

The Conformal Parametrization of Nonnegatively Curved Surfaces Is Not a Homeomorphism at Integer Smoothness

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Abstract

Let $M_0 = \mathbb{C}$ and $M_1 = S^2$, with complete metrics g_κ of constant curvature $\kappa \in \{0, 1\}$. Every complete smooth metric of nonnegative curvature on M_κ can be written uniquely as $\varphi^*(e^{-2u}g_\kappa)$, where φ is an orientation-preserving diffeomorphism fixing 0 and 1 (and ∞ if $\kappa = 1$) and u is a smooth function. Work of Belegardek, Hu and Banach shows that the map $(u, \varphi) \mapsto \varphi^*(e^{-2u}g_\kappa)$ is a homeomorphism when metrics and functions carry the topology of $C^{k+\alpha}$ convergence on compact sets and diffeomorphisms that of $C^{k+1+\alpha}$ convergence, with $0 < \alpha < 1$. Belegardek asked whether this remains true for $\alpha = 0$, and expected that it does not. We show that it fails for every integer $k \geq 0$ and on both surfaces: the map is a continuous bijection whose inverse is discontinuous at every point, also on the subspace of positively curved metrics. For $k = 0$ this follows from an elementary spiral construction. For $k \geq 1$ we use diffeomorphisms $\varphi_\varepsilon = z + \varepsilon z^{k+1} H_\varepsilon(|z|^2)$ whose Beltrami coefficient is approximately $\varepsilon z^{k+1}/\bar{z}$, regularized at the scale $e^{-1/\varepsilon}$. They converge to the identity in C^k , while their $(k+1)$ -st derivative at 0 tends to $-2(k+1)!$. The main work is to choose the conformal factors so that the metrics converge in C^k and keep nonnegative curvature; for $k = 1$ this needs an additional C^1 -small conformal correction. The conformal factors then fail to converge in C^k as well. This is an unrefereed note.

1 Introduction

Let $M_0 = \mathbb{C}$ and $M_1 = S^2 = \mathbb{C} \cup \{\infty\}$, with the metrics $g_0 = |dz|^2$ and $g_1 = 4|dz|^2/(1+|z|^2)^2$ of constant curvature $\kappa \in \{0, 1\}$. Let $\mathcal{D}(M_\kappa)$ be the group of smooth orientation-preserving diffeomorphisms of M_κ that fix 0 and 1 if $\kappa = 0$, and 0, 1 and ∞ if $\kappa = 1$. Let $\mathcal{R}_{\geq 0}(M_\kappa)$ (resp. $\mathcal{R}_{> 0}(M_\kappa)$) be the set of complete smooth Riemannian metrics on M_κ with Gauss curvature $K \geq 0$ (resp. $K > 0$), and let $\mathcal{O}_{\geq 0}(M_\kappa)$ (resp. $\mathcal{O}_{> 0}(M_\kappa)$) be the set of smooth functions u with $e^{-2u}g_\kappa$ in $\mathcal{R}_{\geq 0}(M_\kappa)$ (resp. $\mathcal{R}_{> 0}(M_\kappa)$). Since $K(e^{-2u}g_\kappa) = e^{2u}(\kappa + \Delta_{g_\kappa} u)$, the set $\mathcal{O}_{\geq 0}(S^2)$ consists of the u with $\Delta_{g_1} u \geq -1$; by a theorem of Belegardek and Hu [BH15, Theorem 1.1], $\mathcal{O}_{\geq 0}(\mathbb{C})$ consists of the smooth subharmonic u with $\alpha(u) := \lim_{r \rightarrow \infty} \max_{|z|=r} u(z)/\log r \leq 1$.

By the uniformization theorem, and for $\kappa = 0$ the theorem of Blanc and Fiala [BF41] that a complete nonnegatively curved plane is conformally equivalent to $(\mathbb{C}, g_0)$, every $g \in \mathcal{R}_{\geq 0}(M_\kappa)$ can be written uniquely as $g = \varphi^*(e^{-2u}g_\kappa)$ with $\varphi \in \mathcal{D}(M_\kappa)$ and $u \in \mathcal{O}_{\geq 0}(M_\kappa)$ [BH15, BB18, Bel17]. Thus

$$\Pi_\kappa: \mathcal{O}_{\geq 0}(M_\kappa) \times \mathcal{D}(M_\kappa) \rightarrow \mathcal{R}_{\geq 0}(M_\kappa), \quad \Pi_\kappa(u, \varphi) = \varphi^*(e^{-2u}g_\kappa), \quad (1)$$

is a bijection, and it restricts to a bijection $\mathcal{O}_{> 0} \times \mathcal{D} \rightarrow \mathcal{R}_{> 0}$. Following [BB18], for $0 \leq \gamma \leq \infty$ the C^γ topology on a set of smooth maps, functions or tensors is the topology of uniform C^γ convergence on

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compact sets, and a superscript records it: $\mathcal{R}_{\geq 0}^\gamma(M_\kappa)$, $\mathcal{O}_{\geq 0}^\gamma(M_\kappa)$ and $\mathcal{D}^{\gamma+1}(M_\kappa)$ carry the C^γ , C^γ and $C^{\gamma+1}$ topologies. With these topologies Π_κ is a continuous bijection $\mathcal{O}_{\geq 0}^\gamma(M_\kappa) \times \mathcal{D}^{\gamma+1}(M_\kappa) \rightarrow \mathcal{R}_{\geq 0}^\gamma(M_\kappa)$ [BB18, (8.3), (9.2)]. It is a homeomorphism when $\gamma = \infty$ or $\gamma \notin \mathbb{Z}$: for $M_0 = \mathbb{C}$ this is [BH15, Theorem 4.1] (for $C^{k+\gamma}$ with $0 < \gamma < 1$, as stated in arXiv v3 of [BH15]; by [BB18, proof of Theorem 8.4] the published statement wrongly assumed integer exponents, cf. the erratum [BH16]), and for $M_1 = S^2$ it is [BB18, Theorem 8.4]. The key analytic input is the theorem of Earle and Schatz [ES70] that solutions of the Beltrami equation depend continuously, in Hölder norms, on the Beltrami coefficient. In the Oberwolfach report [Bel17, p. 149], which writes the map with the pushforward φ_* in place of φ^* (this makes no difference; see the paragraph after Theorem 1.1), Belegarde asked whether the map is a homeomorphism, explained that it is one for the $C^{r,\alpha}$ topologies with $0 < \alpha < 1$, and wrote “we expect this to fail when $\alpha = 0$ ”; the same expectation is stated in [BB18, Section 1]. So the open case is that of integer smoothness. (If the parameters carry the C^∞ topology and the metrics the C^k topology with k finite, Π_0 is not a homeomorphism for a trivial reason [BH15, Section 1]; the question concerns the matched topologies above.) The question is record OWR-15208-008 of the UnsolvedMath dataset [ula26].

We confirm this expectation.

Theorem 1.1. *Let $\kappa \in \{0, 1\}$ and let $k \geq 0$ be an integer. The continuous bijection*

$$\Pi_\kappa: \mathcal{O}_{\geq 0}^k(M_\kappa) \times \mathcal{D}^{k+1}(M_\kappa) \rightarrow \mathcal{R}_{\geq 0}^k(M_\kappa)$$

is not a homeomorphism: its inverse is discontinuous at every point. More precisely, for every $g = \Pi_\kappa(u, \varphi) \in \mathcal{R}_{\geq 0}(M_\kappa)$ there are metrics $g_l = \Pi_\kappa(u_l, \varphi_l) \in \mathcal{R}_{\geq 0}(M_\kappa)$, $l \in \mathbb{N}$, such that $g_l \rightarrow g$ in the C^k topology and the following hold.

- (a) *If $k = 0$: each g_l is isometric to g and coincides with g outside a fixed compact subset of the chart $\mathbb{C} \subset M_\kappa$, $u_l = u$, and φ_l does not converge to φ in the C^1 topology.*
- (b) *If $k \geq 1$: $\varphi_l \rightarrow \varphi$ in the C^k topology, but φ_l does not converge to φ in the C^{k+1} topology, and u_l does not converge to u in the C^k topology. If g is not flat, then moreover every g_l coincides with g outside a fixed compact subset of the chart $\mathbb{C} \subset M_\kappa$.*

If $g \in \mathcal{R}_{> 0}(M_\kappa)$, then all g_l can be chosen in $\mathcal{R}_{> 0}(M_\kappa)$. Consequently the restriction $\mathcal{O}_{> 0}^k(M_\kappa) \times \mathcal{D}^{k+1}(M_\kappa) \rightarrow \mathcal{R}_{> 0}^k(M_\kappa)$ is not a homeomorphism either, and its inverse is discontinuous at every point.

For $k \geq 1$ the loss is sharp: the inverse is continuous into $\mathcal{O}^{k-1+\beta} \times \mathcal{D}^{k+\beta}$ for every $\beta \in (0, 1)$ (Remark 7.1). Since u_l fails to converge on its own, the answer stays negative for any topology on $\mathcal{D}(M_\kappa)$ if $k \geq 1$, and at flat metrics also for $k = 0$ (Remark 7.2 and Proposition 7.3).

Pullback and pushforward. The Oberwolfach report writes the parametrization with the push-forward: there $g = \varphi_*(e^{-2u}g_\kappa)$, the map is $(u, \varphi) \mapsto \varphi_*(e^{-2u}g_\kappa)$, and the homeomorphism property is stated for g , u and φ_* varying in the $C^{r,\alpha}$ topology [Bel17, p. 149]. We follow [BH15, BB18], which use the pullback. The two formulations are equivalent. Indeed $\varphi_* = (\varphi^{-1})^*$, so the map of the report is $\Pi_\kappa \circ (\text{id} \times \text{inv})$ with $\text{inv}(\varphi) = \varphi^{-1}$, and inv is a homeomorphism of $\mathcal{D}^j(M_\kappa)$ onto itself for every integer $j \geq 1$, because $\mathcal{D}^j(M_\kappa)$ is a topological group [BB18, Sections 8 and 9]. (For the sequences constructed below, which differ from a fixed diffeomorphism only on a fixed compact subset of $\mathbb{C}$, the continuity of inv that is needed is elementary.) Hence Theorem 1.1 holds verbatim for the map of the report, with φ^{-1} and φ_l^{-1} in place of φ and φ_l and with the same u and u_l . Finally, $\varphi_*(e^{-2u}g_\kappa)$ involves the differential of φ^{-1} , so letting φ_* vary in the $C^{r,\alpha}$ topology corresponds to $\varphi \in C^{r+1,\alpha}$ (compare [BB18, Section 1]); for $\alpha = 0$ and $r = k$ this is the topology of $\mathcal{D}^{k+1}(M_\kappa)$.

