

Arbitrary Local-Cohomology Lengths between Consecutive Ideal Powers

Alper Ferudun*

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Abstract

For every field k and every integer $i \geq 0$, we construct a fixed complete regular local ring $(R, \mathfrak{m})$ of dimension $2i + 2$ and a fixed prime complete-intersection ideal I with the following property. Every sequence $(a_n)_{n \geq 1}$ of nonnegative integers is realized by ideals J_n satisfying $I^{n+1} \subseteq J_n \subseteq I^n$ and $\text{length}_R H_{\mathfrak{m}}^i(R/J_n) = a_n$. All lower local cohomology vanishes, and all transition maps below degree $i + 1$ are zero. Thus even adjacent-power bounds impose no asymptotic restriction on these lengths for arbitrary decreasing cofinal families. The construction inserts a one-column module presentation into each successive homogeneous layer; its length is computed directly in the top Čech module of a power-series ring. The families need not be multiplicative. Accordingly, the result concerns the unrestricted-family formulation and makes no assertion about ordinary, symbolic, or Frobenius powers.

1 The scope of the realization problem

Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$, and let (J_n) be decreasing and cofinal with (I^n) . One can ask what asymptotic behavior is possible for the finite lengths $\text{length}_R H_{\mathfrak{m}}^i(R/J_n)$. This formulation appears in Dao's Oberwolfach contribution [1, Question 1, p. 3240], whose motivating examples are structured powers. Here we isolate what follows from the decreasing and cofinal assumptions alone.

The answer is unrestricted: every nonnegative integer sequence occurs, even when the family lies between consecutive powers of a fixed prime ideal in a regular local ring. Throughout this note a *decreasing family* does not include the additional condition $J_m J_n \subseteq J_{m+n}$.

Theorem 1. *Fix a field k and an integer $i \geq 0$. There exist a complete regular local ring $(R, \mathfrak{m})$, with residue field k and dimension $d = 2i + 2$, and a prime complete-intersection ideal I such that the following holds. For every sequence $a_n \in \mathbb{Z}_{\geq 0}$, there are ideals J_n , $n \geq 1$, satisfying*

$$I^{n+1} \subseteq J_n \subseteq I^n, \tag{1}$$

$$\text{length}_R H_{\mathfrak{m}}^i(R/J_n) = a_n, \tag{2}$$

$$H_{\mathfrak{m}}^q(R/J_n) = 0 \quad (0 \leq q < i). \tag{3}$$

Moreover, $\dim(R/J_n) = i + 1$, and the natural maps

$$H_{\mathfrak{m}}^q(R/J_{n+1}) \longrightarrow H_{\mathfrak{m}}^q(R/J_n)$$

are zero for every $q < i + 1$. The ring R and the ideal I are independent of the sequence.

*Mercury Software GmbH. Email: alper@mercurycodelab.com.

In particular, every quotient in the construction has finite local cohomology below its dimension. The adjacent-power inclusions already give $J_{n+1} \subseteq J_n$ and cofinality. No choice of a rapidly growing subsequence of powers is involved.

2 A module with a prescribed finite local-cohomology length

Set $s = i + 1$ and

$$A = k[[u_1, \dots, u_i, v]], \quad \mathfrak{n} = (u_1, \dots, u_i, v).$$

For $a \geq 0$, let

$$c_a = (u_1, \dots, u_i, v^a)^\top \in A^s, \quad M_a = A^s / A c_a.$$

When $i = 0$, this means $A = k[[v]]$ and $M_a = A/(v^a)$.

Lemma 2. *For every $a \geq 0$,*

$$H_{\mathfrak{n}}^q(M_a) = 0 \quad (0 \leq q < i), \quad \text{length}_A H_{\mathfrak{n}}^i(M_a) = a.$$

Proof. Since A is a domain and $v^a \neq 0$, there is an exact sequence

$$0 \longrightarrow A \xrightarrow{c_a} A^s \longrightarrow M_a \longrightarrow 0. \quad (4)$$

The ring A is regular local of dimension s , so $H_{\mathfrak{n}}^q(A) = 0$ for $q < s$. For example, this follows from the Koszul complexes on powers of the regular sequence $u_1, \dots, u_i, v$, whose direct limit is the Čech complex. Writing $E = H_{\mathfrak{n}}^s(A)$, the long exact sequence of (4) gives the asserted lower vanishing and

$$H_{\mathfrak{n}}^{s-1}(M_a) \simeq \ker(E \xrightarrow{c_a} E^s) = 0 :_E (u_1, \dots, u_i, v^a). \quad (5)$$

This exact segment also applies when $s = 1$.

