with the pair (i, j) , $\lambda_j + \beta\lambda_i \geq 0$, hold since $\text{Pyr}_i = P_i$. For $j \neq i$ we get $\lambda_j \geq -\beta\lambda_i \geq 0$, hence $\lambda_i + \beta\lambda_j \geq \lambda_i - \beta^2\lambda_i = (\beta^2 - 1)(-\lambda_i) \geq 0$, because $\lambda_i \leq 0$ and $\beta \geq 1$; and for $j, k \neq i$, $\lambda_k + \beta\lambda_j \geq 0$ since $\lambda_j, \lambda_k \geq 0$. So $L_0 \subseteq Q$. Conversely let $x \in Q$. If all $\lambda_i(x) \geq 0$, then $x \in \Delta$. Otherwise $\lambda_i(x) < 0$ for some i , and then $\lambda_j(x) \geq -\beta\lambda_i(x) > 0$ for all $j \neq i$, so $x \in P_i = \text{Pyr}_i$. Thus $L_0 = Q$; it is an intersection of half-spaces and a finite union of simplices, hence a convex polytope. It contains Δ and the a_i , and it is a union of convex hulls of subsets of these points, so it is their convex hull.

(b) K_0 is an intersection of L_0 with half-spaces, hence a convex polytope, and $\Delta \subseteq K_0 \subseteq L_0$. On Pyr_i we have $\lambda_k \geq 0$ for $k \neq i$ (shown in (a)), so only the constraint with index i is active there, and for $x = (1-s)y + sa_i$ it reads $-\frac{sh}{n+1} \geq -\frac{s_0 h}{n+1}$, i.e. $s \leq s_0$. Hence $K_0 \cap \text{Pyr}_i = T_i$, and $K_0 = \Delta \cup \bigcup_i T_i$. $\square$

For $h > \frac{n+1}{n-1}$ the union L_0 is not convex: the midpoint of a_0 and a_1 then has two negative barycentric coordinates and lies in no piece. At $h = \frac{n+1}{n-1}$ we have $\beta = 1$, so the inequalities for (i, j) and (j, i) coincide: the facets $\text{conv}((F_i \cap F_j) \cup \{a_i\})$ and $\text{conv}((F_i \cap F_j) \cup \{a_j\})$ of adjacent pyramids are coplanar, and L_0 has $n(n+1)/2$ facets (45 for $n = 9$) instead of $n(n+1)$.

Now let Δ be a regular simplex with centroid 0 and inradius 1. This is no loss of generality for us: Lemma 5.1 and the ratio $\det A(K_0)/\det A(L_0)$ are invariant under affine maps, and the construction commutes with them. If $v_0, \dots, v_n$ are the vertices, then $\sum_i v_i = 0$, $|v_i| = n$, $\langle v_i, v_j \rangle = -n$ for $i \neq j$, $c_i = -v_i/n$, $|c_i| = 1$, $a_i = (1+h)c_i$, and Pyr_i has height h over F_i . The point $(1-s)y + sa_i$ of Pyr_i lies at height sh above F_i , so T_i is the slab of Pyr_i between its base and the parallel hyperplane at the fraction s_0 of its height; it carries the fraction $1 - (1-s_0)^n$ of the volume of Pyr_i .

Lemma 5.2. *Let Δ be regular with centroid 0 and inradius 1, $0 < h \leq \frac{n+1}{n-1}$ and $0 < s_0 < 1$. Then K_0 and L_0 have centroid 0, $A(L_0) = \sigma_L^2 I_n$ and $A(K_0) = \sigma_K^2 I_n$, where $\sigma_L^2 = \sigma^2(1)$, $\sigma_K^2 = \sigma^2(s_0)$, and for $0 < t \leq 1$*

$$J(t) = \int_0^t [(n-1)(1-s)^2 + (1+hs)^2](1-s)^{n-1} ds, \quad \sigma^2(t) = \frac{\frac{n}{n+2} + hJ(t)}{1 + h(1 - (1-t)^n)}.$$

Consequently $\det A(K_0)/\det A(L_0) = (\sigma_K^2/\sigma_L^2)^n$.