The analytic mechanism behind Theorem 1.1 is classical. Writing a metric as $\lambda|dz + \mu d\bar{z}|^2$, the diffeomorphism φ solves the Beltrami equation $\bar{\partial}\varphi = \mu\partial\varphi$, and the passage from μ to $\partial\varphi$ is governed by the Beurling transform. This operator is bounded on the Hölder spaces $C^{k+\alpha}$ with $0 < \alpha < 1$ (see [AIM09, Chapter 4]), but not on C^k , just as Schauder estimates fail at integer exponents (see [GT01, Chapter 4]). For $k = 0$ the failure is visible already for isometric metrics: spiral diffeomorphisms (Section 3) converge uniformly to the identity with pullback metrics converging in C^0 , while their differential at 0 is a fixed rotation. For $k \geq 1$ the maps $\varphi_\varepsilon = \text{id} + \psi_\varepsilon$ of Section 4 have Beltrami coefficient $\approx \varepsilon z^{k+1}/\bar{z}$ near 0, cut off at the scale $\delta = e^{-1/\varepsilon}$. Their $(k+1)$ -st derivative at 0 contains the term $\varepsilon \log \delta^2 = -2$. The main difficulty is to keep the curvature nonnegative. For $k \geq 2$ this follows from C^2 convergence. For $k = 1$ the curvature of the pulled-back metrics is of order $-\varepsilon/\delta$ near 0, and we add a radial, C^1 -small conformal correction whose Laplacian is large and positive near 0 (Section 5.2).

What is new. We found no earlier treatment of the integer case. The case $k = 0$ is an elementary observation, and the analytic mechanism is classical, as explained above. What we add is a construction for $k \geq 1$ that stays inside the complete metrics of nonnegative curvature (for $k = 1$ this needs the correction of Section 5.2), and the fact that the inverse is discontinuous at every point, for both surfaces.

Organization. Section 2 collects notation and four lemmas. Section 3 treats $k = 0$. Section 4 constructs the maps φ_ε and proves the needed estimates. Section 5 controls the curvature. Section 6 proves Theorem 1.1, Section 7 contains remarks, and Section 8 reports numerical sanity checks, which are not part of the proofs.

2 Notation and preliminaries

We write $z = x + iy$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\Delta = \partial_x^2 + \partial_y^2 = 4\partial\bar{\partial}$. For a function f (complex valued, or a tuple of functions) and $j \geq 0$ let $|D^j f(z)| = \max_{|\alpha|=j} |\partial^\alpha f(z)|$, the maximum over multi-indices $\alpha \in \mathbb{N}^2$, and for a set $Q \subset \mathbb{C}$ let $\|f\|_{C^k(Q)} = \max_{j \leq k} \sup_Q |D^j f|$, as in [BB18, (2.1)]. Then $|\partial^i \bar{\partial}^j f| \leq |D^{i+j} f|$, and by the Leibniz rule $|D^j(fg)| \leq \sum_{i \leq j} \binom{j}{i} |D^i f| |D^{j-i} g|$. Tensors on M_κ are handled through their coefficients in the chart z of $\mathbb{C} \subset M_\kappa$; all our sequences of metrics, functions and maps differ from a fixed one only on a fixed compact subset of this chart, except in Section 6.3, where $M_\kappa = \mathbb{C}$. $\mathbb{D}$ denotes the closed unit disc.

Beltrami form. Every Riemannian metric on a planar domain can be written uniquely as $\lambda|dz + \mu d\bar{z}|^2$ with $\lambda > 0$ and $|\mu| < 1$. If $\mu = a + ib$, its coefficients in $E dx^2 + 2F dx dy + G dy^2$ are

$$E = \lambda(1 + |\mu|^2 + 2a), \quad F = 2\lambda b, \quad G = \lambda(1 + |\mu|^2 - 2a). \quad (2)$$

Lemma 2.1. *Let φ be an orientation-preserving local diffeomorphism between planar domains with $\partial\varphi \neq 0$, and let $\mu_\varphi = \bar{\partial}\varphi/\partial\varphi$. Then $\varphi^*|dw|^2 = |\partial\varphi|^2|dz + \mu_\varphi d\bar{z}|^2$. If U, U', w are smooth real functions with $U' \circ \varphi = U + \log|\partial\varphi| + w$, then*

$$\varphi^*(e^{-2U'}|dw|^2) = e^{-2U-2w}|dz + \mu_\varphi d\bar{z}|^2.$$

Proof. $d\varphi = \partial\varphi dz + \bar{\partial}\varphi d\bar{z} = \partial\varphi(dz + \mu_\varphi d\bar{z})$, and $\varphi^*(e^{-2U'}|dw|^2) = e^{-2U' \circ \varphi}|d\varphi|^2$. $\square$

Lemma 2.2. *There is a universal constant C with the following property. Let μ be smooth with $|\mu| \leq 1/2$ on a planar domain, and let $\gamma = |dz + \mu d\bar{z}|^2$. Then*

$$\begin{aligned} K_\gamma &= 4 \operatorname{Re}(\partial^2 \mu) + R_1, & |R_1| &\leq C(|\mu| |D^2 \mu| + |D\mu|^2), \\ \Delta_\gamma f &= \Delta f + R_2, & |R_2| &\leq C(|\mu| |D^2 f| + |D\mu| |Df|) \end{aligned}$$

for every smooth real f , where Δ_γ is the Laplace–Beltrami operator of γ . Moreover $K(e^{-2f}\gamma) = e^{2f}(K_\gamma + \Delta_\gamma f)$.

Proof. The last formula is the conformal change formula $K(e^{2\sigma}\gamma) = e^{-2\sigma}(K_\gamma - \Delta_\gamma \sigma)$ with $\sigma = -f$. By (2) with $\lambda = 1$ we have $E = 1 + |\mu|^2 + 2a$, $F = 2b$, $G = 1 + |\mu|^2 - 2a$ and $EG - F^2 = (1 - |\mu|^2)^2$. Brioschi's formula expresses K_γ as $(EG - F^2)^{-2}$ times the difference of the determinants of

$$\begin{pmatrix} -\frac{1}{2}E_{yy} + F_{xy} - \frac{1}{2}G_{xx} & \frac{1}{2}E_x & F_x - \frac{1}{2}E_y \\ F_y - \frac{1}{2}G_x & E & F \\ \frac{1}{2}G_y & F & G \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{1}{2}E_y & \frac{1}{2}G_x \\ \frac{1}{2}E_y & E & F \\ \frac{1}{2}G_x & F & G \end{pmatrix}.$$

Expanding along the first rows gives $K_\gamma = (-\frac{1}{2}E_{yy} + F_{xy} - \frac{1}{2}G_{xx})/(EG - F^2) + q/(EG - F^2)^2$, where q is a sum of products of two first derivatives of E, F, G with coefficients that are polynomials in E, F, G . Since $|\mu| \leq \frac{1}{2}$, the functions E, F, G are bounded, $EG - F^2 \geq \frac{9}{16}$ and $|DE| + |DF| + |DG| \leq C|D\mu|$; hence $|q| \leq C|D\mu|^2$. Next, $-\frac{1}{2}E_{yy} + F_{xy} - \frac{1}{2}G_{xx} = a_{xx} - a_{yy} + 2b_{xy} - \frac{1}{2}\Delta|\mu|^2$, and $a_{xx} - a_{yy} + 2b_{xy} = \operatorname{Re}((\partial_x - i\partial_y)^2 \mu) = 4 \operatorname{Re}(\partial^2 \mu)$. The bounds $|\Delta|\mu|^2| \leq C(|\mu| |D^2 \mu| + |D\mu|^2)$, $|4 \operatorname{Re}(\partial^2 \mu)| \leq 4|D^2 \mu|$ and $|(1 - |\mu|^2)^{-2} - 1| \leq C|\mu|^2$ give the formula for K_γ . For the Laplacian, $\Delta_\gamma f = (1 - |\mu|^2)^{-1} \partial_i ((\delta^{ij} + N^{ij}) \partial_j f)$, where $\sqrt{\det \gamma} \gamma^{-1} = I + N$ and

$$N = \frac{1}{1 - |\mu|^2} \begin{pmatrix} 2|\mu|^2 - 2a & -2b \\ -2b & 2|\mu|^2 + 2a \end{pmatrix}, \quad |N| \leq C|\mu|, \quad |DN| \leq C|D\mu|.$$

Expanding the derivative gives the bound for R_2 . □

Lemma 2.3. *Let $g = \Pi_\kappa(u, \varphi)$ and $h = \Pi_\kappa(u, \operatorname{id}) = e^{-2u} g_\kappa$. Let $h_l = \Pi_\kappa(u_l, \psi_l)$ and put $g_l = \varphi^* h_l$. Then $g_l = \Pi_\kappa(u_l, \psi_l \circ \varphi)$, and g_l is isometric to h_l . Moreover $g_l \rightarrow g$ in the C^k topology if $h_l \rightarrow h$ in the C^k topology, and for every $j \geq 0$, $\psi_l \circ \varphi \rightarrow \varphi$ in the C^j topology if and only if $\psi_l \rightarrow \operatorname{id}$ in the C^j topology.*

Proof. $(\psi \circ \varphi)^* = \varphi^* \psi^*$, and $\psi_l \circ \varphi$ lies in $\mathcal{D}(M_\kappa)$. Pulling back by the fixed diffeomorphism φ is continuous in the C^k topology, since φ maps compact sets to compact sets and its derivatives are bounded on them. Right composition with φ or φ^{-1} is continuous in the C^j topology for the same reason (chain rule). □

We also use two elementary facts. First, a smooth metric on $\mathbb{C}$ that coincides with a complete metric outside a compact set is complete, since the two metrics are uniformly equivalent. Second, if $\varphi' \in \mathcal{D}(M_\kappa)$, u' is smooth and $g' = \varphi'^*(e^{-2u'} g_\kappa)$ is complete with $K \geq 0$ (resp. $K > 0$), then $e^{-2u'} g_\kappa$ is isometric to g' , so $u' \in \mathcal{O}_{\geq 0}(M_\kappa)$ (resp. $\mathcal{O}_{> 0}$) and $\Pi_\kappa^{-1}(g') = (u', \varphi')$.

