

The Affine Cone over the Second Secant Variety of $(\mathbb{P}^1)^5$ Is Gorenstein: On Three Questions of Michałek

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Abstract

In his abstract in Oberwolfach Report 19/2016, Michałek asked whether the affine cone over the second secant variety σ_2 of the Segre product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is Gorenstein. We prove that it is. The homogeneous coordinate ring R of $\sigma_2((\mathbb{P}^1)^5) \subset \mathbb{P}^{31}$, the variety of binary tensors of border rank at most two, is Cohen–Macaulay and Gorenstein, with canonical module $R(-4)$, Hilbert series $(1 + 20t + 210t^2 + 772t^3 + 1250t^4 + 772t^5 + 210t^6 + 20t^7 + t^8)/(1 - t)^{12}$, degree 3256 and Castelnuovo–Mumford regularity 8. The proof combines the toric chart of Michałek, Oeding and Zwiernik, on which the variety is a product of an affine space and a Gorenstein toric variety; the decomposition of the coordinate ring due to Raicu, from which we derive the Hilbert series in all degrees; and standard tools, namely Kawamata–Viehweg vanishing for varieties with rational singularities and Serre duality. The identity $\omega = \mathcal{O}(-4)$ for the canonical sheaf is also proved directly, by gluing the toric charts. Michałek, Oeding and Zwiernik had shown that the variety is Gorenstein locally; Oeding (2016) conjectured the arithmetic property. To our knowledge the theorem and its proof are new; no proof that this cone is Gorenstein was found in the literature accessible to us, and a search that finds nothing is not a proof of novelty. The same abstract asks two further questions. Whether the convex hull of the lattice points in any ball is a normal polytope: the answer is no, by a companion note (a ball in $\mathbb{R}^6$, and examples in every dimension at least six). Whether, for a homogeneous polynomial f and a generic linear subspace L , the closure of $\nabla f(L)$ misses $L^\perp$: the answer is yes over $\mathbb{C}$, as the source itself records, attributing the observation to B. Teissier; we add a short self-contained proof and an example showing that it fails in characteristic p dividing $\deg f$, and claim nothing new for this part. This is an unrefereed note.

1 Introduction

1.1 The questions

For a projective variety $X \subset \mathbb{P}^N$ the second secant variety $\sigma_2(X)$ is the closure of the union of the lines joining two points of X . When X is the Segre variety of a product of projective spaces, $\sigma_2(X)$ is the variety of tensors of border rank at most two. In his abstract “Toric structures in nontoric varieties” in the report of the Oberwolfach workshop on toric geometry held from 27 March to 2 April 2016 (joint work with L. Oeding and P. Zwiernik), Michałek recalls the classification of the (locally) Gorenstein varieties $\sigma_2(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n})$ from [MOZ15]: apart from those that fill their ambient space, they are $\sigma_2(\mathbb{P}^a \times \mathbb{P}^a)$, the three cases $(a_1, a_2, a_3) \in \{(1, 1, 3), (1, 3, 3), (3, 3, 3)\}$, and $\sigma_2((\mathbb{P}^1)^5)$ [Mic16, p. 1000, Theorem 2]. He notes that the affine cones in the three-factor family had been shown to be Gorenstein by Oeding [Oed16], and asks [Mic16, p. 1000, Conjecture 3]:

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“Is the affine cone over $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ Gorenstein?”

The abstract then lists two questions presented in the lecture [Mic16, p. 1000, Conjecture 5]: (1) whether the convex hull of all lattice points in any ball is a normal polytope, with a reference to [BGM16, Question 7.2(b)]; and (2) whether, for a homogeneous polynomial f in $n + 1$ variables and a generic linear subspace L of $\mathbb{P}^n$, the intersection $L^\perp \cap \overline{\nabla f(L)}$ is always empty, with a reference to a conjecture of Michałek, Sturmfels, Uhler and Zwiernik [MSUZ16, Section 5]. On p. 1001 the abstract records that B. Teissier observed in Oberwolfach that the answer to (2) is positive, as a special case of his results [Tei73, Tei82], refers to [Huh12] for a recent application and a new proof, and says that (1) still awaits an answer.