Proof. Symmetry. Every permutation of the vertices extends to a linear map of $\mathbb{R}^n$ (the v_i span $\mathbb{R}^n$ and $\sum_i v_i = 0$), which is orthogonal because it preserves the Gram matrix of the v_i . These maps permute the barycentric coordinates, so they preserve Δ , the set of the a_i , and $Q = L_0$ and K_0 , which are defined symmetrically in the coordinates. Hence the centroid of each body is fixed by all of them; a fixed point has equal barycentric coordinates, so it is 0. The covariance matrix A of either body commutes with every such map S , because SX has the same law as X . The representation of the symmetric group on $\mathbb{R}^n$ obtained in this way is irreducible: if $W \neq 0$ is an invariant subspace and $0 \neq w = \sum_i \alpha_i v_i \in W$ with $\sum_i \alpha_i = 0$, then $\alpha_i \neq \alpha_j$ for some i, j , and the transposition S of v_i and v_j gives $w - Sw = (\alpha_i - \alpha_j)(v_i - v_j) \in W$; applying permutations, all differences $v_k - v_l$ lie in W , and they span $\mathbb{R}^n$. The eigenspaces of A are invariant, so A is scalar, $A = \sigma^2 I_n$ with $\sigma^2 = \mathbb{E}|X|^2/n$.

Second moments. If X is uniform in a k -simplex with vertices $u_0, \dots, u_k$, then $\mathbb{E}XX^\top = (\sum_i u_i u_i^\top + (\sum_i u_i)(\sum_i u_i)^\top)/((k+1)(k+2))$ (the barycentric coordinates of X are Dirichlet(1, ..., 1) distributed, with $\mathbb{E}\mu_i \mu_j = (1 + \delta_{ij})/((k+1)(k+2))$). Hence $\mathbb{E}_\Delta |X|^2 = (n+1)n^2/((n+1)(n+2)) = \frac{n^2}{n+2}$, and for Y uniform in F_i , using $\sum_{j \neq i} v_j = -v_i$, $\mathbb{E}|Y|^2 = (n \cdot n^2 + n^2)/(n(n+1)) = n$. Since F_i lies in the hyperplane $\{\langle x, c_i \rangle = 1\}$ and $\mathbb{E}Y = c_i$, we get $\mathbb{E}|Y - c_i|^2 = n-1$ and $Y - c_i \perp c_i$. On Pyr_i write $x = (1-s)y + sa_i$ with $y \in F_i$, $s \in [0, 1]$. Then $x = (1-s)(y - c_i) + (1+hs)c_i$, so $|x|^2 = (1-s)^2|y - c_i|^2 + (1+hs)^2$, and $dx = h(1-s)^{n-1} ds dy$ (dy the $(n-1)$ -dimensional measure on F_i): the directions along F_i are scaled by $1-s$, and $\partial x/\partial s = a_i - y$ has normal component h . Therefore, writing $|F|$ for the $(n-1)$ -dimensional volume of a facet,

$$\int_{\text{Pyr}_i \cap \{s \leq t\}} |x|^2 dx = h|F|J(t), \quad V_d(\text{Pyr}_i \cap \{s \leq t\}) = h|F| \frac{1 - (1-t)^n}{n}.$$

Moreover $V_d(\Delta) = \frac{n+1}{n}|F|$ (Δ is the union of the $n+1$ cones of height 1 from 0 over its facets) and $\int_\Delta |x|^2 dx = V_d(\Delta) \frac{n^2}{n+2}$. For $B_t = \Delta \cup \bigcup_i (\text{Pyr}_i \cap \{s \leq t\})$, whose pieces overlap in null sets, adding up gives $\mathbb{E}_{B_t} |X|^2 = n[\frac{n}{n+2} + hJ(t)]/[1 + h(1 - (1-t)^n)] = n\sigma^2(t)$. Since $B_1 = L_0$ and $B_{s_0} = K_0$ by Lemma 5.1, the claim follows. $\square$

Proposition 5.3 (the certified pair). *Let $n = 9$, $h = \frac{5}{4} = \frac{n+1}{n-1}$ and $s_0 = \frac{1}{11}$, so that L_0 is the convex hull of the regular simplex and the apexes of pyramids of height $\frac{5}{4}$ (in units of the inradius) on its*

facets, and K_0 keeps of each pyramid only the slab between its base and the parallel hyperplane at $\frac{1}{11}$ of its height (the bottom $\frac{1}{11}$ of the height, which carries the fraction $1 - (\frac{10}{11})^9 = 0.5759\dots$ of the volume of the pyramid). Then $V_d(L_0) = \frac{9}{4}V_d(\Delta)$, $V_d(K_0) = \frac{16221529219}{9431790764}V_d(\Delta) = 1.7199\dots V_d(\Delta)$,

$$\sigma_L^2 = \frac{149}{176}, \quad \sigma_K^2 = \frac{300657950289351}{345453686247824}, \quad \frac{\sigma_K^2}{\sigma_L^2} = \frac{300657950289351}{292457950289351} = 1.0280382188\dots,$$

and

$$R := \frac{\det A(K_0)}{\det A(L_0)} = \left(\frac{300657950289351}{292457950289351} \right)^9 = 1.2825766950\dots > 1.$$