Lemma 2.4. *Let $k \geq 1$, $Q = \overline{\mathbb{D}}$, let F be smooth near Q , and let $F_l \rightarrow F$ and $\Phi_l \rightarrow \operatorname{id}$ in $C^k(Q)$, where $\Phi_l(Q) \subset Q$. Then $F_l \circ \Phi_l \rightarrow F$ in $C^k(Q)$.*

Proof. Write $F_l \circ \Phi_l - F = (F_l - F) \circ \Phi_l + (F \circ \Phi_l - F)$. By the Faà di Bruno formula, $\|G \circ \Phi_l\|_{C^k(Q)} \leq C_k(1 + \|\Phi_l\|_{C^k(Q)})^k \|G\|_{C^k(Q)}$, so the first term tends to 0. In the Faà di Bruno expansion of $D^j(F \circ \Phi_l)$, $j \leq k$, the factors $(D^i F) \circ \Phi_l$ converge uniformly to $D^i F$ (uniform continuity), $D\Phi_l \rightarrow I$ and $D^i \Phi_l \rightarrow 0$ for $2 \leq i \leq k$; hence $D^j(F \circ \Phi_l) \rightarrow D^j F$ uniformly. □

3 The case $k = 0$: spirals

Fix a smooth nondecreasing $S: \mathbb{R} \rightarrow [0, 1]$ with $S = 0$ on $(-\infty, 0]$, $S = 1$ on $[1, \infty)$ and $0 \leq S' \leq C_S$. For $0 < \varepsilon < 1$ let

$$a_\varepsilon(r) = S(\varepsilon \log \frac{1}{2r}) \quad (r > 0), \quad a_\varepsilon(0) = 1, \quad \varphi_\varepsilon(z) = z e^{ia_\varepsilon(|z|)}.$$

Proposition 3.1. *Let $h = e^{-2u}g_\kappa \in \mathcal{R}_{\geq 0}(M_\kappa)$. Then $\varphi_\varepsilon \in \mathcal{D}(M_\kappa)$, $\varphi_\varepsilon = \text{id}$ on $\{|z| \geq \frac{1}{2}\}$, and $\varphi_\varepsilon(z) = e^i z$ on $\{|z| \leq \frac{1}{2}e^{-1/\varepsilon}\}$; in particular $|\partial\varphi_\varepsilon(0) - 1| = |e^i - 1| > 0.95$. The metric $h_\varepsilon = \varphi_\varepsilon^* h = \Pi_\kappa(u, \varphi_\varepsilon)$ is isometric to h , coincides with h outside $\{|z| < \frac{1}{2}\}$, and $\|h_\varepsilon - h\|_{C^0(\{|z| \leq 1/2\})} \leq C\varepsilon$, with C depending only on u and C_S .*

Proof. The function $a_\varepsilon(|z|)$ is smooth on $\mathbb{C} \setminus \{0\}$ and equals 1 on $\{|z| \leq \frac{1}{2}e^{-1/\varepsilon}\}$, so it is smooth; it vanishes on $\{|z| \geq \frac{1}{2}\}$. Since $|\varphi_\varepsilon(z)| = |z|$, the map $z \mapsto ze^{-ia_\varepsilon(|z|)}$ is a smooth inverse. Hence φ_ε is a diffeomorphism of $\mathbb{C}$, equal to the identity near 1 and near ∞ , fixing 0. With $r = |z|$ and $t = ra'_\varepsilon(r) = -\varepsilon S'(\varepsilon \log \frac{1}{2r})$, so that $|t| \leq C_S\varepsilon$, one computes

$$\partial\varphi_\varepsilon = e^{ia_\varepsilon}(1 + \frac{it}{2}), \quad \bar{\partial}\varphi_\varepsilon = e^{ia_\varepsilon} \frac{it}{2} \frac{z}{\bar{z}}, \quad |\partial\varphi_\varepsilon|^2 - |\bar{\partial}\varphi_\varepsilon|^2 = 1.$$

So φ_ε preserves orientation, and $\varphi_\varepsilon \in \mathcal{D}(M_\kappa)$. Moreover $|\mu_\varphi| \leq C_S\varepsilon/2$ and $||\partial\varphi_\varepsilon|^2 - 1| \leq C_S^2\varepsilon^2/4$. Since $S(x) \leq C_S x$ for $x \geq 0$, for $|z| \leq \frac{1}{2}$ we get $|\varphi_\varepsilon(z) - z| \leq |z|a_\varepsilon(|z|) \leq C_S\varepsilon|z| \log \frac{1}{2|z|} \leq C_S\varepsilon/(2e)$. In the chart, $h = e^{-2U}|dz|^2$ with $U = u$ if $\kappa = 0$ and $U = u + \log(1 + |z|^2) - \log 2$ if $\kappa = 1$. By Lemma 2.1, $h_\varepsilon = e^{-2U \circ \varphi_\varepsilon} |\partial\varphi_\varepsilon|^2 |dz + \mu_\varphi d\bar{z}|^2$, and by (2) its coefficients differ from those of h by at most $C(|U \circ \varphi_\varepsilon - U| + ||\partial\varphi_\varepsilon|^2 - 1| + |\mu_\varphi|) \leq C\varepsilon$ on $\{|z| \leq \frac{1}{2}\}$. Finally $h_\varepsilon = \varphi_\varepsilon^*(e^{-2u}g_\kappa) = \Pi_\kappa(u, \varphi_\varepsilon)$ is isometric to h . $\square$

4 Log-twisted maps

In this section and the next, $k \geq 1$ is fixed, as are numbers $0 < a < b < 1$ and a smooth $\chi: [0, \infty) \rightarrow [0, 1]$ with $\chi = 1$ on $[0, a^2]$ and $\chi = 0$ on $[b^2, \infty)$. For $0 < \varepsilon \leq \frac{1}{4}$ let $\delta = e^{-1/\varepsilon}$ and

$$H(s) = - \int_s^\infty \frac{\chi(t) dt}{t + \delta^2} \quad (s \geq 0), \quad \psi(z) = \psi_\varepsilon(z) = \varepsilon z^{k+1} H(|z|^2), \quad \varphi_\varepsilon = \text{id} + \psi_\varepsilon.$$

We write $\rho = \rho(z) = |z| + \delta$. In this section C, C_j denote constants that depend only on k, a, b, χ (and on j), never on ε .

Lemma 4.1. *The function ψ is smooth, $\psi = 0$ on $\{|z| \geq b\}$, and*

$$\bar{\partial}\psi = \varepsilon \chi(|z|^2) \frac{z^{k+2}}{|z|^2 + \delta^2}, \tag{3}$$

$$\partial\psi = \varepsilon z^k \left[(k+1)H(|z|^2) + \frac{|z|^2 \chi(|z|^2)}{|z|^2 + \delta^2} \right]. \tag{4}$$

On $[0, a^2]$ we have $H(s) = \log(s + \delta^2) - c_\delta$, where $c_\delta = \log(a^2 + \delta^2) + \int_{a^2}^{b^2} \chi(t) dt / (t + \delta^2)$ satisfies $|c_\delta| \leq C_0$, with C_0 depending only on a, b .

Proof. $H'(s) = \chi(s)/(s + \delta^2)$ and $H = 0$ on $[b^2, \infty)$. On $[0, a^2]$ the formula for H follows by integration; it extends smoothly to $s > -\delta^2$, so $H(|z|^2)$ is smooth near 0, and elsewhere it is smooth because $|z|^2 > 0$. Using $\bar{\partial}z^{k+1} = 0$, $\bar{\partial}|z|^2 = z$, $\partial z^{k+1} = (k+1)z^k$ and $\partial|z|^2 = \bar{z}$ gives (3) and (4). Finally $|\log(a^2 + \delta^2)| \leq 2|\log a| + 1$ and $0 \leq \int_{a^2}^{b^2} \chi dt / (t + \delta^2) \leq 2\log(b/a)$. $\square$

For $m \in \mathbb{R}$ let S^m be the class of smooth $F: \mathbb{R}^2 \rightarrow \mathbb{C}$ with $|D^j F(\zeta)| \leq C_j(1 + |\zeta|)^{m-j}$ for all $j \geq 0$.

Lemma 4.2. *Polynomials of degree at most m lie in S^m ; $(1 + |\zeta|^2)^\sigma \in S^{2\sigma}$ for $\sigma \in \mathbb{R}$; $S^m \cdot S^{m'} \subset S^{m+m'}$; and $\ell(\zeta) = \log(1 + |\zeta|^2)$ satisfies $|D^j \ell(\zeta)| \leq C_j(1 + |\zeta|)^{-j}$ for $j \geq 1$. If $F \in S^m$ and $\delta > 0$, then $f(z) = \delta^m F(z/\delta)$ satisfies $|D^j f(z)| \leq C_j(|z| + \delta)^{m-j}$.*

Proof. The first claim is clear. By induction on j , $D^j(1 + |\zeta|^2)^\sigma$ is a linear combination of terms $P(\zeta)(1 + |\zeta|^2)^{\sigma-i}$ with $i \leq j$ and P a polynomial of degree at most $2i - j$; since $1 + |\zeta|^2 \geq \frac{1}{2}(1 + |\zeta|)^2$, this gives the second claim. The third follows from the Leibniz rule, and the fourth from $D\ell = 2\zeta(1 + |\zeta|^2)^{-1} \in S^{-1}$ (componentwise). Finally $D^j f(z) = \delta^{m-j}(D^j F)(z/\delta)$ and $\delta(1 + |z|/\delta) = |z| + \delta$. $\square$

Lemma 4.3 (Estimates). *There is $\varepsilon_0 \in (0, \frac{1}{4}]$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $\mu_\varepsilon := \bar{\partial}\psi/(1 + \partial\psi)$ the following hold.*

- (i) On $\{|z| \leq a\}$: $|D^j \bar{\partial}\psi| \leq C_j \varepsilon \rho^{k-j}$ for all $j \geq 0$.
- (ii) On $\{|z| \leq a\}$: $|D^j \partial\psi| \leq C_j \varepsilon \rho^{k-j}(1 + \log \frac{1}{\rho})$ for $0 \leq j \leq k$, and $|D^j \partial\psi| \leq C_j \varepsilon \rho^{k-j}$ for $j \geq k+1$.
- (iii) On $\{a \leq |z| \leq b\}$: $|D^j \psi| \leq C_j \varepsilon$ for all $j \geq 0$.
- (iv) $|\partial\psi| + |\bar{\partial}\psi| \leq \frac{1}{4}$ on $\mathbb{C}$, and $\|\psi\|_{C^k(\mathbb{C})} \leq C\varepsilon$.
- (v) On $\{|z| \leq a\}$: $|D^j \mu_\varepsilon| \leq C_j \varepsilon \rho^{k-j}$ for $0 \leq j \leq k$. On $\{a \leq |z| \leq b\}$: $|D^j \mu_\varepsilon| \leq C_j \varepsilon$ for all j . Hence $\|\mu_\varepsilon\|_{C^k(\mathbb{C})} \leq C\varepsilon$.
- (vi) If $k = 1$, then on $\{|z| \leq a\}$: $|D^2 \mu_\varepsilon| \leq C\varepsilon/\rho$ and $|D^2(\mu_\varepsilon - \bar{\partial}\psi)| \leq C\varepsilon$.

