

A Negative Answer to Wolansky's Question on Mutually Dominating Multiphase Measure Spaces

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Abstract

In an Oberwolfach report of 2017, Wolansky introduced a domination order between vector-valued measure spaces (X, σ) and (Y, η) with k components, defined by one Markov kernel that transports every σ_i to η_i , and asked whether two atomless spaces that dominate each other must admit deterministic maps $T : X \rightarrow Y$ and $S : Y \rightarrow X$ with $T_{\#}\sigma_i = \eta_i$ and $S_{\#}\eta_i = \sigma_i$. We show that the answer is no as posed, already for $k = 2$ and compact atomless spaces. Mutual domination is equality of the colour laws (Blackwell equivalence), and a deterministic transport has to respect colours, so it must carry colour fibres onto colour fibres. This gives explicit examples on $[0, 1]$ and $[0, 1]^2$ in which only one of the two maps exists, and an example on $[0, 1]$ in which neither exists. On the positive side, for standard Borel spaces with equal colour laws, T exists whenever the colour fibres of X are atomless, for instance for atomless spaces with $k = 1$ or with finitely many colours. We give necessary conditions in general and an exact classification for finite spaces. The results are consequences of the classical comparison of experiments (Blackwell, Le Cam, Torgersen), and we claim no novelty beyond answering this question with explicit atomless examples and the atomless-fibre positive case. This is an unrefereed note.

1 Introduction

Let $k \geq 1$. A k -phase space (X, σ) is a measurable space X with k finite positive measures $\sigma = (\sigma_1, \dots, \sigma_k)$, not all zero. We write $|\sigma| = \sigma_1 + \dots + \sigma_k$ for the total mass measure, and call (X, σ) *atomless* if $|\sigma|$ has no atoms (equivalently, no σ_i has atoms). Following Wolansky [Wol17], a *multiphase transport* from (X, σ) to (Y, η) is a measurable family $x \mapsto P_x$ of probability measures on Y such that

$$\int_X P_x(B) \sigma_i(dx) = \eta_i(B) \quad (1 \leq i \leq k, B \subseteq Y \text{ measurable}). \quad (1)$$

If one exists, (X, σ) *dominates* (Y, η) , written $(X, \sigma) \succeq (Y, \eta)$. A *deterministic transport* is a measurable map $T : X \rightarrow Y$ with $T_{\#}\sigma_i = \eta_i$ for all i ; it is the case $P_x = \delta_{T(x)}$ of (1). The report poses the following open question [Wol17, p. 388].

Problem 1.1 (Wolansky). *Let (X, σ) and (Y, η) be atomless k -phase spaces with $(X, \sigma) \succeq (Y, \eta)$ and $(Y, \eta) \succeq (X, \sigma)$. Do there exist deterministic transports $T : X \rightarrow Y$ and $S : Y \rightarrow X$, that is, $T_{\#}\sigma_i = \eta_i$ and $S_{\#}\eta_i = \sigma_i$ for all i ?*

(The report prints the second domination as $(\eta, Y) \succeq (\sigma, Y)$ and the index range as $i = 1, \dots, N$; we read Y as X and N as k . It imposes no topological hypothesis on X and Y : the counterexamples

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below live on compact metric spaces, the positive results on standard Borel spaces.) The problem is recorded as OWR-15212-005 in the UnsolvedMath corpus [ula26]. For $k = 1$ and atomless standard Borel spaces the answer is yes (Corollary 4.3).

Answer. No as posed; yes if the colour fibres are atomless (for example, for atomless spaces with $k = 1$ or with finitely many colours). To state this precisely we need two notions. Put $m = |\sigma|$ and $\kappa = d\sigma/dm : X \rightarrow \Delta_k$, where Δ_k is the simplex of probability vectors in $\mathbb{R}^k$. The *colour law* of (X, σ) is $\nu_\sigma = \kappa_\# m$, and the *colour fibres* are the conditional measures m_p of m given $\kappa = p$. For (Y, η) we write likewise $n = |\eta|$, $\lambda = d\eta/dn$, $\nu_\eta = \lambda_\# n$ and n_p .

Theorem 1.2. (a) (Lemma 2.2.) For standard Borel spaces, $(X, \sigma) \succeq (Y, \eta) \succeq (X, \sigma)$ if and only if $\nu_\sigma = \nu_\eta$.

(b) (Theorem 3.1.) For every $k \geq 2$ there are atomless k -phase spaces on $[0, 1]$ and $[0, 1]^2$ that dominate each other, such that either a deterministic S exists but no deterministic T , or neither T nor S exists.

(c) (Theorem 4.2.) If X, Y are standard Borel, $\nu_\sigma = \nu_\eta$ and the colour fibres of X are atomless, then a Borel T exists; if the fibres of both spaces are atomless, then T and S both exist. This covers atomless spaces with $k = 1$ or with finitely many colours. In general, if T exists then $T_\# m_p = n_p$ for the colour fibres m_p of X and n_p of Y , and if T and S both exist then the fibres have the same atom masses with the same multiplicities. For finite spaces the existence of T and S is classified exactly (Proposition 4.5).

The mechanism is simple. A deterministic T with $T_\# \sigma_i = \eta_i$ has to preserve the colour (Lemma 2.3), so it maps the points of X of colour p into the points of Y of colour p . Total masses can be atomless on both sides while the fibres are not: in Example 1 below the fibres of X are points and those of Y are intervals, so the experiment on Y is the experiment on X together with independent noise, which a deterministic map cannot create.