The three questions form record OWR-14601-005 of the UnsolvedMath collection [ula26], where the problem is listed as open. The literature note of the record says that the secant-cone question is answered affirmatively for $(\mathbb{P}^1)^5$ and lists [MOZ15] as a source. What [MOZ15] prove for this variety is a local statement: $\sigma_2((\mathbb{P}^1)^5)$ is covered by open sets isomorphic to $\mathbb{A}^5 \times T_{2,5}$ [MOZ15, Theorem 4.5], and the toric variety $T_{2,5}$ is Gorenstein with terminal singularities [MOZ15, Corollary 5.7], so that every local ring of $\sigma_2((\mathbb{P}^1)^5)$ is Gorenstein; the general classification [MOZ15, Theorem 7.18], which is Theorem 2 of the source, is likewise obtained from the local toric cones. The question of the source concerns the affine cone, and it is posed right after that result.

Throughout, $X = \text{Seg}((\mathbb{P}^1)^5) \subset \mathbb{P}^{31}$, $Y = \sigma_2(X)$, and $R = \mathbb{C}[x]/I(Y)$ is the homogeneous coordinate ring of Y , that is, the coordinate ring of the affine cone $\hat{Y} \subset \mathbb{C}^{32}$ of binary 5-tensors of border rank at most two. We read “the affine cone over Y is Gorenstein” as “ R is a Gorenstein ring”; since R is graded, this is the same as asking that the local ring of $\hat{Y}$ at its vertex be Gorenstein [BH98, Section 3.6]. In the terminology of [Oed16, KMZ20], Y is then *arithmetically Gorenstein*. This is stronger than the local property of Y ; it includes the Cohen–Macaulay property of R .

1.2 Results

Theorem A. *Let R be the homogeneous coordinate ring of $Y = \sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{31}$.*

- (a) *R is a Gorenstein ring. It is Cohen–Macaulay of Krull dimension 12, and its canonical module is $\omega_R \cong R(-4)$. Hence the affine cone over Y is Gorenstein.*
- (b) *The Hilbert series of R is*

$$\sum_{r \geq 0} \dim_{\mathbb{C}} R_r t^r = \frac{1 + 20t + 210t^2 + 772t^3 + 1250t^4 + 772t^5 + 210t^6 + 20t^7 + t^8}{(1 - t)^{12}},$$

so $\deg Y = 3256$, the a -invariant of R is -4 , and the Castelnuovo–Mumford regularity of R is 8 (that of $I(Y)$ is 9).

- (c) *Y is projectively normal, $\omega_Y \cong \mathcal{O}_Y(-4)$, and $H^i(Y, \mathcal{O}_Y(k)) = 0$ for $0 < i < 11$ and all $k \in \mathbb{Z}$.*

Theorem A answers Conjecture 3 of [Mic16] in the affirmative and confirms the conjecture of [Oed16, Remark 3.15]. Its Cohen–Macaulay part is the case $\sigma_2((\mathbb{P}^1)^5)$ of Oeding’s conjecture that all secant varieties of Segre products are arithmetically Cohen–Macaulay [Oed16, Conjecture 1.1]. In [Oed16, Table 1] the “socle degree” of an arithmetically Gorenstein variety is the regularity of its ideal (it is 5 for 4×4 matrices of rank at most 2); the entry “?” given there for $\sigma_2((\mathbb{P}^1)^5)$ is therefore 9.

The proof occupies Sections 2 to 5. In the coordinates of *secant cumulants*, Michałek, Oeding and Zwiernik showed that $Y \cap \{x_\emptyset \neq 0\}$ is $\mathbb{A}^5 \times T$ for a normal affine toric variety T , which is Gorenstein [MOZ15, Theorem 4.5 and Corollary 5.7]; we re-prove the facts about T that we need (Proposition 2.2). The complement $H_0 = Y \cap \{x_\emptyset = 0\}$ is an irreducible and reduced hyperplane