Proof. Choose ε_0 so small that $\delta \leq (1 - a)/2$. Then on $\{|z| \leq a\}$ we have $\delta \leq \rho < 1$, and therefore

$$\varepsilon \log \frac{1}{\rho} \leq \varepsilon \log \frac{1}{\delta} = 1, \quad \rho^m(1 + \log \frac{1}{\rho}) \leq 1 \quad (m \geq 1), \quad (5)$$

the second because $t \mapsto t(1 + \log \frac{1}{t})$ increases on $(0, 1]$ to the value 1.

- (i) On $\{|z| \leq a\}$, $\bar{\partial}\psi = \varepsilon \delta^k F_1(z/\delta)$ with $F_1(\zeta) = \zeta^{k+2}(1 + |\zeta|^2)^{-1} \in S^k$, and Lemma 4.2 applies.
- (ii) On $\{|z| \leq a\}$, (4) and Lemma 4.1 give

$$\partial\psi = \varepsilon(k+1)z^k \Lambda - \varepsilon(k+1)c_\delta z^k + \varepsilon \delta^k F_2(z/\delta), \quad \Lambda = \log(|z|^2 + \delta^2), \quad F_2(\zeta) = \frac{\zeta^k |\zeta|^2}{1 + |\zeta|^2} \in S^k.$$

The last two terms are bounded by $C_j \varepsilon \rho^{k-j}$, because $|D^i z^k| \leq C|z|^{k-i} \leq C\rho^{k-i}$ for $i \leq k$, $D^i z^k = 0$ for $i > k$, and $|c_\delta| \leq C_0$. For the first term, $\Lambda = \log \delta^2 + \ell(z/\delta)$, so $|D^m \Lambda| \leq C_m \rho^{-m}$ for $m \geq 1$ by Lemma 4.2; and $|\Lambda| \leq 2 \log \frac{1}{\rho} + \log 2$, since $\rho^2/2 \leq |z|^2 + \delta^2 \leq \rho^2 < 1$. By the Leibniz rule $D^j(z^k \Lambda)$ is a combination of the products $D^i z^k \cdot D^{j-i} \Lambda$ with $i \leq \min(j, k)$. The product with $i = j$ occurs only if $j \leq k$; it contains the undifferentiated logarithm and is bounded by $C|z|^{k-j}|\Lambda| \leq C\rho^{k-j}(1 + \log \frac{1}{\rho})$. The products with $i < j$ are bounded by $C\rho^{k-i}\rho^{-(j-i)} = C\rho^{k-j}$. This proves (ii). For $j \geq k+1$ no logarithm survives, because $D^i z^k = 0$ for $i > k$.

(iii) For $s \in [a^2, b^2]$ we have $H(s) = -\int_s^{b^2} \chi(t) dt/(t + \delta^2)$, and all s -derivatives of this function are bounded on $[a^2, b^2]$ uniformly in $\delta \in (0, \frac{1}{2}]$, since $t \geq a^2 > 0$. Hence $|D^j \psi| \leq C_j \varepsilon$ on $\{a \leq |z| \leq b\}$.

(iv) On $\{|z| \leq a\}$, (i), (ii) and (5) give $|\bar{\partial}\psi| \leq C\varepsilon \rho^k \leq C\varepsilon$ and $|\partial\psi| \leq C\varepsilon \rho^k(1 + \log \frac{1}{\rho}) \leq C\varepsilon$ (here $k \geq 1$ is used). Together with (iii), $|\partial\psi| + |\bar{\partial}\psi| \leq C\varepsilon \leq \frac{1}{4}$ after shrinking ε_0 . Next, on $\{|z| \leq a\}$, $|\psi| \leq \varepsilon|z|^{k+1}(|\Lambda| + C_0) \leq C\varepsilon \rho(1 + \log \frac{1}{\rho}) \leq C\varepsilon$. For $1 \leq j \leq k$, every derivative $\partial^\alpha \psi$ with $|\alpha| = j$ is a derivative of order $j-1$ of $\partial_x \psi = \partial\psi + \bar{\partial}\psi$ or of $\partial_y \psi = i(\partial\psi - \bar{\partial}\psi)$, and by (i), (ii) and (5), $|D^{j-1} \partial\psi| + |D^{j-1} \bar{\partial}\psi| \leq C\varepsilon \rho^{k-j+1}(1 + \log \frac{1}{\rho}) \leq C\varepsilon$. With (iii) this gives $\|\psi\|_{C^k(\mathbb{C})} \leq C\varepsilon$.

(v) Let $f = \partial\psi$ and $R = (1+f)^{-1}$, so that $|f| \leq \frac{1}{4}$ and $\mu_\varepsilon = \bar{\partial}\psi \cdot R$. By the Faà di Bruno formula, for $m \geq 1$, $D^m R$ is a linear combination of terms $(1+f)^{-1-r} D^{m_1} f \cdots D^{m_r} f$ with $r \geq 1$, $m_i \geq 1$ and $m_1 + \cdots + m_r = m$. On $\{|z| \leq a\}$ and for $m \leq k$, (ii) and (5) give

$$|D^m f| \leq C\varepsilon(1 + \log \frac{1}{\rho})\rho^{k-m_i} \leq 2C\rho^{k-m_i} \leq 2C\rho^{-m_i}, \quad (6)$$

hence $|D^m R| \leq C\rho^{-m}$ for $0 \leq m \leq k$. This is where $\varepsilon \log(1/\rho) \leq 1$ is used: each factor $D^{m_i} f$ carries its own factor ε , which absorbs its logarithm. By the Leibniz rule, $D^j \mu_\varepsilon$ is a combination of the products $D^i \bar{\partial}\psi \cdot D^{j-i} R$, and by (i), for $j \leq k$, $|D^j \mu_\varepsilon| \leq C \sum_{i \leq j} \varepsilon \rho^{k-i} \rho^{-(j-i)} = C\varepsilon \rho^{k-j}$. In particular $D^k \mu_\varepsilon$ contains no logarithm: each of its terms has exactly one factor $D^i \bar{\partial}\psi$, which is free of logarithms and carries the factor ε , and the factors $D^m f$, which may contain $\log \frac{1}{\rho}$, are bounded by (6). On $\{a \leq |z| \leq b\}$ the bound follows from (iii) and $|f| \leq \frac{1}{4}$, and $\mu_\varepsilon = 0$ on $\{|z| \geq b\}$.

(vi) Let $k = 1$ and $|z| \leq a$. By (i), (ii) and (5),

$$\begin{aligned} |\bar{\partial}\psi| &\leq C\varepsilon\rho, & |D\bar{\partial}\psi| &\leq C\varepsilon, & |D^2\bar{\partial}\psi| &\leq C\varepsilon/\rho, \\ |f| &\leq C\varepsilon\rho(1 + \log \frac{1}{\rho}), & |Df| &\leq C\varepsilon(1 + \log \frac{1}{\rho}) \leq 2C, & |D^2f| &\leq C\varepsilon/\rho. \end{aligned}$$

Hence $|R| \leq 2$, $|DR| \leq C|Df|$ and $|D^2R| \leq C(|D^2f| + |Df|^2) \leq C(\varepsilon/\rho + 1)$. Therefore $|D^2\mu_\varepsilon| \leq C(|D^2\bar{\partial}\psi| + |D\bar{\partial}\psi||DR| + |\bar{\partial}\psi||D^2R|) \leq C\varepsilon/\rho$. Next, $\mu_\varepsilon - \bar{\partial}\psi = -\bar{\partial}\psi \cdot f \cdot R$, and $D^2(\bar{\partial}\psi f R)$ is a combination of the products $D^{i_1} \bar{\partial}\psi D^{i_2} f D^{i_3} R$ with $i_1 + i_2 + i_3 = 2$. Using the bounds above and $\varepsilon(1 + \log \frac{1}{\rho}) \leq 2$, each of the six products is at most $C\varepsilon$; for example $|D^2\bar{\partial}\psi||f||R| \leq C\varepsilon^2(1 + \log \frac{1}{\rho}) \leq C\varepsilon$ and $|D\bar{\partial}\psi||Df||R| \leq C\varepsilon$. $\square$

Lemma 4.4. *For $0 < \varepsilon \leq \varepsilon_0$, φ_ε is a diffeomorphism of $\mathbb{C}$ that equals the identity on $\{|z| \geq b\}$, maps $\{|z| \leq b\}$ onto itself and fixes 0 and 1; it extends to a diffeomorphism of S^2 fixing ∞ . Thus $\varphi_\varepsilon \in \mathcal{D}(M_\kappa)$ for $\kappa = 0, 1$. Moreover $\bar{\partial}\varphi_\varepsilon/\partial\varphi_\varepsilon = \mu_\varepsilon$ and $\|\varphi_\varepsilon - \text{id}\|_{C^k(\mathbb{C})} \leq C\varepsilon$.*

Proof. The real differential of ψ at a point is the real-linear map $v \mapsto \partial\psi v + \bar{\partial}\psi \bar{v}$, whose norm is $|\partial\psi| + |\bar{\partial}\psi| \leq \frac{1}{4}$ by Lemma 4.3(iv). Hence $|\varphi_\varepsilon(p) - \varphi_\varepsilon(q)| \geq \frac{3}{4}|p - q|$, and the Jacobian $|1 + \partial\psi|^2 - |\bar{\partial}\psi|^2 \geq (\frac{3}{4})^2 - (\frac{1}{4})^2$ is positive. So φ_ε is an injective, orientation-preserving local diffeomorphism, and it is proper because it is the identity outside a compact set. Its image is therefore open and closed, i.e. all of $\mathbb{C}$. It is the identity on $\{|z| \geq b\}$, so by bijectivity it maps $\{|z| \leq b\}$ onto itself. It fixes 0 because $\psi(0) = 0$, and 1 because $b < 1$. The last two claims follow from $\partial\varphi_\varepsilon = 1 + \partial\psi$, $\bar{\partial}\varphi_\varepsilon = \bar{\partial}\psi$ and Lemma 4.3(iv). $\square$

Lemma 4.5. *If $0 < \varepsilon \leq \min(\varepsilon_0, 1/C_0)$, then*

$$\partial^{k+1}(\varphi_\varepsilon - \text{id})(0) = \varepsilon(k+1)!(\log \delta^2 - c_\delta) = -(k+1)!(2 + \varepsilon c_\delta), \quad \partial^k(\log |\partial\varphi_\varepsilon|)(0) = \frac{1}{2} \partial^{k+1}(\varphi_\varepsilon - \text{id})(0).$$

Consequently, for every closed disc $Q \ni 0$, $\|\varphi_\varepsilon - \text{id}\|_{C^{k+1}(Q)} \geq (k+1)!$ and $\|\log |\partial\varphi_\varepsilon|\|_{C^k(Q)} \geq (k+1)!/2$.