Relation to known theory. Both lemmas below are classical in disguise. Fix the state space $\{1, \dots, k\}$. If $\sigma \succeq \eta$ then $\eta_i(Y) = \sigma_i(X) =: w_i$ (take $B = Y$ in (1); components with $w_i = 0$ can be dropped), and (1) says exactly that the experiment $(\eta_i/w_i)_i$ is a garbling (Markov image) of $(\sigma_i/w_i)_i$, that is, X is at least as informative as Y in the sense of Blackwell [Bla51, Bla53]. The colour is the posterior probability vector of the parameter for the prior proportional to (w_i) , so the colour law is Blackwell's *standard measure* of the experiment [Bla51, Sect. 4] up to the fixed change of variables between priors, and mutual domination is Blackwell equivalence. Lemma 2.2 is contained in the part of the theory that does not need the Sherman–Stein converse: the reduction of an experiment to its standard experiment, whose outcome is the posterior, that is, the colour [Bla51, Thm 3]; the description of the sufficiency order by mean-preserving transformations [Bla51, Thm 6], [Bla53, Thm 2]; and the fact that sufficiency implies the convex-function inequality [Bla51, Thm 8], [Bla53, Thm 3]. The converse, that the convex-function inequality implies domination, is the Sherman–Stein–Blackwell theorem [She51, Bla53, Tor70, Tor91, FGPR23]; we do not use it. The report quotes the criterion from [Bla51], where it is credited to Bohnenblust, Shapley and Sherman (private communication). A deterministic transport T is a statistic, and Lemma 2.3 is a version of the fact that a statistic whose induced experiment is at least as informative as the original one is sufficient (the posterior depends on the observation only through the statistic) [HS49, Cam64], [FGPR23, Thm 5.4]. Example 1 of Theorem 3.1 is, moreover, an instance of [Bla51, Thm 11]: combining an experiment with an independent experiment whose distributions do not depend on the parameter does not change the standard measure. So Theorem 1.2(b) is a folklore-level consequence of sufficiency and Blackwell equivalence, the familiar difference between equivalence and isomorphism of experiments. We claim no novelty for

this mechanism or for Lemmas 2.2 and 2.3; what we add is the answer to Problem 1.1 with explicit atomless examples and the fibre analysis of when deterministic transports exist.

2 Preliminaries

Let $\Delta_k = \{p \in [0, 1]^k : \sum_i p_i = 1\}$. For a k -phase space (X, σ) with $m = |\sigma|$ we have $\sigma_i \leq m$, so $\sigma_i = \kappa_i m$ for measurable $\kappa_i \geq 0$ with $\sum_i \kappa_i = 1$. We fix a version $\kappa = (\kappa_1, \dots, \kappa_k) : X \rightarrow \Delta_k$, the *colour map* (written $d\sigma/d|\sigma|$ in [Wol17]). The *colour law* $\nu_\sigma = \kappa_\# m$ is a finite Borel measure on Δ_k and does not depend on the version. For (Y, η) we write $n = |\eta|$, $\lambda = d\eta/dn : Y \rightarrow \Delta_k$, and $\nu_\eta = \lambda_\# n$. If $T_\# \sigma_i = \eta_i$ for all i , summation gives $T_\# m = n$.

A map T from X is called *m-measurable* if it is measurable for the m -completion of the σ -algebra of X . Borel maps are m -measurable. Non-existence statements below are proved for m -measurable T , so they are stronger; existence statements produce Borel maps. If Y is standard Borel, an m -measurable T agrees m -a.e. with a Borel map, which is again a deterministic transport. In a standard Borel space a measure is atomless if and only if all points have measure zero.

Remark 2.1. The report does not say which norm is meant in $|\sigma|$; we use the total mass $\sum_i \sigma_i$. Domination is defined by kernels and does not depend on this choice, and neither does equality of colour laws. For a norm N on $\mathbb{R}^k$ the total variation is $N(\kappa) m$ and the colour is $\kappa/N(\kappa)$, so the colour law is $\nu_\sigma^N = (p \mapsto p/N(p))_\# (N(p) \nu_\sigma(dp))$, and conversely $\nu_\sigma = (u \mapsto u/\|u\|_1)_\# (\|u\|_1 \nu_\sigma^N(du))$. So $\nu_\sigma = \nu_\eta$ if and only if $\nu_\sigma^N = \nu_\eta^N$, e.g. for the Euclidean norm.

Lemma 2.2 (Blackwell equivalence). *Let (X, σ) and (Y, η) be k -phase spaces.*

- (a) *If $(X, \sigma) \succeq (Y, \eta)$ then $m(X) = n(Y)$ and $\int F \circ \lambda dn \leq \int F \circ \kappa dm$ for every continuous convex $F : \Delta_k \rightarrow \mathbb{R}$.*
- (b) *If $(X, \sigma) \succeq (Y, \eta)$ and $(Y, \eta) \succeq (X, \sigma)$ then $\nu_\sigma = \nu_\eta$.*
- (c) *If Y is standard Borel and $\nu_\sigma = \nu_\eta$ then $(X, \sigma) \succeq (Y, \eta)$. So for standard Borel X and Y , mutual domination is equivalent to $\nu_\sigma = \nu_\eta$.*