Proof. The volumes follow from the volume formulas in the proof of Lemma 5.2, and the rest from Lemma 5.2 by integrating polynomials with rational coefficients. For instance, with $u = 1 - s$, $J(1) = \int_0^1 [8u^2 + (\frac{9}{4} - \frac{5}{4}u)^2]u^8 du = \frac{8}{11} + \frac{81}{16 \cdot 9} - \frac{90}{16 \cdot 10} + \frac{25}{16 \cdot 11} = \frac{153}{176}$, so $\sigma_L^2 = (\frac{9}{11} + \frac{5}{4} \cdot \frac{153}{176}) / \frac{9}{4} = \frac{149}{176}$; σ_K^2 is obtained in the same way from $J(\frac{1}{11})$. Our program (Section 8) evaluates these integrals in exact rational arithmetic and recomputes all values independently from an exact decomposition of L_0 and K_0 into simplices, without the symmetry argument. $\square$

The same construction already gives $\det A(K_0) > \det A(L_0)$ in low dimensions: for $(n, h, s_0) = (3, \frac{2}{9}, \frac{2}{5})$, $(4, \frac{4}{9}, \frac{1}{4})$ and $(5, \frac{5}{8}, \frac{1}{5})$ the ratios are $1.013123\dots$, $1.048589\dots$ and $1.094243\dots$ (exact rational values in Section 8). For $n = 3$ this is an explicit instance of the failure of monotonicity of $\det A(\cdot)$ in $\mathbb{R}^3$, whose existence is part of [Rad12, Theorem 3].

Lemma 5.4 (fourth moment of K_0). *For the pair of Proposition 5.3, let Z be uniformly distributed in K_0 . Then $\mathbb{E}[|Z|^2 Z Z^\top] = \frac{1}{9}\mathbb{E}|Z|^4 I_9$ with*

$$\mathbb{E}|Z|^4 = \frac{675609637612497091767}{8694378375485234432} = 77.70649\dots,$$

and therefore

$$\Lambda_0 := \frac{\lambda_{\max}(\mathbb{E}[|Z|^2 Z Z^\top])}{\sigma_K^4} = \frac{\mathbb{E}|Z|^4}{9\sigma_K^4} = 11.3985358\dots$$

Moreover $|Z| \leq 9$, so the cruder bound $\Lambda_0 \leq q_0 := 81/\sigma_K^2 = 93.068\dots$ also follows from Lemma 3.4.

Proof. The matrix $\mathbb{E}[|Z|^2 Z Z^\top]$ commutes with the orthogonal maps used in the proof of Lemma 5.2, so it is scalar, equal to $\frac{1}{9}\mathbb{E}|Z|^4$ times I_9 . The fourth moment is computed from the same decomposition as in Lemma 5.2. On Δ , writing $X = \sum_i \lambda_i v_i$ with $\lambda \sim \text{Dirichlet}(1, \dots, 1)$, we have $|X|^2 = n(n+1) \sum_i \lambda_i^2 - n$, and $\mathbb{E} \sum_i \lambda_i^2 = \frac{2}{n+2}$, $\mathbb{E}(\sum_i \lambda_i^2)^2 = \frac{4(n+6)}{(n+2)(n+3)(n+4)}$. On Pyr_i , $|x|^4 = (1-s)^4 W^2 + 2(1-s)^2(1+hs)^2 W + (1+hs)^4$ with $W = |y - c_i|^2 = n(n+1) \sum_{j \neq i} \mu_j^2 - (n+1)$, $\mu \sim \text{Dirichlet}(1, \dots, 1)$ on n coordinates, $\mathbb{E} \sum \mu_j^2 = \frac{2}{n+1}$ and $\mathbb{E}(\sum \mu_j^2)^2 = \frac{4(n+5)}{(n+1)(n+2)(n+3)}$; and s is independent of y , with density proportional to $(1-s)^{n-1}$ on $[0, s_0]$. Mixing with the volume weights of Lemma 5.2 gives the stated value, which our program also obtains from the exact simplex decomposition. Finally $K_0 \subseteq \text{conv}$ of the vertices of Δ and the points $(1-s_0)v_j + s_0 a_i$, whose norms are at most 9, and Lemma 3.4 gives $\Lambda_0 \leq 9^2/\sigma_K^2$. $\square$