Proof. On $\{|z| \leq a\}$ we have $f := \partial\psi = z^k q(|z|^2)$ with $q(s) = \varepsilon[(k+1)(\log(s + \delta^2) - c_\delta) + s/(s + \delta^2)]$, which is smooth for $s > -\delta^2$. By the Leibniz rule, $\partial^k(z^k q(|z|^2))(0) = k!q(0)$, since every other term contains a factor $\partial^i z^k$ with $i < k$, which vanishes at 0; and $\bar{\partial}^k(z^k q(|z|^2)) = z^k \bar{\partial}^k q(|z|^2)$ vanishes at 0. Hence $\partial^{k+1}\psi(0) = \partial^k f(0) = k!q(0) = \varepsilon(k+1)!(\log \delta^2 - c_\delta)$, and $\log \delta^2 = -2/\varepsilon$. Next, $\log(1+f) = f + f^2\Theta(f)$ with Θ holomorphic on $\{|f| < 1\}$. Since $f^2 = z^{2k}q^2$, every term in the Leibniz expansions of $\partial^k(f^2\Theta(f))$ and $\bar{\partial}^k(f^2\Theta(f))$ contains a derivative of z^{2k} of order at most $k < 2k$, which vanishes at 0. Thus $\partial^k \log(1+f)(0) = k!q(0)$ and $\bar{\partial}^k \log(1+f)(0) = 0$. Since $\log |\partial\varphi_\varepsilon| = \frac{1}{2}(\log(1+f) + \overline{\log(1+f)})$ and $\partial^k \bar{F} = \overline{\partial^k F}$, we get $\partial^k \log |\partial\varphi_\varepsilon|(0) = \frac{1}{2}k!q(0)$. The lower bounds follow from $|\partial^k F| \leq |D^k F|$ and $|2 + \varepsilon c_\delta| \geq 1$. $\square$

Thus $\mu_\varepsilon \rightarrow 0$ and $\varphi_\varepsilon \rightarrow \text{id}$ in C^k , while $\partial\varphi_\varepsilon \not\rightarrow 1$ in C^k : the estimate $\|\partial\varphi - 1\|_{C^k} \lesssim \|\mu\|_{C^k}$ fails.

5 Keeping the curvature nonnegative

Let U be a smooth real function on a neighbourhood of $\overline{\mathbb{D}}$ with

$$\Delta U \geq c_0 > 0 \quad \text{on } \overline{\mathbb{D}}, \quad (7)$$

and let $h = e^{-2U}|dz|^2$, so that $K(h) = e^{2U}\Delta U > 0$ on $\overline{\mathbb{D}}$.

Proposition 5.1. *Let $k \geq 1$. There are smooth real functions w_ε on $\mathbb{C}$, supported in $\{|z| \leq b\}$, which vanish if $k \geq 2$ and do not depend on U , such that the following holds for some $\varepsilon_1 > 0$ depending on k, a, b, χ , the auxiliary data of Section 5.2 and U . For $0 < \varepsilon \leq \varepsilon_1$:*

- (a) $\|w_\varepsilon\|_{C^k(\mathbb{C})} \leq C\varepsilon$, with C independent of U ;
- (b) the metric $g_\varepsilon = e^{-2U-2w_\varepsilon}|dz + \mu_\varepsilon d\bar{z}|^2$ satisfies $K(g_\varepsilon) > 0$ on $\overline{\mathbb{D}}$, and $g_\varepsilon = h$ on $\{b \leq |z| \leq 1\}$;
- (c) $\|g_\varepsilon - h\|_{C^k(\overline{\mathbb{D}})} \leq C_U \varepsilon$, with C_U depending on k, a, b, χ and $\|U\|_{C^k(\overline{\mathbb{D}})}$, and, if $k = 1$, also on the auxiliary data $\chi_w, \beta, a_w, b_w, a_\beta, b_\beta$ of Section 5.2 and on A , through w_ε .

Proof of (c) and of the last claim in (b). On $\{|z| \geq b\}$, $\mu_\varepsilon = 0$ and $w_\varepsilon = 0$. By (2), the coefficients of $g_\varepsilon - h$ are $e^{-2U}[e^{-2w_\varepsilon}(1 + |\mu_\varepsilon|^2 \pm 2\operatorname{Re} \mu_\varepsilon) - 1]$ and $2e^{-2U-2w_\varepsilon}\operatorname{Im} \mu_\varepsilon$. So (c) follows from Lemma 4.3(v), part (a) and the Leibniz and Faà di Bruno formulas. $\square$

5.1 The case $k \geq 2$

Here $w_\varepsilon = 0$. Since $k \geq 2$, part (c) gives $\|g_\varepsilon - h\|_{C^2(\overline{\mathbb{D}})} \rightarrow 0$. By Brioschi's formula, K is a continuous function of the 2-jet of the coefficients on the set where $EG - F^2 > 0$, and the 2-jets of h on $\overline{\mathbb{D}}$ form a compact subset of this set. Hence $K(g_\varepsilon) \rightarrow K(h)$ uniformly on $\overline{\mathbb{D}}$, and $K(g_\varepsilon) > 0$ on $\overline{\mathbb{D}}$ for small ε , because $K(h) \geq e^{2\min U} c_0$.

5.2 The case $k = 1$: a positive correction

For $k = 1$, C^1 convergence does not control the curvature. Indeed, near 0 the leading term $4\operatorname{Re}(\partial^2 \bar{\partial} \psi)$ of $K(|dz + \mu_\varepsilon d\bar{z}|^2)$ has size about ε/δ and is negative on half of every small circle. We compensate with a radial conformal factor. Fix numbers

$$0 < a_w < b_w \leq a_\beta < b_\beta \leq b, \quad a_w \leq a,$$

a smooth $\chi_w: [0, \infty) \rightarrow [0, 1]$ with $\chi_w = 1$ on $[0, a_w^2]$ and $\chi_w = 0$ on $[b_w^2, \infty)$, and a smooth $\beta \geq 0$ supported in $[a_\beta^2, b_\beta^2]$ with $\int \beta = 1$ (the auxiliary data). Let $A = 16$. For $0 < \varepsilon \leq \varepsilon_0$ define

$$\begin{aligned} p(s) &= \chi_w(s) \frac{s + 2\delta^2}{(s + \delta^2)^{3/2}} - c_w \beta(s), \quad c_w = \int_0^\infty \chi_w(s) \frac{s + 2\delta^2}{(s + \delta^2)^{3/2}} ds, \\ W'(s) &= \frac{A\varepsilon}{4s} \int_0^s p(t) dt, \quad W(s) = - \int_s^\infty W'(t) dt, \quad w_\varepsilon(z) = W(|z|^2). \end{aligned}$$

Lemma 5.2. *The function w_ε is smooth, radial and supported in $\{|z| \leq b_\beta\}$, and, with C depending only on the auxiliary data:*

- (i) $\Delta w_\varepsilon = A\varepsilon p(|z|^2) \geq -CA\varepsilon$ on $\mathbb{C}$;
- (ii) on $\{|z| \leq a_w\}$, $w_\varepsilon = A\varepsilon \sqrt{|z|^2 + \delta^2} + \text{const}$, hence $\Delta w_\varepsilon = A\varepsilon(|z|^2 + 2\delta^2)(|z|^2 + \delta^2)^{-3/2}$ and $|D^2 w_\varepsilon| \leq CA\varepsilon/\rho$;

(iii) $|w_\varepsilon| + |Dw_\varepsilon| \leq CA\varepsilon$ on $\mathbb{C}$, and $\|w_\varepsilon\|_{C^2(\{|z| \geq a_w\})} \leq CA\varepsilon$.

Proof. Differentiation shows $\int_0^s (t + 2\delta^2)(t + \delta^2)^{-3/2} dt = 2s(s + \delta^2)^{-1/2}$. Hence $0 \leq c_w \leq 2b_w$, and $\int_0^\infty p = c_w - c_w \int \beta = 0$. Let $Q(s) = \int_0^s p$. Since $p = 0$ on $[b_\beta^2, \infty)$, Q vanishes there, so W' and W vanish on $[b_\beta^2, \infty)$. On $[0, a_w^2]$ we have $Q(s) = 2s(s + \delta^2)^{-1/2}$, $W'(s) = \frac{A\varepsilon}{2}(s + \delta^2)^{-1/2}$ and $W(s) = A\varepsilon\sqrt{s + \delta^2} + \text{const}$; this formula extends smoothly to $s > -\delta^2$, so w_ε is smooth, and (ii) follows by a direct computation and Lemma 4.2 (applied to $(1 + |\zeta|^2)^{1/2} \in S^1$). For a radial function $W(|z|^2)$ we have $\bar{\partial}W = W'z$ and $\Delta W = 4(W' + sW'') = 4(sW')'$, which equals $A\varepsilon p$ because $sW' = \frac{A\varepsilon}{4}Q$; and $p \geq -c_w \sup \beta \geq -2b_w \sup \beta$. This proves (i). For (iii): if $s \leq a_\beta^2$ then $\beta = 0$ on $[0, s]$, so $0 \leq Q(s) \leq 2s(s + \delta^2)^{-1/2}$ and $0 \leq W'(s) \leq \frac{A\varepsilon}{2}(s + \delta^2)^{-1/2}$; if $s \geq a_\beta^2$ then $|Q(s)| \leq 2c_w$ and $|W'(s)| \leq A\varepsilon b_w/a_\beta^2$. Since $|Dw_\varepsilon| = 2|W'(s)||z|$ and $|w_\varepsilon(z)| \leq \int_{|z|^2}^{b_\beta^2} |W'|$, the first bound in (iii) follows. On $[a_w^2, \infty)$ the functions p and Q and their derivatives are bounded uniformly in δ , hence so are W' and W'' (up to the factor $A\varepsilon$), which gives the second bound. $\square$

Lemma 5.3. For $z \in \mathbb{C}$ and $\delta > 0$,

$$4 \left| \partial^2 \frac{z^3}{|z|^2 + \delta^2} \right| \leq 8 \frac{|z|^2 + 2\delta^2}{(|z|^2 + \delta^2)^{3/2}}.$$

Proof. A direct computation gives $\partial^2(z^3/(|z|^2 + \delta^2)) = z(2|z|^4 + 6\delta^2|z|^2 + 6\delta^4)/(|z|^2 + \delta^2)^3$. Put $|z|^2 = \sigma\delta^2$. The claim becomes $\sigma^{1/2}(\sigma^2 + 3\sigma + 3) \leq (\sigma + 2)(\sigma + 1)^{3/2}$, and after squaring it follows from the identity $(\sigma + 2)^2(\sigma + 1)^3 - \sigma(\sigma^2 + 3\sigma + 3)^2 = \sigma^4 + 4\sigma^3 + 7\sigma^2 + 7\sigma + 4$. $\square$

Proof of Proposition 5.1 for $k = 1$. Part (a) is Lemma 5.2(iii). Let $\gamma = |dz + \mu_\varepsilon d\bar{z}|^2$. Since $|\mu_\varepsilon| \leq 2|\bar{\partial}\psi| \leq \frac{1}{2}$, Lemma 2.2 gives

$$e^{-2(U+w_\varepsilon)}K(g_\varepsilon) = K_\gamma + \Delta_\gamma(U + w_\varepsilon) = 4\text{Re}(\partial^2\mu_\varepsilon) + \Delta U + \Delta w_\varepsilon + R_1 + R_2, \quad (8)$$

with $|R_1| \leq C(|\mu_\varepsilon||D^2\mu_\varepsilon| + |D\mu_\varepsilon|^2)$ and $|R_2| \leq C(|\mu_\varepsilon||D^2(U + w_\varepsilon)| + |D\mu_\varepsilon||D(U + w_\varepsilon)|)$.