Proof. (a) Let $\pi(dx, dy) = m(dx)P_x(dy)$. Its first marginal is m , because P_x is a probability measure. Summing (1) over i shows that its second marginal is n ; in particular $m(X) = n(Y)$. For measurable $B \subseteq Y$,

$$\int \kappa_i(x) \mathbf{1}_B(y) d\pi = \int \kappa_i(x) P_x(B) m(dx) = \int P_x(B) \sigma_i(dx) = \eta_i(B) = \int \lambda_i(y) \mathbf{1}_B(y) d\pi.$$

Hence for every affine ℓ on Δ_k we have $\int \ell(\kappa) \mathbf{1}_B(y) d\pi = \int \ell(\lambda) \mathbf{1}_B(y) d\pi$. A continuous convex F on the compact convex set Δ_k is the supremum of its affine minorants, and countably many suffice (the minorants are parametrised by a subset of $\mathbb{R}^{k+1}$, which is separable). Let $\rho(B) = \int F(\kappa) \mathbf{1}_B(y) d\pi$ and let $\psi = d\rho/dn$ (the y -marginal of π is n and F is bounded). For each minorant ℓ , $\rho(B) \geq \int \ell(\kappa) \mathbf{1}_B d\pi = \int_B \ell(\lambda) dn$ for all B , so $\psi \geq \ell \circ \lambda$ n -a.e. Taking the supremum over the countable family gives $\psi \geq F \circ \lambda$ n -a.e., and therefore $\int F \circ \kappa dm = \rho(Y) = \int \psi dn \geq \int F \circ \lambda dn$.

(b) Let f be a polynomial on $\mathbb{R}^k$. The Hessian of f is bounded on Δ_k , so for large K the polynomial $f + K|p|^2$ has a positive semidefinite Hessian on Δ_k and is convex along every segment in Δ_k . The same holds for $K|p|^2$. Applying (a) in both directions to both functions gives $\int F d\nu_\sigma = \int F d\nu_\eta$ for $F = f + K|\cdot|^2$ and $F = K|\cdot|^2$. Subtracting, $\int f d\nu_\sigma = \int f d\nu_\eta$ for all polynomials f . Polynomials are dense in $C(\Delta_k)$, so $\nu_\sigma = \nu_\eta$.

(c) Let $\nu = \nu_\eta = \lambda_\# n$. Since Y is standard Borel there is a Borel kernel $p \mapsto n_p$ of probability measures on Y with $n = \int n_p \nu(dp)$ and $n_p(\lambda = p) = 1$ for ν -a.e. p (disintegration theorem, e.g.

[Kal21]). Put $P_x = n_{\kappa(x)}$. Then $x \mapsto P_x(B)$ is measurable, being a Borel kernel composed with κ . The set of p with $n_p(\lambda = p) < 1$ is ν -null, so its preimage under κ is m -null (as $\kappa_{\#}m = \nu$) and does not affect the integrals. Using $\kappa_{\#}m = \nu$ and $\lambda_i = p_i$ n_p -a.s.,

$$\begin{aligned} \int P_x(B) \sigma_i(dx) &= \int n_{\kappa(x)}(B) \kappa_i(x) m(dx) = \int p_i n_p(B) \nu(dp) \\ &= \int \left(\int_B \lambda_i dn_p \right) \nu(dp) = \int_B \lambda_i dn = \eta_i(B). \end{aligned} \quad \square$$

In (c) only Y needs to be standard Borel; X needs a measurable colour map. The next lemma is the key to all non-existence results.

Lemma 2.3 (Rigidity). *Let (X, σ) and (Y, η) be k -phase spaces with $(Y, \eta) \succeq (X, \sigma)$, and let $T : X \rightarrow Y$ be m -measurable with $T_{\#}\sigma_i = \eta_i$ for all i . Then $T_{\#}m = n$ and $\lambda \circ T = \kappa$ m -almost everywhere. Symmetrically, if $(X, \sigma) \succeq (Y, \eta)$ and $S : Y \rightarrow X$ is n -measurable with $S_{\#}\eta_i = \sigma_i$ for all i , then $\kappa \circ S = \lambda$ n -almost everywhere.*

Proof. We have $T_{\#}m = n$, so $\lambda \circ T$ is defined m -a.e. Using $\sigma_i = \kappa_i m$, $\eta_i = \lambda_i n$ and $T_{\#}\sigma_i = \eta_i$,

$$\int \kappa_i (\lambda_i \circ T) dm = \int \lambda_i \circ T d\sigma_i = \int \lambda_i d\eta_i = \int \lambda_i^2 dn, \quad \int (\lambda_i \circ T)^2 dm = \int \lambda_i^2 dn.$$

Summing over i , $\int |\kappa - \lambda \circ T|^2 dm = \int |\kappa|^2 dm - \int |\lambda|^2 dn$. By Lemma 2.2(a) for $(Y, \eta) \succeq (X, \sigma)$ with the convex function $F = |\cdot|^2$, $\int |\kappa|^2 dm \leq \int |\lambda|^2 dn$. So the left side is ≤ 0 , hence 0. The second statement is the first with the roles of the spaces exchanged. $\square$

3 Three counterexamples

Here Leb is Lebesgue measure, $\{t\} = t - \lfloor t \rfloor$, and $k = 2$. We put $\theta(y) = \{2y\}$ and $\vartheta(x) = \{3x\}$ on $[0, 1)$. The maps θ and ϑ preserve Leb: for Borel $A \subseteq [0, 1)$, $\text{Leb}(\theta^{-1}A) = \frac{1}{2} \text{Leb}(A) + \frac{1}{2} \text{Leb}(A)$ (two branches of slope 2), and similarly for ϑ (three branches of slope 3). We put $\theta(1) = \vartheta(1) = 0$.