6 Products: the second question

Theorem 6.1. *Let $K_0 \subseteq L_0$ be convex bodies in $\mathbb{R}^{d_0}$, let T_0 be an invertible affine map such that $T_0 K_0$ has centroid 0 and $A(T_0 K_0) = \frac{1}{3}I_{d_0}$, let Z be uniformly distributed in $T_0 K_0$, and put*

$$\Lambda_0 := \lambda_{\max}(\mathbb{E}[|Z|^2 Z Z^\top]) / (\frac{1}{3})^2, \quad C_0 := \max(8, 4 + 2\Lambda_0), \quad R_0 := \frac{\det A(K_0)}{\det A(L_0)}.$$

Then for every $d \geq 1$, with $m = \lfloor d/d_0 \rfloor$,

$$c^*(d) \geq \left[(d+1) \prod_{r=1}^d \left(1 + \frac{C_0}{r} \right) \right]^{-1/2} R_0^{m/2}, \quad (9)$$

and consequently $\liminf_{d \rightarrow \infty} c^*(d)^{1/d} \geq R_0^{1/(2d_0)}$.

The number Λ_0 does not depend on the choice of T_0 , which is unique up to an orthogonal map (see the remark after Lemma 3.4).

Proof. Write $d = md_0 + j$ with $0 \leq j < d_0$ and put

$$K := (T_0 K_0)^m \times [-1, 1]^j, \quad L := (T_0 L_0)^m \times [-1, 1]^j \quad (\subseteq \mathbb{R}^d).$$

These are convex bodies with $K \subseteq L$. The uniform distribution on K is the product of the uniform distributions on the factors, which are independent blocks with mean 0 and covariance $\frac{1}{3}I$; for the blocks $T_0 K_0$ the quantity of Lemma 3.4 is Λ_0 , and for the intervals it is $\frac{9}{5}$, with $4 + 2 \cdot \frac{9}{5} < 8$. By Lemma 3.4 the uniform distribution on K satisfies (H) with constant C_0 , and Lemma 3.2 gives $\mathbb{E}V_K \geq [(d+1) \prod_r (1 + C_0/r)]^{-1/2} (\mathbb{E}V_K^2)^{1/2}$. By the Cauchy–Schwarz inequality $\mathbb{E}V_L \leq (\mathbb{E}V_L^2)^{1/2}$. By Lemma 2.1, and because the covariance matrix of a product body is block diagonal,

$$\frac{\mathbb{E}V_K^2}{\mathbb{E}V_L^2} = \frac{\det A(K)}{\det A(L)} = \left(\frac{\det A(T_0 K_0)}{\det A(T_0 L_0)} \right)^m = R_0^m,$$

the last step by the affine invariance of the ratio (Section 2). Hence $c^*(d) \geq \mathbb{E}V_K/\mathbb{E}V_L \geq [(d+1) \prod_r (1 + C_0/r)]^{-1/2} R_0^{m/2}$. For the second claim note that $\log \prod_{r=1}^d (1 + C_0/r) \leq C_0 \sum_{r=1}^d \frac{1}{r} \leq C_0(1 + \log d)$ and $m/d \rightarrow 1/d_0$. $\square$

For $d < d_0$ (so $m = 0$) the bound (9) is smaller than 1 and says nothing. The bound grows exponentially, but it starts far below 1; where it crosses 1 is decided by the following elementary lemma.

Lemma 6.2 (thresholds). *Let $C > 0$, $R > 1$, $d_0 \geq 1$, and $B(d) := [(d+1) \prod_{r=1}^d (1 + C/r)]^{-1/2} R^{\lfloor d/d_0 \rfloor/2}$ for $d \geq 0$.*

- (a) $B(d+1) < B(d)$ whenever d_0 does not divide $d+1$.
- (b) For each $j \in \{0, \dots, d_0 - 1\}$ the set $\{m \geq 0 : B(md_0 + j) > 1\}$ is either empty or of the form $\{m \geq M_j\}$.
- (c) If $M \geq 1$, $B(d_0(M-1)) \leq 1$ and $B(d_0M + d_0 - 1) > 1$, then $B(d) > 1$ if and only if $d \geq d_0M$.