section (Lemma 4.2), so $\omega_Y \cong \mathcal{O}_Y(a)$, and Serre duality with the Hilbert polynomial gives $a = -4$; the Hilbert function is computed in all degrees from Raicu’s decomposition of R [Rai12, Theorem 4.1] (Proposition 3.1). A second proof of $\omega_Y \cong \mathcal{O}_Y(-4)$ glues explicit generators on the 32 coordinate charts (Section 4.3). Since Y has rational singularities [MOZ15, Corollary 4.10] and ω_Y is anti-ample, Kawamata–Viehweg vanishing and Serre duality kill the intermediate cohomology of all $\mathcal{O}_Y(k)$. So the section ring $\bigoplus_k H^0(Y, \mathcal{O}_Y(k))$ is Cohen–Macaulay, it equals R by the Hilbert function, and the symmetric h -vector gives the Gorenstein property [Sta78, Theorem 4.4].

For question (1) the answer is negative: the companion note [Fer26] gives a ball in $\mathbb{R}^6$ whose lattice hull is not normal, and examples in every dimension $n \geq 6$ (Section 6). For question (2) the answer is positive over $\mathbb{C}$, in each of the readings of $\nabla f(L)$ discussed in Section 7 (Theorem C); in characteristic p dividing $\deg f$ it fails (Remark 7.1).

1.3 What is known, what is new, and what is not claimed

Part (a). Oeding’s theorem [Oed16, Theorem 3.14] states that $\sigma_2(\mathbb{P}^a \times \mathbb{P}^a)$ (a classical case), $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3)$, $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3)$ and $\sigma_2(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ are arithmetically Gorenstein; he listed $\sigma_2((\mathbb{P}^1)^5)$ in [Oed16, Table 1] as conjecturally arithmetically Gorenstein, with socle degree “?”, and wrote that he was not able to prove this case [Oed16, Remark 3.15]. Among the known cases of his conjecture that all secant varieties of Segre products are arithmetically Cohen–Macaulay [Oed16, Conjecture 1.1], listed in [Oed16, Section 2], second secant varieties appear only for Segre products of at most four factors; for more factors the list gives only the local result of [MOZ15]. An arithmetically Gorenstein variety is locally Gorenstein; so [MOZ15, Theorem 7.18] and [Oed16, Theorem 3.14] left $\sigma_2((\mathbb{P}^1)^5)$ as the only second secant variety of a Segre product, not filling its ambient space, whose arithmetic Gorenstein property was undecided. Khadam, Michałek and Zwiernik state that the classification of arithmetically Gorenstein secant varieties remains open even for Segre products, and ask which secant varieties of Segre–Veronese varieties are arithmetically Gorenstein [KMZ20, Question 4.7]; they do not name $\sigma_2((\mathbb{P}^1)^5)$. Daleo and Hauenstein designed a numerical test of the arithmetic Gorenstein property and applied it to $\sigma_3((\mathbb{P}^1)^4)$ and to the third secant variety of the fourth Veronese embedding of $\mathbb{P}^2$, which the test found arithmetically Gorenstein, and to $\sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2)$, whose h -vector is not symmetric [DH16]; $\sigma_2((\mathbb{P}^1)^5)$ is not among their examples. No proof that this cone is Gorenstein was found in the literature accessible to us; a search that finds nothing is not a proof of novelty (“Scope and priority” in Section 8).

The ingredients are not new. The secant cumulants, the chart $\mathbb{A}^5 \times T$ and the toric variety T are due to Michałek, Oeding and Zwiernik [MOZ15], and so are the rational singularities of Y [MOZ15, Corollary 4.10] and the Gorenstein property of T [MOZ15, Corollary 5.7], which we re-verify combinatorially; the Hilbert function comes from Raicu’s theorem [Rai12, Theorem 4.1]; Kawamata–Viehweg vanishing, Grauert–Riemenschneider vanishing, Serre duality, the comparison of local and sheaf cohomology and Stanley’s theorem are standard. What is new, to our knowledge, is Theorem A with its proof: the closed form of the Hilbert series (Proposition 3.1), the identification $\omega_Y \cong \mathcal{O}_Y(-4)$ (Section 4), and the deduction of projective normality and of the Gorenstein property (Section 5).