Theorem 3.1. *Each of the following pairs of 2-phase spaces is atomless, lives on compact metric spaces with their Borel σ -algebras, and satisfies $(X, \sigma) \succeq (Y, \eta) \succeq (X, \sigma)$.*

- (1) $X = [0, 1]$, $\sigma = (x dx, (1 - x) dx)$; $Y = [0, 1]^2$, $\eta = (y_1 dy, (1 - y_1) dy)$. A Borel S exists, but there is no m -measurable T .
- (2) $X = Y = [0, 1]$, $\sigma = (x dx, (1 - x) dx)$, $\eta = (\theta(y) dy, (1 - \theta(y)) dy)$. A Borel S exists, but there is no m -measurable T .
- (3) $X = Y = [0, 1]$, $\sigma = (\vartheta(x) dx, (1 - \vartheta(x)) dx)$, $\eta = (\theta(y) dy, (1 - \theta(y)) dy)$. There is no m -measurable T and no n -measurable S .

For every $k \geq 2$ the same holds after repeating the second component $k - 2$ times (the extra components repeat the conditions for $i = 2$ in (1) and in $T_{\#}\sigma_i = \eta_i$).

In (2) and (3) we have $|\sigma| = |\eta| = \text{Leb}$. The colour fibres (with their conditional masses) are: in (1) a point against an interval; in (2) a point of mass 1 against two points of mass $\frac{1}{2}$; in (3) three points of mass $\frac{1}{3}$ against two points of mass $\frac{1}{2}$.

Proof of (1). Domination. For $x \in [0, 1]$ let $P_x = \delta_x \otimes \text{Leb}$, that is, $P_x(B) = \text{Leb}\{y_2 : (x, y_2) \in B\}$; this is a Borel kernel. By Fubini, $\int P_x(B) x dx = \int_B y_1 dy = \eta_1(B)$, and in the same way for $1 - x$. Conversely let $Q_y = \delta_{y_1}$. Then $\int Q_y(A) y_1 dy = \int_0^1 \mathbf{1}_A(y_1) y_1 dy_1 = \sigma_1(A)$, and the same for the second component. The last identity says that $S(y) = y_1$ satisfies $S_{\#}\eta_i = \sigma_i$.

No T . Suppose $T = (T_1, T_2) : [0, 1] \rightarrow [0, 1]^2$ is m -measurable with $T_{\#}\sigma_i = \eta_i$. Here $m = \text{Leb}$, $n = \text{Leb}^2$, $\kappa(x) = (x, 1 - x)$ and $\lambda(y) = (y_1, 1 - y_1)$. By Lemma 2.3, $\lambda(T(x)) = \kappa(x)$ a.e., so $T_1(x) = x$ for a.e. x . Choose a Borel function $\tilde{T}_2$ that equals T_2 a.e. (possible because T_2 is Lebesgue measurable) and let $G = \{(y_1, \tilde{T}_2(y_1))\}$, a Borel set. By Fubini, $\text{Leb}^2(G) = \int \text{Leb}\{y_2 = \tilde{T}_2(y_1)\} dy_1 = 0$, so $n(G) = 0$. But $T(x) \in G$ for a.e. x , so $(T_{\#}m)(G) = 1$. This contradicts $T_{\#}m = n$. $\square$

Proof of (2). Domination. Let $P_x = \frac{1}{2}(\delta_{x/2} + \delta_{(x+1)/2})$. Substituting $y = x/2$ and $y = (x+1)/2$,

$$\int P_x(B) x dx = \frac{1}{2} \int_0^1 (\mathbf{1}_B(\frac{x}{2}) + \mathbf{1}_B(\frac{x+1}{2})) x dx = \int_{B \cap [0, \frac{1}{2})} 2y dy + \int_{B \cap [\frac{1}{2}, 1)} (2y-1) dy = \int_B \theta dy = \eta_1(B).$$

The component $1 - x$ is the same computation. Let $Q_y = \delta_{\theta(y)}$. Since $\theta_{\#} \text{Leb} = \text{Leb}$, $\int Q_y(A) \theta(y) dy = \int_A p dp = \sigma_1(A)$, likewise for $1 - p$. Hence $S = \theta$ is a deterministic transport $Y \rightarrow X$.