Proof. (a) If $\lfloor (d+1)/d_0 \rfloor = \lfloor d/d_0 \rfloor$, then $B(d+1)/B(d) = [\frac{d+2}{d+1}(1 + \frac{C}{d+1})]^{-1/2} < 1$. (b) Let $g(m) = \log B(md_0 + j)$. Its increment $g(m+1) - g(m)$ equals $\frac{1}{2} \log R - \frac{1}{2} \log \frac{md_0 + j + d_0 + 1}{md_0 + j + 1} - \frac{1}{2} \sum_{r=md_0 + j + 1}^{md_0 + j + d_0} \log(1 + C/r)$, which is non-decreasing in m ; and $g(0) \leq 0$, since $B(j) \leq B(0) = 1$ by (a). If $g(m) > 0 \geq g(m+1)$ for some m , then the increment at m is negative, so all earlier increments are negative and $g(m) \leq g(0) \leq 0$, a contradiction. (c) By (b) with $j = 0$, $B(d_0m) \leq 1$ for $0 \leq m \leq M-1$, and by (a) $B(d) \leq B(d_0 \lfloor d/d_0 \rfloor) \leq 1$ for all $d < d_0M$. By (b) with $j = d_0 - 1$, $B(d_0m + d_0 - 1) > 1$ for all $m \geq M$, and by (a) $B(d) \geq B(d_0 \lfloor d/d_0 \rfloor + d_0 - 1) > 1$ for all $d \geq d_0M$. $\square$

Proof of Theorem 1.2. Apply Theorem 6.1 to the pair of Proposition 5.3, with $d_0 = 9$. Its covariance is scalar, so T_0 is a multiple of the identity (after a translation), and $\Lambda_0 = 11.3985358\dots$ is the number of Lemma 5.4. Thus $C_0 = 4 + 2\Lambda_0 = 26.7970716\dots \leq 26.8 \leq 27$ and $R_0 = R = 1.2825766950\dots \geq 1.28257669$; since the right-hand side of (9) decreases in C_0 and increases in R_0 , we obtain the first inequality of (2), and the second one follows from $\prod_{r=1}^d (1 + 27/r) = \binom{d+27}{27}$. By Theorem 6.1, $\liminf_d c^*(d)^{1/d} \geq R^{1/18} = (\sigma_K^2/\sigma_L^2)^{1/2}$, which is (3).

For the thresholds we use Lemma 6.2(c) with $d_0 = 9$. For the middle term of (2) ($C = 26.8$, $R = 1.28257669$) and $M = 725$, the two hypotheses $B(6516) \leq 1$ and $B(6533) > 1$ are inequalities between rational numbers: $B(d) > 1$ if and only if $R^{\lfloor d/9 \rfloor} > (d+1) \prod_{r=1}^d (1 + C/r)$. Our program verifies them in exact integer arithmetic (Section 8); so this term exceeds 1 if and only if $d \geq 6525$. In the same way, the right-hand side of (2) ($C = 27$) exceeds 1 if and only if $d \geq 6570$ ($M = 730$). $\square$

Proof of Corollary 1.3. Combine (3) with Theorem 1.1: $\log c^*(d) \geq cd$ for all large d with $c = \log 1.01392 > 0$, and $\log c^*(d) \leq C'd$ for all $d \geq 1$. $\square$

Remark 6.3. (a) With the exact values $C_0 = 26.79707\dots$ and $R = 1.28257669502\dots$ the bound (9) also exceeds 1 if and only if $d \geq 6525$. Its d -th root is 0.9456 at $d = 900$, 1.003318 at $d = 9000$ and 1.012497 at $d = 90000$; the limit 1.0139221956... is approached slowly, because of the factor $\prod_r (1 + C_0/r) \approx d^{C_0}$. (b) Without Lemma 5.4, the bounded case of Lemma 3.4 gives $\Lambda_0 \leq q_0 = 93.068\dots$, hence $C_0 \leq 4 + 2q_0 = 190.1367\dots$; with this constant and the exact R , (9) exceeds 1 if and only if $d \geq 44694$. The rate (3) is the same. All thresholds stated in this note are certified as in the proof of Theorem 1.2.