On $\{|z| \leq a_w\}$. Here $\chi = 1$, so $\bar{\partial}\psi = \varepsilon z^3/(|z|^2 + \delta^2)$. By Lemma 4.3(v),(vi), $|\mu_\varepsilon| \leq C\varepsilon\rho$, $|D\mu_\varepsilon| \leq C\varepsilon$, $|D^2\mu_\varepsilon| \leq C\varepsilon/\rho$ and $|D^2(\mu_\varepsilon - \bar{\partial}\psi)| \leq C\varepsilon$; by Lemma 5.2, $|Dw_\varepsilon| \leq CA\varepsilon$ and $|D^2w_\varepsilon| \leq CA\varepsilon/\rho$. Hence $|R_1| \leq C\varepsilon^2$ and $|R_2| \leq C'\varepsilon$, where C' depends also on $\|U\|_{C^2(\mathbb{D})}$. By Lemma 5.3,

$$4\text{Re}(\partial^2\mu_\varepsilon) \geq -4\varepsilon \left| \partial^2 \frac{z^3}{|z|^2 + \delta^2} \right| - 4|D^2(\mu_\varepsilon - \bar{\partial}\psi)| \geq -8\varepsilon \frac{|z|^2 + 2\delta^2}{(|z|^2 + \delta^2)^{3/2}} - C\varepsilon.$$

Since $\Delta w_\varepsilon = 16\varepsilon(|z|^2 + 2\delta^2)(|z|^2 + \delta^2)^{-3/2}$ there, (8) and (7) give $e^{-2(U+w_\varepsilon)}K(g_\varepsilon) \geq c_0 - C''\varepsilon$.

On $\{a_w \leq |z| \leq b\}$. Here $\rho \geq a_w$, so Lemma 4.3(v),(vi) give $|\mu_\varepsilon| + |D\mu_\varepsilon| + |D^2\mu_\varepsilon| \leq C\varepsilon/a_w$, and Lemma 5.2 gives $\|w_\varepsilon\|_{C^2(\{|z| \geq a_w\})} \leq CA\varepsilon$ and $\Delta w_\varepsilon \geq -CA\varepsilon$. Hence $e^{-2(U+w_\varepsilon)}K(g_\varepsilon) \geq c_0 - C'''\varepsilon$.

On $\{b \leq |z| \leq 1\}$ we have $g_\varepsilon = h$. So $K(g_\varepsilon) > 0$ on $\mathbb{D}$ once $\varepsilon \leq \varepsilon_1 := \min(\varepsilon_0, c_0/(2C''), c_0/(2C'''))$. $\square$

6 Proof of Theorem 1.1

6.1 The case $k = 0$

Let $g = \Pi_\kappa(u, \varphi)$ and $h = e^{-2u}g_\kappa$. With $\varepsilon_l = 1/(l+1)$, Proposition 3.1 gives $h_l = \Pi_\kappa(u, \varphi_{\varepsilon_l}) \in \mathcal{R}_{\geq 0}(M_\kappa)$, isometric to h (so in $\mathcal{R}_{> 0}$ if h is), equal to h outside $\{|z| < \frac{1}{2}\}$, with $h_l \rightarrow h$ in C^0 , while $\partial\varphi_{\varepsilon_l}(0) = e^i$ for all l , so $\varphi_{\varepsilon_l} \not\rightarrow \text{id}$ in C^1 . Lemma 2.3 transfers this to $g_l = \varphi^*h_l = \Pi_\kappa(u, \varphi_{\varepsilon_l} \circ \varphi)$, which is isometric to g and equals g outside $\varphi^{-1}(\{|z| < \frac{1}{2}\})$.

6.2 The case $k \geq 1$, g not flat

By Lemma 2.3 we may assume $\varphi = \text{id}$, i.e. $g = h = e^{-2u}g_\kappa$; the general case follows by pulling back with φ . In the chart, $h = e^{-2V}|dz|^2$ with $V = u$ if $\kappa = 0$ and $V = u + \log(1 + |z|^2) - \log 2$ if $\kappa = 1$. Since $K(h) \geq 0$ and h is not flat, the open set $\{K(h) > 0\}$ is non-empty; choose a point $p \in \mathbb{C} \setminus \{0, 1\}$ in it and $r_0 > 0$ such that the closed disc $B = \{|z - p| \leq r_0\}$ does not contain 0 or 1 and $K(h) > 0$ on B . In the coordinate $\zeta = (z - p)/r_0$ we have $h = e^{-2U}|d\zeta|^2$ with $U(\zeta) = V(p + r_0\zeta) - \log r_0$, and $\Delta U = r_0^2(\Delta V)(p + r_0\zeta) > 0$ on $\overline{\mathbb{D}}$, so (7) holds for some $c_0 > 0$. For $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1, 1/C_0)$ let Φ_ε , μ_ε and w_ε be as in Sections 4 and 5 (in the variable ζ), and define U'_ε by

$$U'_\varepsilon \circ \Phi_\varepsilon = U + \log |\partial \Phi_\varepsilon| + w_\varepsilon.$$

This is a smooth function that equals U outside $\{|\zeta| \leq b\}$. Let $\varphi_\varepsilon(z) = p + r_0\Phi_\varepsilon((z - p)/r_0)$, extended by the identity outside B , and let u_ε be the smooth function on M_κ that equals u outside B and satisfies $u_\varepsilon - u = (U'_\varepsilon - U)((z - p)/r_0)$ on B . Then $\varphi_\varepsilon \in \mathcal{D}(M_\kappa)$ by Lemma 4.4, and by Lemma 2.1 the metric $g_\varepsilon := \Pi_\kappa(u_\varepsilon, \varphi_\varepsilon)$ equals, in the coordinate ζ on B ,

$$\Phi_\varepsilon^*(e^{-2U'_\varepsilon}|d\zeta|^2) = e^{-2U-2w_\varepsilon}|d\zeta + \mu_\varepsilon d\bar{\zeta}|^2,$$

and $g_\varepsilon = h$ outside B . By Proposition 5.1, $K(g_\varepsilon) > 0$ on B and $K(g_\varepsilon) = K(h) \geq 0$ elsewhere; g_ε is complete because it equals h outside B . Hence $g_\varepsilon \in \mathcal{R}_{\geq 0}(M_\kappa)$ (and $g_\varepsilon \in \mathcal{R}_{> 0}$ if $h \in \mathcal{R}_{> 0}$), $u_\varepsilon \in \mathcal{O}_{\geq 0}(M_\kappa)$, and $\Pi_\kappa^{-1}(g_\varepsilon) = (u_\varepsilon, \varphi_\varepsilon)$. Now let $\varepsilon = \varepsilon_l \rightarrow 0$.

- $g_{\varepsilon_l} \rightarrow h$ in the C^k topology, by Proposition 5.1(c) (changing coordinates from ζ to z multiplies the coefficients by r_0^{-2} and their j -th derivatives by r_0^{-2-j}), and $g_{\varepsilon_l} - h$ is supported in B .
- $\varphi_{\varepsilon_l} \rightarrow \text{id}$ in C^k , since $\varphi_\varepsilon - \text{id} = r_0\psi_\varepsilon((z - p)/r_0)$ and $\|\psi_\varepsilon\|_{C^k} \leq C\varepsilon$ (Lemma 4.3(iv)).
- $\varphi_{\varepsilon_l} \not\rightarrow \text{id}$ in C^{k+1} , since $|\partial^{k+1}(\varphi_\varepsilon - \text{id})(p)| = r_0^{-k}|\partial^{k+1}\psi_\varepsilon(0)| \geq r_0^{-k}(k+1)!$ by Lemma 4.5.
- $u_{\varepsilon_l} \not\rightarrow u$ in C^k . Otherwise, along a subsequence, $U'_{\varepsilon_l} \rightarrow U$ in $C^k(\overline{\mathbb{D}})$. Since $\Phi_{\varepsilon_l} \rightarrow \text{id}$ in $C^k(\overline{\mathbb{D}})$ and $\Phi_{\varepsilon_l}(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$, Lemma 2.4 gives $U'_{\varepsilon_l} \circ \Phi_{\varepsilon_l} \rightarrow U$ in $C^k(\overline{\mathbb{D}})$, that is, $\log |\partial \Phi_{\varepsilon_l}| + w_{\varepsilon_l} \rightarrow 0$. As $w_{\varepsilon_l} \rightarrow 0$ in C^k (Proposition 5.1(a)), this contradicts $\|\log |\partial \Phi_{\varepsilon_l}|\|_{C^k(\overline{\mathbb{D}})} \geq (k+1)!/2$ (Lemma 4.5).

This proves (b) for non-flat g . Every metric in $\mathcal{R}_{\geq 0}(S^2)$ is non-flat by the Gauss–Bonnet theorem, so this completes the case $\kappa = 1$.

6.3 The case $\kappa = 0$, $k \geq 1$, g flat

By Lemma 2.3 we may assume $g = h = e^{-2u}g_0$. Since h is complete and flat and $\mathbb{C}$ is simply connected, $(\mathbb{C}, h)$ is isometric to the Euclidean plane. An isometry $\iota: (\mathbb{C}, h) \rightarrow (\mathbb{C}, g_0)$ is conformal for g_0 , hence $\iota(z) = \lambda z + c$ or $\lambda \bar{z} + c$, and $h = \iota^*g_0 = |\lambda|^2g_0$. So u is a constant c_* .