The top Čech module E has the inverse-monomial k -basis

$$u_1^{-b_1} \dots u_i^{-b_i} v^{-c}, \quad b_1, \dots, b_i, c \geq 1.$$

Indeed, in the quotient of the localization at $u_1 \dots u_i v$ by the sum of the localizations missing one variable, precisely these monomials survive. Each representative has finitely many surviving terms, even though its numerator may be a power series.

Multiplication by u_j kills the basis elements with $b_j = 1$ and sends the remaining basis elements to distinct basis elements. Thus simultaneous annihilation by $u_1, \dots, u_i$ forces all $b_j = 1$. Multiplication by v^a then kills exactly the elements with $1 \leq c \leq a$. Hence the kernel in (5) has basis

$$u_1^{-1} \dots u_i^{-1} v^{-c}, \quad 1 \leq c \leq a.$$

Its k -dimension, and therefore its A -length, is a . For $a = 0$ this basis is empty and the kernel is zero. $\square$

This is the Čech version of the local-duality calculation for $A/(u_1, \dots, u_i, v^a)$. The useful feature is that the presentation always has the same number of generators.

3 Proof of the realization theorem

Take

$$R = A[[x_0, \dots, x_i]], \quad \mathfrak{m} = (\mathfrak{n}, x_0, \dots, x_i), \quad I = (x_0, \dots, x_i).$$

The ring has dimension $2i + 2$, and I is prime and generated by a regular sequence. These choices depend only on k and i .

For $n \geq 1$, distinguish the s degree- n monomials

$$e_{n,j} = x_0^{n-1} x_j, \quad 0 \leq j \leq i.$$

Let W_n be the set of all other degree- n monomials in the x variables. Given the desired sequence (a_n) , define

$$g_n = \sum_{j=1}^i u_j e_{n,j-1} + v^{a_n} e_{n,i}, \quad (6)$$

$$J_n = I^{n+1} + (W_n) + (g_n). \quad (7)$$

Empty sums and empty sets have their usual meanings. All the added generators have x -degree n , proving (1). In particular,

$$J_{n+1} \subseteq I^{n+1} \subseteq J_n$$

for every possible choice of a_n and a_{n+1} .

The quotient R/I^{n+1} is free over A on the x -monomials of degree at most n . The new generators in (7) affect only the degree- n layer. Multiplying such a generator by any x_j gives zero modulo I^{n+1} ; its arbitrary power-series multiples consequently reduce to its A -multiples. After killing W_n , the remaining degree- n layer has basis $e_{n,0}, \dots, e_{n,i}$, and its single relation is precisely c_{a_n} . We obtain the A -module decomposition

$$R/J_n \simeq A^{N_n} \oplus M_{a_n}, \quad N_n = \binom{n+s-1}{s}. \quad (8)$$

Here N_n counts the monomials of x -degree less than n . Only an A -linear decomposition is asserted.

Every x_j is nilpotent on R/J_n . Thus every term in its Čech complex involving localization at an x_j is zero, and

$$H_{\mathfrak{m}}^q(R/J_n) \simeq H_{\mathfrak{n}}^q(R/J_n)$$

as A -modules. The free summand in (8) has zero local cohomology below degree s . Lemma 2 therefore proves (2) and (3). Finite k -dimension equals R -length, since R has residue field k . The sandwich implies $\sqrt{J_n} = I$, so each quotient has dimension $s = i + 1$.

Finally, the natural map $R/J_{n+1} \rightarrow R/J_n$ factors through R/I^{n+1} . This intermediate module is free over A , and the same Čech argument shows that its local cohomology vanishes in degrees below s . Functoriality proves that every transition map in the theorem is zero. This completes the proof.