Remark 6.4 (Vempala's covariance version). Let $c_V(d) := \sup\{\det A(K)/\det A(L) : K \subseteq L \text{ convex bodies in } \mathbb{R}^d\}$, the optimal constant in the covariance version of the question [Rad12, Theorem 3 and Proposition A.4]. Products of pairs show $c_V(d_1 + d_2) \geq c_V(d_1)c_V(d_2)$, and $1 \leq c_V(d) \leq (L_d/L_{B^d})^{2d}$, because $V_d(K) \leq V_d(L)$ and $L_L \geq L_{B^d}$ (the case $k = 2$ of [Rad12, Theorem 6], together with Lemma 2.1). By Fekete's lemma and [KL25, Theorem 1.2], $\gamma_V := \lim_d c_V(d)^{1/d} = \sup_d c_V(d)^{1/d}$ exists and is finite, and Proposition 5.3 gives $\gamma_V \geq R^{1/9} = 1.0280382\dots$. Applying Theorem 6.1 to pairs whose ratio $(\det A(K_0)/\det A(L_0))^{1/d_0}$ approaches γ_V (every convex body is bounded, so $\Lambda_0 < \infty$) gives $\liminf_d c^*(d)^{1/d} \geq \gamma_V^{1/2}$. So any pair with a larger ratio per dimension improves (3).

Remark 6.5 (Monte Carlo evidence, not used). For the pair of Proposition 5.3 itself, four independent Monte Carlo estimates with $2 \cdot 10^6$ to $4 \cdot 10^6$ random simplices per body gave $\mathbb{E}V_{K_0}/\mathbb{E}V_{L_0} = 1.109 \pm 0.002$, 1.108 ± 0.002 , 1.110 ± 0.001 and 1.111 ± 0.002 (one standard error), while the second moments agreed with Lemma 2.1 within 1.2 standard errors. This suggests that the first moment already fails to be monotone for this pair by about 11%, and that $c^*(9) \geq 1.1$; it is evidence, not a proof, and nothing in this note depends on it.

7 Open questions

1. The exact rate. Does $\lim_d c^*(d)^{1/d}$ exist, and what is it? We only know $1.01392 \leq \liminf \leq \limsup \leq \sqrt{2\pi e} \sup_n L_n$, and the exact value of $c^*(d)$ is unknown for every $d \geq 3$.

2. First versus second moment. For a convex body L put $\theta(L) = (\mathbb{E}V_L^2)^{1/2}/\mathbb{E}V_L$, an affine invariant. If $\sup\{\theta(L) : L \subset \mathbb{R}^d\} = e^{o(d)}$, then the Cauchy–Schwarz inequality for K and Remark 6.4 give $\lim_d c^*(d)^{1/d} = \gamma_V^{1/2}$. By Lemma 3.2 this would follow if all isotropic convex bodies satisfied (H) with one constant C_0 . For $P = I$ and $c = 0$, condition (H) is the inequality of the variance conjecture, which is recorded in another abstract of the same Oberwolfach report (record OWR-15215-005 of [ula26]). Rademacher's Khinchine-type inequality [Rad12, Lemma A.3] only gives $\theta(L) \leq c^{d+1}$.

3. Better pairs and small dimensions. The rate (3) comes from one explicit pair. A floating-point exploration of the parameters of Section 5 for $3 \leq n \leq 13$, which is not certified, found the largest per-dimension rate $R^{1/(2n)}$ at $n = 9$ (about 1.013925, with $h = \frac{5}{4}$ and $s_0 \approx 0.0925$, against 1.0139222 for $s_0 = \frac{1}{11}$); non-symmetric pairs were not searched. A better lower bound for γ_V , that is, a pair with a larger ratio per dimension (Remark 6.4), would improve (3). For moderate dimensions, a rigorous first-moment computation for polytopes, for instance a proof of $c^*(9) \geq 1.1$ as suggested by Remark 6.5, would give explicit non-trivial values, which our method does not provide below $d = 6525$.

8 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved in the text. The computations certify finite facts, namely the rational numbers in Proposition 5.3 and Lemma 5.4 and the integer inequalities behind the thresholds in Theorem 1.2 and Remark 6.3, and they provide independent checks. The reproducibility package (folder `reproducibility/` of the source archive) has three parts.