No T . Suppose T is m -measurable with $T_{\#}\sigma_i = \eta_i$. By Lemma 2.3, $\theta(T(x)) = x$ a.e., so $T(x) \in \{x/2, (x+1)/2\}$. Let $A = \{x : T(x) < \frac{1}{2}\}$ (measurable). For Borel $B \subseteq [0, \frac{1}{2})$ we get $T^{-1}(B) = A \cap 2B$ up to a null set, and $T_{\#}m = n$ gives $\text{Leb}(B) = \text{Leb}(A \cap 2B)$. With $E = 2B$, which is an arbitrary Borel subset of $[0, 1)$, this says $\text{Leb}(A \cap E) = \frac{1}{2} \text{Leb}(E)$ for all E . By the Lebesgue density theorem $\mathbf{1}_A = \frac{1}{2}$ a.e., which is absurd. $\square$

Proof of (3). Domination. Let $P_x = \frac{1}{2}(\delta_{\vartheta(x)/2} + \delta_{(\vartheta(x)+1)/2})$, the kernel of (2) composed with ϑ . Since $\vartheta_{\#} \text{Leb} = \text{Leb}$ and the colour components are $\vartheta(x)$ and $1 - \vartheta(x)$, the computation of (2) gives $\int P_x(B) \sigma_i(dx) = \eta_i(B)$. Conversely let $Q_y = \frac{1}{3} \sum_{j=0}^2 \delta_{(\theta(y)+j)/3}$. Using $\theta_{\#} \text{Leb} = \text{Leb}$ and substituting $x = (p+j)/3$,

$$\int Q_y(A) \theta(y) dy = \frac{1}{3} \sum_{j=0}^2 \int_0^1 \mathbf{1}_A(\frac{p+j}{3}) p dp = \sum_{j=0}^2 \int_{A \cap [\frac{j}{3}, \frac{j+1}{3})} (3x-j) dx = \int_A \vartheta dx = \sigma_1(A),$$

and the same for the second component.

No T . Suppose T is m -measurable with $T_{\#}\sigma_i = \eta_i$. By Lemma 2.3, $\theta(T(x)) = \vartheta(x)$ a.e., so $T(x) \in \{\vartheta(x)/2, (\vartheta(x)+1)/2\}$. Let $A = \{T < \frac{1}{2}\}$. For Borel $B \subseteq [0, \frac{1}{2})$, $T^{-1}(B) = A \cap \vartheta^{-1}(2B)$ a.e., and $T_{\#}m = n$ gives $\text{Leb}(B) = \text{Leb}(A \cap \vartheta^{-1}(2B))$. For Borel $E \subseteq [0, 1)$,

$$\text{Leb}(A \cap \vartheta^{-1}E) = \sum_{j=0}^2 \text{Leb}\left(A \cap \frac{E+j}{3}\right) = \int_E h dp, \quad h(p) = \frac{1}{3} \sum_{j=0}^2 \mathbf{1}_A(\frac{p+j}{3}) \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}.$$

Thus $\int_E h = \frac{1}{2} \text{Leb}(E)$ for every E , so $h = \frac{1}{2}$ a.e., which is impossible.

No S . Suppose S is n -measurable with $S_{\#}\eta_i = \sigma_i$. By Lemma 2.3 (second statement), $\vartheta(S(y)) = \theta(y)$ a.e., so $S(y) = (\theta(y) + j(y))/3$ with $j(y) \in \{0, 1, 2\}$. Let $A_0 = \{S < \frac{1}{3}\}$. For Borel $B \subseteq [0, \frac{1}{3})$, $S^{-1}(B) = A_0 \cap \theta^{-1}(3B)$ a.e., and $S_{\#}n = m$ gives $\text{Leb}(B) = \text{Leb}(A_0 \cap \theta^{-1}(3B))$. For Borel $E \subseteq [0, 1)$, $\text{Leb}(A_0 \cap \theta^{-1}E) = \int_E g dp$ with $g(p) = \frac{1}{2} \sum_{l=0}^1 \mathbf{1}_{A_0}(\frac{p+l}{2}) \in \{0, \frac{1}{2}, 1\}$. So $\int_E g = \frac{1}{3} \text{Leb}(E)$ for all E , hence $g = \frac{1}{3}$ a.e., which is impossible. $\square$

Hence Problem 1.1 has a negative answer, already for $k = 2$ and with Lebesgue measure on both sides.

Remark 3.2 (Not a failure of approximation). Let $T_N(x) = (x, \{Nx\})$ in Example 1. Then $\int f(T_N) d\sigma_i \rightarrow \int f d\eta_i$ for $i = 1, 2$ and every continuous f on $[0, 1]^2$. To see this, put $x = (j + s)/N$ on each interval $[j/N, (j + 1)/N)$:

$$\int f(T_N x) x dx = \sum_j \frac{1}{N} \int_0^1 f\left(\frac{j+s}{N}, s\right) \frac{j+s}{N} ds$$

is a Riemann sum in the first variable and exact in the second, and it tends to $\int_0^1 \int_0^1 f(u, s) u du ds$ by uniform continuity (likewise for $1 - x$). So η_i is a weak limit of $(T_N)_\# \sigma_i$, although the exact equation $T_\# \sigma_i = \eta_i$ has no solution. For $i = 1$ and $f(y) = y_2$ or $f(y) = y_1 y_2$ the error is exactly $1/(12N)$ (exact arithmetic; see the Verification paragraph).

4 Colour fibres: positive results and necessary conditions

In this section (X, σ) and (Y, η) are standard Borel k -phase spaces with $\nu_\sigma = \nu_\eta =: \nu$, so that they dominate each other by Lemma 2.2(c). By the disintegration theorem there are Borel probability kernels $p \mapsto m_p$ on X and $p \mapsto n_p$ on Y with $m = \int m_p \nu(dp)$, $n = \int n_p \nu(dp)$, and $m_p(\kappa = p) = 1 = n_p(\lambda = p)$ for ν -a.e. p ; they are unique up to ν -null sets. These are the *colour fibres*. By the remark in Section 2 we may take deterministic transports Borel.