Let $v = \frac{1}{2}\log(1 + |z|^2)$ and, for $0 < t \leq 1$, $U_t = c_* + tv$ and $h_t = e^{-2U_t}g_0$. Then $\Delta U_t = 2t(1 + |z|^2)^{-2} > 0$, so $K(h_t) > 0$; h_t is complete, because a curve from $\{|z| \leq R\}$ to infinity has h_t -length at least $e^{-c_*} \int_R^\infty (1 + r^2)^{-t/2} dr = \infty$; and $h_t \rightarrow g$ in the C^∞ topology as $t \rightarrow 0$. On $\overline{\mathbb{D}}$ we have $\Delta U_t \geq t/2$. For $l \in \mathbb{N}$ put $t_l = 1/l$ and apply Sections 4 and 5 to $U = U_{t_l}$, centred at 0 (which is allowed: φ_ε fixes 0, and it is the identity near 1 because $b < 1$), with a parameter ε_l chosen so that

$$0 < \varepsilon_l \leq \min(\varepsilon_0, 1/C_0, \varepsilon_1(U_{t_l}), 1/l) \quad \text{and} \quad C_{U_{t_l}}\varepsilon_l \leq 1/l.$$

This gives $\varphi_l = \text{id} + \psi_{\varepsilon_l}$, $w_l = w_{\varepsilon_l}$, u_l defined by $u_l \circ \varphi_l = U_{t_l} + \log |\partial \varphi_l| + w_l$, and $g_l = \Pi_0(u_l, \varphi_l)$, which equals $e^{-2U_{t_l}-2w_l}|dz + \mu_{\varepsilon_l}d\bar{z}|^2$ on $\overline{\mathbb{D}}$ and h_{t_l} outside $\overline{\mathbb{D}}$. As in Section 6.2, $g_l \in \mathcal{R}_{> 0}(\mathbb{C})$ and $\Pi_0^{-1}(g_l) = (u_l, \varphi_l)$. Note that $\varepsilon_l \leq 1/l$, so $\varepsilon_l \rightarrow 0$; the second and third claims below use this.

- $g_l \rightarrow g$ in the C^k topology: for a compact $Q \subset \mathbb{C}$, $\|g_l - g\|_{C^k(Q)} \leq \|g_l - h_{t_l}\|_{C^k(\mathbb{D})} + \|h_{t_l} - g\|_{C^k(Q)} \leq 1/l + o(1)$.
- $\varphi_l \rightarrow \text{id}$ in C^k but not in C^{k+1} : $\|\varphi_l - \text{id}\|_{C^k} \leq C\varepsilon_l$, where C does not depend on l because ψ_ε does not depend on U ; and $|\partial^{k+1}(\varphi_l - \text{id})(0)| \geq (k+1)!$ by Lemma 4.5.
- $u_l \not\rightarrow c_*$ in C^k : otherwise, along a subsequence, $u_l \circ \varphi_l \rightarrow c_*$ in $C^k(\mathbb{D})$ by Lemma 2.4; since also $U_{t_l} \rightarrow c_*$ and $\|w_l\|_{C^k} \leq C\varepsilon_l \rightarrow 0$ with C independent of l (Proposition 5.1(a)), we would get $\log |\partial \varphi_l| \rightarrow 0$ in $C^k(\mathbb{D})$, contradicting Lemma 4.5.

Pulling back by φ (Lemma 2.3) completes the proof of (b), and of Theorem 1.1. $\square$

7 Remarks

Remark 7.1 (Sharpness). Let $k \geq 1$ and $\beta \in (0, 1)$. The identity map $\mathcal{R}_{\geq 0}^k(M_\kappa) \rightarrow \mathcal{R}_{\geq 0}^{k-1+\beta}(M_\kappa)$ is continuous, because on a closed disc $\|f\|_{C^{k-1+\beta}} \leq C\|f\|_{C^k}$. Since $k-1+\beta \notin \mathbb{Z}$, the map $\Pi_\kappa^{-1} : \mathcal{R}_{\geq 0}^{k-1+\beta} \rightarrow \mathcal{O}_{\geq 0}^{k-1+\beta} \times \mathcal{D}^{k+\beta}$ is continuous [BH15, Theorem 4.1], [BB18, Theorem 8.4 and (9.2)]. Hence Π_κ^{-1} is continuous from $\mathcal{R}_{\geq 0}^k$ into $\mathcal{O}_{\geq 0}^{k-1+\beta} \times \mathcal{D}^{k+\beta}$: only the endpoint exponents are lost. Consistently, in Theorem 1.1(b) we have $\varphi_l \rightarrow \varphi$ in C^k . For $k = 0$ there is no such statement.

Remark 7.2 (Other topologies on the diffeomorphisms). For $k \geq 1$, the sequences of Theorem 1.1(b) have $g_l \rightarrow g$ in C^k while $u_l \not\rightarrow u$ in C^k . So the map $g \mapsto u$ (the first component of Π_κ^{-1}) from $\mathcal{R}_{\geq 0}^k$ to $\mathcal{O}_{\geq 0}^k$ is discontinuous at every point, and Π_κ is not a homeomorphism for any topology on $\mathcal{D}(M_\kappa)$. For $k = 0$ the spirals leave u unchanged, and their pullbacks $\varphi_\varepsilon^* g_\kappa$ converge to g_κ in C^0 . At flat metrics the following example shows that the u -component is discontinuous for $k = 0$ as well.

Proposition 7.3 (Smoothed cones). *For $0 < \varepsilon \leq \frac{1}{2}$ let $\eta = e^{-1/\varepsilon}$, $\beta = \varepsilon/(1-\varepsilon)$, $\delta = \eta^{1+\beta}$, $A(s) = (s + \eta^2)/(1 + \eta^2)$ and*

$$\varphi_\varepsilon(z) = z A(|z|^2)^{\beta/2}, \quad u_\varepsilon(w) = \frac{\varepsilon}{2} \log(|w|^2 + \delta^2).$$

Then $u_\varepsilon \in \mathcal{O}_{>0}(\mathbb{C})$, $\varphi_\varepsilon \in \mathcal{D}(\mathbb{C})$, and $g_\varepsilon = \Pi_0(u_\varepsilon, \varphi_\varepsilon)$ satisfies $\sup_{\mathbb{C}} |g_\varepsilon - g_0| \leq C\varepsilon$ (coefficientwise, with a universal C), so $g_\varepsilon \rightarrow g_0$ uniformly on $\mathbb{C}$. However $u_\varepsilon(0) = -1/(1-\varepsilon)$ and $|\partial \varphi_\varepsilon(0)| \leq e^{-1}$; in particular $u_\varepsilon \not\rightarrow 0$ in C^0 and $\varphi_\varepsilon^ g_0 \not\rightarrow g_0$ in C^0 .*

Proof. $\Delta u_\varepsilon = 2\varepsilon\delta^2(|w|^2 + \delta^2)^{-2} > 0$, and $e^{-2u_\varepsilon} g_0 = (|w|^2 + \delta^2)^{-\varepsilon} |dw|^2$ is complete since $\varepsilon < 1$ (radial lengths diverge); so $u_\varepsilon \in \mathcal{O}_{>0}(\mathbb{C})$. The map φ_ε is radial, $z \mapsto zF(|z|^2)$ with $F = A^{\beta/2}$ smooth and positive, and $r \mapsto rF(r^2)$ is a strictly increasing bijection of $[0, \infty)$ (its logarithmic derivative is $1 + \beta t > 0$, where $t = |z|^2/(|z|^2 + \eta^2)$); hence φ_ε is a diffeomorphism of $\mathbb{C}$, and it fixes 0 and 1. One computes $\partial \varphi_\varepsilon = F(1 + \frac{\beta t}{2})$ and $\bar{\partial} \varphi_\varepsilon = F \frac{\beta t}{2} \frac{z}{\bar{z}}$, so $|\bar{\partial} \varphi_\varepsilon| < |\partial \varphi_\varepsilon|$; thus φ_ε preserves orientation, $\varphi_\varepsilon \in \mathcal{D}(\mathbb{C})$, and by Lemma 2.1 $g_\varepsilon = \lambda dz + \mu d\bar{z}^2$ with $|\mu| = \frac{\beta t/2}{1 + \beta t/2} \leq \beta/2 \leq \varepsilon$ and

$$\log \lambda = -\varepsilon \log(sA^\beta + \delta^2) + \beta \log A + 2 \log(1 + \frac{\beta t}{2}), \quad s = |z|^2.$$

We use $\beta - \varepsilon - \varepsilon\beta = 0$. If $s \geq \eta^2$, then $A \geq \eta^2$, so $sA^\beta \geq \eta^{2+2\beta} = \delta^2$, and $1 \leq (s + \eta^2)/s \leq 2$; writing $\log(sA^\beta + \delta^2) = \log s + \beta \log A + \log(1 + \delta^2/(sA^\beta))$ gives $\log \lambda = \varepsilon \log(A/s) - \varepsilon \log(1 + \delta^2/(sA^\beta)) + 2 \log(1 + \frac{\beta t}{2}) = O(\varepsilon)$. If $s \leq \eta^2$, then $A/\eta^2 \in [(1 + \eta^2)^{-1}, 2]$ and $sA^\beta \leq 2^\beta \delta^2$, so $\log(sA^\beta + \delta^2) = \log \delta^2 + O(1)$ and $\beta \log A = \beta \log \eta^2 + O(\beta)$, whence $\log \lambda = 2 \log \eta (\beta - \varepsilon - \varepsilon\beta) + O(\varepsilon) = O(\varepsilon)$. The implied constants are universal. So $|\lambda - 1| \leq C\varepsilon$ and $|\mu| \leq \varepsilon$, and (2) gives the uniform bound. Finally $u_\varepsilon(0) = \varepsilon \log \delta = -(1 + \beta) = -1/(1 - \varepsilon)$ and $|\partial \varphi_\varepsilon(0)| = (\eta^2/(1 + \eta^2))^{\beta/2} \leq \eta^\beta = e^{-1/(1-\varepsilon)}$. $\square$

By Lemma 2.3 (and multiplication by the constant e^{-2c_*}) the same holds at every flat metric on $\mathbb{C}$.

Remark 7.4 (Variants and scope). (i) [BB18] also considers the map (8.1) there, $\mathcal{D}^{\gamma+1}(S^2) \times \Gamma^\gamma(S^2) \rightarrow \mathcal{R}^\gamma(S^2)$, $(\varphi, u) \mapsto \varphi^*(e^{-2u}g_1)$, on all smooth metrics on S^2 . The constructions of Sections 3 and 4 and Section 6.2, with $w_\varepsilon = 0$, at any point $p \notin \{0, 1, \infty\}$ and without the curvature step, show that its inverse is discontinuous at every point for every integer $\gamma \geq 0$. (ii) We do not treat $\mathbb{RP}^2$, which is also covered by [BB18, Theorem 8.4] for non-integral exponents; an equivariant version of the constructions should work, but we have not written it out. Nor do we treat the spaces $\mathcal{R}_{\geq \lambda}(S^2)$ with $\lambda \neq 0$.