The exact program of this note, `check_note.py` (Python 3, standard library only, about 20 seconds), tests the statements as printed. (i) For the pair of Proposition 5.3 it decomposes L_0 into Δ and the 10 pyramids, and K_0 into Δ and 90 simplices (a staircase triangulation of each truncated pyramid, cross-checked by the signed decomposition into pyramids minus their cut-off tops), and computes in rational arithmetic, from Dirichlet moments of orders up to four: the volumes, the centroids, the covariance matrices (with an exact test that they are scalar), $\sigma_L^2 = \frac{149}{176}$, σ_K^2 , their ratio, the ratio $\det A(K_0)/\det A(L_0)$ in the coordinates $(\lambda_1, \dots, \lambda_9)$ without the symmetry argument (it equals $(\sigma_K^2/\sigma_L^2)^9$ exactly), the closed form of Lemma 5.2, $\mathbb{E}|Z|^4$, the scalar matrix $\mathbb{E}[|Z|^2 Z Z^T]$, Λ_0 , C_0 and q_0 . (ii) The same for the pairs with $n = 3, 4, 5$ listed after Proposition 5.3. (iii) For each of the four explicit bounds ((9) with the exact constants, the two bounds in (2), and the crude bound of Remark 6.3) the two exact integer comparisons required by Lemma 6.2(c), giving the thresholds 6525, 6525, 6570 and 44694, and the decimal digits of $R^{1/18}$ by an integer square root. (iv) Negative controls: with $s_0 = \frac{1}{10}$ the values of σ_K^2 and of the ratio change, and the threshold becomes 6552; moving one apex outwards by one percent destroys the scalar covariance; for $h = \frac{3}{2}$ the midpoint of two apexes lies in no piece of the union. (v) Lemma 5.1 at the certified parameters: the vertices satisfy the inequalities of Q , L_0 has 45 facets, and the descriptions of L_0 and K_0 as Q and as unions of pieces agree on 1500 random rational points. (vi) A floating-point evaluation of (5), (6), Lemma 2.1 for the ball and θ_d for $d \leq 1000$ (Remark 4.1). All checks passed.

The programs with which the results were first obtained (original/): exact closed forms, an exact triangulation of the convex hulls for $n = 3, 4, 5$ with certified supporting hyperplanes, a floating-point integration over the convex hulls for $n = 9$, the exact fourth moment, the Monte Carlo experiments of Remark 6.5, a Monte Carlo test of Lemma 3.4, and the exploration mentioned in Section 7. *The programs of the three verification runs* described below, with their recorded outputs (`verification_run_A/`, `verification_run_B/` and `independent_run_2/`).

Verification. The results were first obtained with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each re-derived the proofs of all lemmas and theorems and wrote its own programs. The first two examined the results before this note was written, the third examined the final text. The first run recomputed the pair in barycentric coordinates from Dirichlet moments, the determinant ratio without the symmetry argument, the fourth moment and the thresholds; it confirmed the convexity of L_0 and K_0 by the volumes of their convex hulls and on 50,000 exact random points, tested Lemmas 3.2 and 3.4 on exact discrete laws, and repeated the Monte Carlo experiment with its own sampler. The second run recomputed the same quantities

by a signed decomposition into simplices for four parameter sets, showed that its programs detect deliberately wrong inputs, confirmed the facet structure of L_0 and the non-convexity of the union for $h = \frac{3}{2}$, compared Miles’s formula with Lemma 2.1 and with Kingman’s closed form for $d \leq 1000$, searched numerically for violations of (H) for products of K_0 (the largest ratio found was 0.27 of the allowed bound), ran the original programs again (all outputs were reproduced exactly), and repeated the literature search. Neither of these two runs found a mathematical error. The corrections they required concern the presentation and the citations: the caveat that the explicit bound is trivial below $d = 6525$, stated in the abstract, the introduction and Theorem 1.2; the thresholds of each form of the bound; the numbering of Rademacher’s proposition and the status of the published text of [Rad12]; the year of [Rei18]; the precise description of K_0 ; a reference for Kingman’s formula; and the coplanar facets of L_0 . All were made. This note was then written, and its program `check_note.py` tests the final statements.