Lemma 4.1 (Atoms). *Let α, β be finite measures on standard Borel spaces with $\alpha(X) = \beta(Y)$, and let $T : X \rightarrow Y$, $S : Y \rightarrow X$ be Borel with $T_\# \alpha = \beta$ and $S_\# \beta = \alpha$. Then α and β have the same atom masses, with the same multiplicities.*

Proof. If x is an atom of α then $\beta(\{T(x)\}) = \alpha(T^{-1}\{T(x)\}) \geq \alpha(\{x\})$, so $T(x)$ is an atom of β of at least the same mass; the same holds for S . Suppose α has atoms and let a be the largest atom mass. The largest atom mass b of β satisfies $b \geq a$, and by symmetry $a \geq b$, so $a = b$. Let $x_1, \dots, x_r$ and $y_1, \dots, y_s$ be the atoms of mass a of α and of β . Each $T(x_j)$ is an atom of mass $\geq a$, hence of mass a . If $T(x_j) = T(x_l) = y$ with $j \neq l$, then $\beta(\{y\}) \geq 2a > b$, impossible; so T is injective on $\{x_j\}$ and $r \leq s$. By symmetry $s \leq r$, so T maps $\{x_j\}$ bijectively onto $\{y_l\}$. Moreover $\alpha(T^{-1}\{y_l\}) = a = \alpha(\{x_j\})$ for $y_l = T(x_j)$, so $T^{-1}\{y_l\}$ differs from $\{x_j\}$ by an α -null set; the same holds for S . Let α', β' be the restrictions of α, β to the complements of $\{x_j\}$ and $\{y_l\}$, and let T', S' agree with T, S off a null set (put a fixed point on the null set). The preceding facts show $T'_\# \alpha' = \beta'$ and $S'_\# \beta' = \alpha'$. If one of α', β' has atoms, the argument applies again; if α' has no atoms then neither does β' , by the first sentence applied to S' . Only finitely many atoms have mass $\geq c$ for each $c > 0$, so every atom is removed after finitely many steps, and the same masses are removed from both sides at each step. $\square$

Theorem 4.2. *Let $(X, \sigma), (Y, \eta)$ be standard Borel k -phase spaces with $\nu_\sigma = \nu_\eta = \nu$ and colour fibres m_p, n_p .*

- (i) (Necessary.) *If a Borel T with $T_\# \sigma_i = \eta_i$ exists, then $T_\# m_p = n_p$ for ν -a.e. p . In particular every atom of m_p is mapped into an atom of n_p of at least the same mass.*
- (ii) (Sufficient.) *If m_p is atomless for ν -a.e. p , then there is a Borel T with $T_\# \sigma_i = \eta_i$ and $\lambda \circ T = \kappa$ m -a.e., whatever the fibres n_p are.*
- (iii) (Necessary for both.) *If Borel T and S both exist, then for ν -a.e. p the measures m_p and n_p have the same atom masses with the same multiplicities (and hence diffuse parts of equal mass).*
- (iv) (Sufficient for both.) *If m_p and n_p are both atomless for ν -a.e. p , then T and S both exist.*

Proof. (i) By Lemma 2.3, $\lambda \circ T = \kappa$ m -a.e. Let f be a bounded Borel function on Δ_k and g a bounded Borel function on Y . Then

$$\int f(\kappa) g(T) dm = \int f(\lambda \circ T) g \circ T dm = \int f(\lambda) g dn.$$

Disintegrating, with $\kappa = p$ m_p -a.s. and $\lambda = p$ n_p -a.s., the left side equals $\int f(p) (\int g \circ T dm_p) \nu(dp)$ and the right side $\int f(p) (\int g dn_p) \nu(dp)$. As f is arbitrary, $\int g \circ T dm_p = \int g dn_p$ for ν -a.e. p . Take g in a countable family of indicators of a generating π -system of Y , so that $T_{\#}m_p = n_p$ ν -a.e. If x is an atom of m_p , then $n_p(\{T(x)\}) = m_p(T^{-1}\{T(x)\}) \geq m_p(\{x\})$.

(ii) Fix Borel injections $\varphi : X \rightarrow [0, 1]$ and $\psi : Y \rightarrow [0, 1]$; then $\psi(Y)$ is Borel and $\psi^{-1} : \psi(Y) \rightarrow Y$ is Borel (Lusin–Souslin theorem, see [Kec95]). Put $G_p(t) = m_p(\varphi \leq t)$ and $H_p(t) = n_p(\psi \leq t)$. These are Borel in p and right-continuous in t , hence jointly Borel. Let $Q_p(u) = \inf\{t \in [0, 1] : H_p(t) \geq u\}$ for $u \in (0, 1]$ and $Q_p(0) = 0$; then $\{(p, u) : Q_p(u) \leq t\} = \{H_p(t) \geq u\}$ shows that Q is jointly Borel. Define $u(x) = G_{\kappa(x)}(\varphi(x))$ and

$$T(x) = \psi^{-1}(Q_{\kappa(x)}(u(x))) \quad \text{if } Q_{\kappa(x)}(u(x)) \in \psi(Y), \quad T(x) = y_0 \text{ otherwise,}$$