4 Consequences and examples

The ordinary powers in our construction satisfy

$$H_{\mathfrak{m}}^q(R/I^n) = 0 \quad (q < i + 1, n \geq 1),$$

because R/I^n is free over A . Thus a change confined to one homogeneous layer suffices to replace these zero lengths by any prescribed sequence.

Corollary 3. *For fixed k and i , the possible finite-length sequences in Theorem 1 are exactly all nonnegative integer sequences. In particular, decreasing cofinal families satisfying (1) need not have a polynomial growth bound or a limit after normalization by $n^{\dim R}$.*

Proof. The necessity of nonnegative integral lengths is automatic, and the theorem proves sufficiency. Choose $a_n = 2^n$ for superpolynomial growth. With $d = 2i + 2$, choose

$$a_n = \begin{cases} 0, & n \text{ even}, \\ n^d, & n \text{ odd}. \end{cases}$$

The normalized liminf is zero and the normalized limsup is one. □

More generally, for any sequence $b_n \geq 0$, the choice $a_n = \lfloor n^d b_n \rfloor$ gives

$$0 \leq b_n - \frac{\text{length}_R H_m^i(R/J_n)}{n^d} < n^{-d}.$$

Thus one can prescribe arbitrary normalized behavior up to this vanishing error. Constant lengths and arbitrary patterns of zero and nonzero lengths are immediate special cases. The zero transition maps explain why this behavior is compatible with a zero inverse limit.

For a concrete positive-degree example, take $i = 1$:

$$\begin{aligned} R &= k[[u, v, x, y]], \quad I = (x, y), \\ J_n &= I^{n+1} + (x^{n-r}y^r : 2 \leq r \leq n) + (ux^n + v^{a_n}x^{n-1}y). \end{aligned}$$

Then $\text{length}_R H_m^1(R/J_n) = a_n$ and $H_m^0(R/J_n) = 0$.

Remark 4. The families in the theorem need not be multiplicative. In degree $i = 0$, the construction is

$$R = k[[v, x]], \quad J_n = (x^{n+1}, v^{a_n}x^n).$$

For $a_1 = 0$ and $a_2 = 1$, one has $J_1 = (x)$ and $J_2 = (x^3, vx^2)$, so $x^2 \in J_1^2 \setminus J_2$. Consequently, the theorem cannot be substituted into a result whose hypotheses require $J_m J_n \subseteq J_{m+n}$.

5 Relation to structured thickenings and verification

Dao and Montaño [2] study ordinary powers, including polynomial upper bounds and rationality phenomena under specified hypotheses. Their monomial results also treat appropriate Noetherian graded families. The examples here do not belong to those classes in general. The ordinary-, symbolic-, and Frobenius-power questions motivating [1] are therefore outside the conclusion.

The distinction between structured and arbitrary thickenings is already important in [3, Remark 3.11 and Examples 5.2–5.3], which gives failures of Kodaira vanishing for arbitrary thickenings. Our assertion is the explicit realization of arbitrary lengths in every chosen cohomological degree, simultaneously along a decreasing family between consecutive powers. The module and Čech ingredients are standard; this note makes no exhaustive priority claim.

An accompanying Python checker performs 84 exact kernel calculations in top-Čech models and 120 Gröbner-basis calculations for finite coefficient truncations of (7), as well as the nonmembership check in Remark 4. These finite checks support reproducibility and are not a substitute for the universal argument. The source archive includes their exact outputs, the source comparison and the detailed self-audit. This preprint was prepared with AI assistance in mathematical exploration, computation, literature comparison and drafting; the author is responsible for its claims. It has not undergone independent peer review.

References

- [1] H. Dao, *Asymptotic Behavior of Local Cohomology of Thickenings of Ideals*, joint work with J. Montaña, I. Smirnov and K.-i. Watanabe, in *Asymptotic Phenomena in Local Algebra and Singularity Theory*, Oberwolfach Report 57/2016, pp. 3240–3241. [doi:10.4171/OWR/2016/57](https://doi.org/10.4171/OWR/2016/57).
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