The third run examined the text of the note and the package line by line, as if nothing had been checked. It read the source and the cited theorems again (the published version of [Rad12] in the copy on the author’s web page, the arXiv versions of [KL25, RR16, KRR18], the published version of [Rei18]), re-derived every proof, and recomputed the pairs of Section 5 for the four parameter sets with its own exact program: the polytopes are taken from their descriptions by inequalities and triangulated by a pulling triangulation whose boundary is certified to be closed, and polynomials are integrated over the simplices by an exact polarization formula, without the decomposition into pyramids and without the symmetry argument. It obtained the rational numbers of Proposition 5.3 and Lemma 5.4, the thresholds 6525, 6525, 6570 and 44694 (and 6552 for $s_0 = \frac{1}{10}$) by exact integer comparisons, and the values of Remark 4.1; it tested Lemmas 3.2 and 3.4 on an exact non-symmetric discrete law, repeated the Monte Carlo experiment with its own sampler, ran all programs of the package again from the source archive (every recorded output was reproduced, up to running times and output format; two recorded outputs of the second run are excerpts of the full output, as the README of the package says), and repeated the literature search. It found no mathematical error and required six corrections, all made: the abstract and Section 1 now say that $\mathbb{E}V$ is within a polynomial factor of $(\mathbb{E}V^2)^{1/2}$ for products of copies of a fixed body, not for all product bodies; the numbering of the published Appendix of [Rad12] is related correctly to that of arXiv v1; [RR16] is cited by section; a sentence of Section 7 on γ_V was corrected; the README of the package was corrected; and this run is recorded here, in the verification report and in `independent_run_2/`. No proof depends on a computation beyond the exact evaluation of the stated rational numbers and integer inequalities.

Scope and priority. The note answers (Q2), read as a question about the optimal constant $c^*(d)$, by showing that $c^*(d)$ grows exponentially (Theorem 1.2 and Corollary 1.3); its explicit bound is non-trivial only for $d \geq 6525$, and the exact value of $c^*(d)$ for $d \geq 3$ and the limit of $c^*(d)^{1/d}$ remain open. For (Q1) it gives a self-contained explicit version (Theorem 1.1) of an answer that is in print. The question is Meckes’s [Mec06], as recorded by Reitzner [Rei17]. The equivalence of (Q1) with the slicing problem is due to Rademacher [Rad12, Appendix A] and is also stated in [RR16, Section 1]; the slicing problem was solved by Klartag and Lehec [KL25]. Lemma 2.1 is Rademacher’s [Rad12, Lemma 4], and Groemer’s theorem [Gro74], Miles’s formula [Mil71] and Kingman’s closed form [Kin69] are classical. New, to our knowledge, are the quantitative exponential lower bound (Theorem 1.2 and Corollary 1.3) and the product bound of Theorem 6.1 on which it rests; the explicit pair of Section 5 differs from the counterexamples of [Rad12, KRR18], and Lemmas 3.2 and 3.4 are elementary. The product idea may be folklore, but no earlier statement of it was found. No earlier quantitative lower bound for the optimal constant $c^*(d)$ was found in the sources searched: the source [Rei17], Meckes’s problem [Mec06], the papers [Rad12, RR16, KRR18, Rei18], the arXiv listings found

by queries on random simplices and monotonicity, on Meckes’s conjecture, on the expected volume of random simplices and on covariance matrices of convex bodies under inclusion, the works citing [Rad12, RR16, KRR18, KL25] in OpenAlex (13, 4, 1 and 19 works, whose titles were screened), zbMATH, and one web search; the three verification runs searched further in the same databases, in Semantic Scholar and with several web searches. A search that finds nothing is not a proof of novelty, and we claim no priority.

What was read: the source [Rei17] in the publisher’s open PDF of the report, pp. 518–520 (and pp. 514–515 for the variance conjecture); Meckes’s problem [Mec06]; [Rad12] completely in the arXiv version v1, and the typeset published version in the copy on the author’s web page, including its Appendix A, where the equivalence is Proposition A.4. The numbers (A1), Theorem A.2, Lemma A.3 and Proposition A.4 of the published Appendix are (10), Theorem 18, Lemma 19 and Proposition 20 in arXiv v1, and the other numbers we cite (Lemma 4 and Theorems 1, 3, 6 and 7) agree in both versions. We read, of [KL25], Section 1 of the arXiv version v1 (the journal version was not compared); the introductions of [RR16] and [KRR18] in their arXiv versions and of [Rei18] in its published open version. We did not read [Gro74, Kla06, Mil71, SW08], whose statements we take from [Rad12, Theorems 6, 7 and A.2], nor [Kin69], of which only the abstract was read (it announces an application to the n -dimensional case of Sylvester’s problem); the closed form (6) was checked numerically against (5). The bibliographic data were checked with Crossref.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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