for a fixed $y_0 \in Y$. This T is Borel. Let p be such that m_p is atomless and $m_p(\kappa = p) = n_p(\lambda = p) = 1$. Since φ is injective, G_p is continuous, so u is uniformly distributed under m_p (probability integral transform), and $\kappa = p$ m_p -a.s. Therefore $Q_p(u)$ has the law of ψ under n_p , which is carried by $\psi(Y)$, and $T_{\#}m_p = n_p$. Hence $T_{\#}m = \int T_{\#}m_p \nu(dp) = n$. Also $\lambda \circ T = p = \kappa$ m_p -a.s., so $\lambda \circ T = \kappa$ m -a.e. Finally, for bounded Borel g ,

$$\int g \circ T d\sigma_i = \int g(T) \kappa_i dm = \int g(T) \lambda_i(T) dm = \int g \lambda_i dn = \int g d\eta_i,$$

that is, $T_{\#}\sigma_i = \eta_i$.

(iii) By (i) for T and, with the roles of the spaces exchanged, for S , for ν -a.e. p we have $T_{\#}m_p = n_p$ and $S_{\#}n_p = m_p$. Apply Lemma 4.1 to $\alpha = m_p$ and $\beta = n_p$. Both have total mass 1, so their diffuse masses coincide.

(iv) Apply (ii) to $(X, \sigma) \rightarrow (Y, \eta)$ and to $(Y, \eta) \rightarrow (X, \sigma)$. □

Corollary 4.3. *Let (X, σ) and (Y, η) be atomless standard Borel k -phase spaces with the same colour law ν . Then T and S both exist in each of the following cases: (a) $k = 1$; (b) ν is purely atomic (for example, κ and λ take finitely many values).*

Proof. If $\nu = \sum_j a_j \delta_{p_j}$ with $a_j > 0$, then $m_{p_j} = m(\cdot \cap \{\kappa = p_j\})/a_j$, which is atomless because m is; likewise for n_{p_j} . Now use Theorem 4.2(iv). For $k = 1$, Δ_k is a point, so ν is a Dirac mass (times $m(X) = n(Y)$). □

Remark 4.4 (What is not claimed). Theorem 4.2 does not characterise the pairs for which T exists when the fibres have atoms: (i) is necessary and (ii) is sufficient, and they do not meet. Likewise the converse of (iii) is open to us for general standard Borel spaces. Both converses need a measurable choice, in p , of a packing or a matching of atoms, which we have not pursued.

For finite spaces everything can be decided.

Proposition 4.5 (Finite spaces). *Let X, Y be finite sets, $\sigma(x), \eta(y) \in [0, \infty)^k \setminus \{0\}$, and $\nu_\sigma = \nu_\eta$. Write $m(x) = |\sigma(\{x\})|$, $n(y) = |\eta(\{y\})|$, $X_p = \{\kappa = p\}$, $Y_p = \{\lambda = p\}$.*

- (a) *A deterministic T exists if and only if for every colour p the numbers $\{m(x) : x \in X_p\}$ can be partitioned into $|Y_p|$ groups whose sums are exactly the numbers $n(y)$, $y \in Y_p$.*

- (b) *Both T and S exist if and only if, for every p , the multisets $\{m(x) : x \in X_p\}$ and $\{n(y) : y \in Y_p\}$ coincide, if and only if there is a bijection $f : X \rightarrow Y$ with $\eta(f(x)) = \sigma(x)$.*

Proof. By Lemma 2.2(c) the spaces dominate each other, and Lemma 2.3 applies. Every point has positive mass, so $\lambda(T(x)) = \kappa(x)$ holds for *every* x , and T maps X_p into Y_p . For $y \in Y_p$ the condition $T_{\#}\sigma = \eta$ at y reads $\sum_{T(x)=y} m(x)p = n(y)p$, i.e. $\sum_{T(x)=y} m(x) = n(y)$, and $T^{-1}(y) \neq \emptyset$ because $n(y) > 0$. So the sets $T^{-1}(y)$, $y \in Y_p$, form a partition as in (a). Conversely, a partition as in (a) defines such a T . This proves (a).

For (b), consider (1) T and S exist; (2) the multisets coincide for every p ; (3) a bijection f with $\eta(f(x)) = \sigma(x)$ exists. If (1) holds, then $T : X_p \rightarrow Y_p$ and $S : Y_p \rightarrow X_p$ are surjective, so $|X_p| = |Y_p|$, T is bijective on X_p and $m(x) = n(T(x))$; this gives (2), and (3) with $f = T$, since $\eta(T(x)) = n(T(x))p = m(x)p = \sigma(x)$. If (2) holds, matching equal masses within each colour class gives (3). If (3) holds, f and f^{-1} are deterministic transports, which is (1). $\square$

For example, two points with $\sigma = (1, 1)$ and one point with $\eta = (2, 2)$ give a map T and no S ; three points with $\sigma = (2, 2)$ and two points with $\eta = (3, 3)$ give neither T nor S (the finite shadow of Example 3).

Verification. The claims were checked as follows.