8 Computations

The proofs above do not use any computation. As sanity checks, the sequences of Sections 3 to 5 were evaluated numerically for the model metrics $U = \frac{1}{2} \log(1 + |z|^2)$ ($\kappa = 0$, $K(h) = 2/(1 + |z|^2)$) and $U = \log(1 + |z|^2) - \log 2$ ($\kappa = 1$, the round sphere), with $a = 0.2$, $b = 0.9$, $a_w = 0.1$, $b_w = 0.3$, $[a_\beta, b_\beta] = [0.35, 0.9]$ and $A = 16$. The cut-offs are steps and a bump in $|z| = \sqrt{s}$: with $S(t) = e^{-1/t}/(e^{-1/t} + e^{-1/(1-t)})$ for $0 < t < 1$, $S = 0$ on $(-\infty, 0]$ and $S = 1$ on $[1, \infty)$,

$$\chi(s) = 1 - S\left(\frac{\sqrt{s} - a}{b - a}\right), \quad \chi_w(s) = 1 - S\left(\frac{\sqrt{s} - a_w}{b_w - a_w}\right), \quad \beta(s) = c_\beta S(\tau)(1 - S(\tau)), \quad \tau = \frac{\sqrt{s} - a_\beta}{b_\beta - a_\beta},$$

with $c_\beta > 0$ chosen so that $\int \beta = 1$. All derivatives were computed by exact truncated Taylor arithmetic (no finite differences) on log-polar grids that resolve $\delta/100 \leq |z| \leq 0.95$; below, $\min K$ denotes the minimum over such a grid. Table 1 shows typical values.

k	κ	ε	$\ \varphi_\varepsilon - \text{id}\ _{C^k}$	$\ g_\varepsilon - h\ _{C^k(\overline{\mathbb{D}})}$	$\partial^{k+1}(\varphi_\varepsilon - \text{id})(0)$	$\min K(g_\varepsilon)$
1	0	0.02 / 0.01	0.0158 / 0.0079	0.893 / 0.395	-3.9503 / -3.9752	0.438 / 0.930
1	1	0.02 / 0.01	0.0158 / 0.0079	3.656 / 1.645	-3.9503 / -3.9752	0.655 / 0.826
2	0	0.02 / 0.01	0.0560 / 0.0280	0.551 / 0.276	-11.851 / -11.926	1.034 / 1.051
2	1	0.02 / 0.01	0.0560 / 0.0280	2.204 / 1.102	-11.851 / -11.926	0.874 / 0.937
3	0	0.02 / 0.01	0.311 / 0.156	3.511 / 1.755	-47.404 / -47.702	1.051 / 1.051
3	1	0.02 / 0.01	0.311 / 0.156	14.04 / 7.021	-47.404 / -47.702	0.922 / 0.961

Table 1: Numerical values (sanity checks) for the maps of Section 4 and the metrics of Section 5. The C^k distances decrease approximately linearly in ε (for $k = 1$ the metric distances contain terms of order ε^2 from e^{-2w_ε} , since $A\varepsilon$ is not small); the values of $\partial^{k+1}(\varphi_\varepsilon - \text{id})(0)$ agree with $-(k+1)!(2 + \varepsilon c_\delta)$ of Lemma 4.5 to all printed digits and tend to $-2(k+1)!$.

For the tested values $\varepsilon \in \{0.02, 0.015, 0.01\}$ the minimum of $K(g_\varepsilon)$ over the grid is positive for $k = 1, 2, 3$ and $\kappa = 0, 1$. For $k = 1$, $\kappa = 0$ and the larger values $\varepsilon = 0.04, 0.03$ a negative minimum occurs in the outer annulus $a_w \leq |z| \leq b$. This is the smallness condition of Proposition 5.1 at work. On that annulus the proof gives $e^{-2(U+w_\varepsilon)}K(g_\varepsilon) \geq \Delta U - C'''\varepsilon$, where C''' depends on the cut-offs, the radii, A and $\|U\|_{C^2(\overline{\mathbb{D}})}$ but not on ε , so $K(g_\varepsilon) > 0$ once ε is below a threshold of order c_0/C''' (this is ε_1 in the proof). Two terms contribute comparably to C''' : the negative part $-A\varepsilon c_w \beta$ of Δw_ε , and the term $4\text{Re}(\partial^2 \bar{\partial} \psi)$ that comes from the transition of χ ; at the negative minimum found in the second independent run described below ($k = 1$, $\kappa = 0$) they are about -27ε and -21ε . Hence the value of ε below which the numerics show $K \geq 0$ depends on the cut-off shapes and on U . The program of item 2 below (its first parameter set) and the second independent run use the same radii but other cut-offs, steps and a bump in $|z|^2$ built from $e^{-1/x}$ (the same in both), and both find a negative

minimum already at $\varepsilon = 0.02$ (about -0.008 and -0.009 for $k = 1$, $\kappa = 0$, in the outer annulus); for these cut-offs the second run finds $C''' \approx 45\text{--}48$ for all tested ε between 0.001 and 0.04 and all tested U , and with steeper cut-offs built from e^{-1/x^2} it finds $C''' \approx 240$ and negative minima at $\varepsilon = 0.02$ and 0.01 (for $k = 1$, and for $k = 2$ with $\kappa = 0$). For smaller ΔU the threshold is smaller: in the chart of Section 6.2 at the point $\frac{1}{2}$ of the round sphere with $r_0 = 0.3$, where $\Delta U \approx 0.16$ at the minimum, that run finds negative values in the outer annulus for $\varepsilon = 0.02, 0.01, 0.005$ and positive ones at $\varepsilon = 0.0025$ ($k = 1$), and for $k = 2$ negative values at $\varepsilon = 0.02$ and 0.01 . This does not contradict Proposition 5.1, which asserts positivity only for $\varepsilon \leq \varepsilon_1$. Without the correction w_ε , the minimum of K for $k = 1$ and $\varepsilon = 0.01$ is about $-2 \cdot 10^{42}$, which shows that the correction of Section 5.2 is needed. The spirals of Section 3 give $\|h_\varepsilon - h\|_{C^0} = 0.49, 0.22, 0.10$ for $\varepsilon = 0.2, 0.1, 0.05$ ($\kappa = 0$), and the smoothed cones of Proposition 7.3 give $\sup_{\mathbb{C}} |\log \lambda|/\varepsilon \in [1.00, 1.18]$ for $0.005 \leq \varepsilon \leq 0.2$.

Verification. The claims were checked as follows.

1. The programs behind Table 1 use exact Taylor-jet arithmetic. They also check Lemma 5.3, the Jacobian of the spirals, and that $(K_\gamma - 4 \operatorname{Re} \partial^2 \mu)/t^2$ stays bounded for $\mu = O(t)$ (Lemma 2.2).
2. A first independent verification run, which was AI-assisted, re-derived every step of the proofs, re-ran these programs with byte-identical outputs, and recomputed $K(g_\varepsilon)$ by a separate method: finite differences and the formula $K(g_\varepsilon) = e^{2V} \Delta_{\varphi_\varepsilon^*|dz|^2} V$ with $V = U + \log |\partial \varphi_\varepsilon| + w_\varepsilon$, in 36 configurations, together with a 60-digit spot check at the scale δ ; it found $K > 0$ throughout. A further program, written for the release, computes $K(g_\varepsilon)$ as the curvature $e^{2F} \Delta F$ of the conformal metric $e^{-2F} |dw|^2$ with $F = V \circ \varphi_\varepsilon^{-1}$, inverting φ_ε by Newton's method and using 120- to 238-digit arithmetic near 0 (down to $|z| = \delta/100$); it uses neither Taylor jets nor Brioschi's formula, and its cut-off functions differ from those above. For $k = 1, 2, 3$, $\kappa = 0, 1$, $\varepsilon \in \{0.0125, 0.008, 0.005\}$ and two sets of cut-offs it finds $K(g_\varepsilon) > 0$ at every grid point; for $\varepsilon = 0.02$ the same holds except for $k = 1$, $\kappa = 0$, where values down to -0.14 occur in the outer annulus, so that $\varepsilon = 0.02$ is not yet small enough for these cut-offs. At $z = -2\delta$, for $k = 1$ and $\varepsilon = 0.02$, it gives $K(g_\varepsilon) \approx 3.6 \cdot 10^{20}$, and $K \approx -4.1 \cdot 10^{20}$ without the correction w_ε . It also checks Lemma 2.2 with the curvature computed from the Riemann tensor, the formulas (3), (4), the values in Lemma 4.5, and Propositions 3.1 and 7.3.
3. A second independent verification run, also AI-assisted, re-derived every step of the proofs and wrote its own programs from the definitions in this note (included in the release): truncated Taylor jets in 56- to 480-digit decimal arithmetic, the radii above, and cut-offs that are steps and a bump in $|z|^2$ built from $e^{-1/x}$ (for comparison also from e^{-1/x^2}). It computes $K(g_\varepsilon)$ from the Christoffel symbols and, separately, as $e^{2V} \Delta_{\varphi_\varepsilon^*|dw|^2} V$; the two agree to a relative error below $2 \cdot 10^{-51}$ at every grid point, and a third code path (finite differences and Brioschi's formula) agrees with them to $2 \cdot 10^{-20}$ at 224 points, including $|z| = \delta/3, \delta, 5\delta$ and all cut-off annuli. It confirms the values in Lemma 4.5, the approximately linear decay of the C^k distances, the inequalities of Lemmas 2.2 and 5.3 on test data, and Propositions 3.1 and 7.3, and it estimated the constant C''' as described above. It also re-ran the programs of items 1 and 2 from the source archive of the release: the outputs of item 1 were reproduced byte for byte, and those of item 2 up to timing fields.

Scope and priority. Theorem 1.1 answers the question of [Bel17, p. 149] in the precise setting of [BB18, (8.3), (9.2), Theorem 8.4]: $\varphi \in \mathcal{D}^{k+1}$, $u \in \mathcal{O}^k$, $g \in \mathcal{R}^k$, all with uniform convergence on compact sets, for $M_0 = \mathbb{C}$ and $M_1 = S^2$ and every integer $k \geq 0$, both for $\mathcal{R}_{\geq 0}$ and for $\mathcal{R}_{> 0}$. The answer is negative. The report writes the map with the pushforward φ_* in place of the pullback φ^* ; as

explained in the paragraph after Theorem 1.1, this does not change the answer, because $\varphi_* = (\varphi^{-1})^*$ and inversion is a homeomorphism of $\mathcal{D}^{k+1}(M_\kappa)$. By Remark 7.2 and Proposition 7.3 the answer stays negative under other natural choices of topology on the diffeomorphism group, except that for $k = 0$ we have treated only flat metrics in that setting. For what we do not treat see Remark 7.4. We read the Oberwolfach report, the arXiv versions of [BB18] (v2) and [BH15] (v3), and later papers of Belegradek; we did not read the erratum [BH16] itself and rely on the account of it in [BB18, proof of Theorem 8.4]. In September 2026 we searched arXiv (Belegradek’s author listing and keyword queries), OpenAlex (all works citing [BH15], [BH16] or [BB18]), zbMATH and the web, and found no treatment of the integer case. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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