1. A self-contained program by the author, `lead_checks.py` (standard library only, about two seconds), works in exact rational arithmetic. It verifies the kernel identities of Examples 2 and 3, in both directions and for both components, on all intervals with endpoints in $\frac{1}{d}\mathbb{Z}$, $d \leq 8$ (960 identities; a wrong kernel is detected as a negative control). It computes the second moments $\int |\kappa|^2 dm = \int |\lambda|^2 dn = \frac{2}{3}$ and the value sets of h and g in Example 3, checks Lemma 4.1 on all 12,647 ordered pairs of integer partitions of $n \leq 12$ (maps in both directions exist if and only if the multisets coincide), and computes the errors in Remark 3.2.
2. A program by the author, `finite_exhaustive.py` (standard library only, about sixteen seconds), enumerates finite spaces with integer vector masses ($k = 2$ with entries ≤ 2 and ≤ 5 points or entries ≤ 3 and ≤ 4 points; $k = 3$ with entries ≤ 2 and ≤ 4 points). For all 43,619 ordered pairs with equal colour law it certifies mutual domination by explicit kernels, brute-forces all maps T and S , and compares the outcome with Proposition 4.5; the assertions pass. For $k = 2$, entries ≤ 3 , at most 4 points, the four cases (T and S ; T only; S only; neither) occur 3875, 882, 882 and 1422 times.
3. A first independent verification run, which was AI-assisted, re-derived the lemmas, the examples and the fibre theorem from an earlier write-up of this note, re-ran both programs with identical output, and wrote separate code, `indep_check.py`. For $k = 2$, vectors in $\{0, \dots, 3\}^2$ and at most 4 points, it decides domination by linear programming rather than by Lemma 2.2, and the existence of T and S by brute force. On 6000 random ordered pairs with equal total mass, mutual domination holds if and only if the colour laws agree, and the existence of T agrees with Proposition 4.5(a) in every case; the outcomes (T and S ; neither; T only; S only) were 115, 33, 31 and 25. The run found one gap, a measurability point in Example 1 (the graph of a merely Lebesgue measurable function), which is repaired in the text.
4. A second independent verification run, also AI-assisted, re-read the source report, re-derived every lemma, example and theorem of the present text, and wrote new code (`run2_finite.py`, `run2_blackwell_k2.py`, `run2_examples.py`). Domination is decided by an exact linear program over the rationals (a hand-written simplex, cross-checked against a standard solver), never by Lemma 2.2; the existence of T and S is decided by brute force over all maps. For $k = 2$

and $k = 3$ with small integer masses, on all 12,348 ordered pairs with equal colour law and on 47,318 sampled ordered pairs with equal total vector and different colour law, mutual domination holds if and only if the colour laws agree; T exists if and only if the packing criterion of Proposition 4.5(a) holds; and T and S both exist if and only if the mass multisets coincide. The outcomes (T and S ; T only; S only; neither) were 8406, 1212, 1212 and 1518; for $k = 2$, entries ≤ 3 , at most 4 points, they are 3875, 882, 882 and 1422, as in item 2. For $k = 2$ exact domination agrees with Blackwell’s integral criterion on 17,407 ordered pairs. The kernel identities of Examples 2 and 3 were checked again in exact arithmetic on all intervals with endpoints in $\frac{1}{d}\mathbb{Z}$, $d \leq 12$, with wrong-kernel negative controls, and the errors of Remark 3.2 for $N \leq 60$. The earlier programs were re-run from the released archive with identical output. The run found no mathematical error; it led to changes in wording and credits.

These computations support the finite classification and the elementary identities; the results for infinite spaces rest on the proofs above.

Scope and priority. Theorem 3.1 answers Problem 1.1 as stated in [Wol17], for atomless compact spaces and every $k \geq 2$; the report imposes no topological hypothesis. Lemmas 2.2 and 2.3 are classical: mutual domination is Blackwell equivalence of experiments, and a deterministic transport with a not less informative image is a sufficient statistic [Bla51, Bla53, She51, Cam64, HS49, Tor70, Tor91, FGPR23]. Theorem 3.1(1) is an experiment together with an independent uniform variable [Bla51, Thm 11], and the examples are instances of the difference between equivalence and isomorphism of experiments. This phenomenon is folklore-level, and we claim no novelty beyond answering this question with explicit atomless examples and the atomless-fibre positive case. The fibre-packing mechanism of Examples 2 and 3, Theorem 4.2 and Proposition 4.5 are not written out in the sources we read, which is a statement about those sources and not a priority claim.

We read the report [Wol17], Section 5 of Gover–Wolansky [GW20], Chapter 5 of Wolansky’s book [Wol20] (Definition 5.3.1 and Theorem 5.3, which give the domination order and the convex-function criterion and credit a version of it to Blackwell), and Gover’s thesis [Gov25], in particular its Sections 3.4 and 3.9. Apart from the report, none of them poses or answers the question. Gover–Wolansky call the domination order a partial order and Gover [Gov25, Claim 3.34] a preorder; by Theorem 3.1 it is not antisymmetric on isomorphism classes of spaces (it is a partial order on classes of spaces with equal colour law). Section 3.9 of [Gov25] characterises, under extra hypotheses, the existence of some transport map T for $X \succeq Y$ by a sequential domination condition; we did not use it. Of the classical literature we read Blackwell’s paper [Bla51] and the statements of the main theorems in [Bla53] (pp. 265–267), and Section 5 of [FGPR23], from which Theorem 5.6 is quoted. We did not see the papers and books of Sherman, Halmos and Savage, Le Cam and Torgersen [She51, HS49, Cam64, Cam86, Tor70, Tor91], and what we say about them is not taken from their text. We searched arXiv, Crossref, OpenAlex and zbMATH, and made three web searches, and found no source that contains the examples or answers the question. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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