Parts (b) and (c). Nothing in Sections 6 and 7 is claimed as new. The negative answer to (1) is the main result of the companion note [Fer26], by the same author, which also treats the 2017 version of this question [Mic17]; Section 6 only states its certificate. Question (2) is answered in the source itself, which attributes the positive answer to Teissier [Mic16, p. 1001]. Michałek, Sturmfels, Uhler and Zwiernik had proved the statement for the set-theoretic image, and for subspaces L of dimension smaller than the codimension of the singular locus of $\{f = 0\}$, where ∇f is regular on L [MSUZ16, Proposition 5.9 and Corollary 5.10]. The proof of Theorem C, a dimension count, is included for the reader’s convenience.

2 The secant variety and its toric chart

2.1 Notation

Let e_0, e_1 be the standard basis of $\mathbb{C}^2$ and $V = (\mathbb{C}^2)^{\otimes 5}$. For $I \subseteq [5] := \{1, \dots, 5\}$ put $e_I = e_{\varepsilon_1} \otimes \dots \otimes e_{\varepsilon_5}$, where $\varepsilon_i = 1$ for $i \in I$ and $\varepsilon_i = 0$ otherwise, and let $(x_I)_{I \subseteq [5]}$ be the dual coordinates; $x_i := x_{\{i\}}$. For $a = (a_1, \dots, a_5) \in \mathbb{C}^5$ let $\alpha(a) = \bigotimes_{i=1}^5 (e_0 + a_i e_1)$, so that $x_I(\alpha(a)) = \prod_{i \in I} a_i$. The Segre variety $X \subset \mathbb{P}(V) = \mathbb{P}^{31}$ consists of the classes of the non-zero tensors $a^1 \otimes \dots \otimes a^5$; it is smooth of dimension 5, and its affine cone $\hat{X} \subset V$ is the set of tensors of rank at most one. The secant variety $Y = \sigma_2(X)$ is the closure of the union of the lines $\langle p, q \rangle$, $p \neq q \in X$; its affine cone $\hat{Y}$ is the closure of the set of tensors $u + v$ with $u, v \in \hat{X}$. Let $\mathbb{C}[x] = \mathbb{C}[x_I : I \subseteq [5]]$, let $I(Y) \subset \mathbb{C}[x]$ be the homogeneous prime ideal of Y , and $R = \mathbb{C}[x]/I(Y)$.

For $K \subseteq [5]$ let $\sigma_K \in GL(V)$ be the tensor product of the maps exchanging e_0 and e_1 in the factors $i \in K$ and of the identity in the other factors. Then $\sigma_K(e_I) = e_{I \Delta K}$, hence $x_I \circ \sigma_K = x_{I \Delta K}$, and σ_K preserves $\hat{X}$, X , $\hat{Y}$ and Y . We put

$$U_K = Y \cap \{x_K \neq 0\}, \quad U_0 := U_\emptyset.$$

Since $x_\emptyset \circ \sigma_K = x_K$, the map σ_K restricts to an isomorphism $U_K \rightarrow U_0$; and $Y = \bigcup_K U_K$, because every point of $\mathbb{P}^{31}$ has a non-zero coordinate.

2.2 Secant cumulants and the chart

On the affine chart $\{x_\emptyset \neq 0\} \subset \mathbb{P}^{31}$, identified with $\mathbb{A}^{31}$ by $x_\emptyset = 1$, Michałek, Oeding and Zwiernik introduce new coordinates in two steps [MOZ15, Section 2]: the central moments $y_i = x_i$ and, for $|I| \geq 2$, $y_I = \sum_{A \subseteq I} (-1)^{|I \setminus A|} x_A \prod_{i \in I \setminus A} x_i$; then the *secant cumulants* $z_i = y_i$ and, for $|I| \geq 2$, $z_I = \sum_{\pi} (-1)^{|\pi|-1} \prod_{B \in \pi} y_B$, where π runs over the partitions of I into blocks of consecutive elements of I (in the order of $[5]$), all of size at least two. Both changes of coordinates are triangular ($y_I = x_I + a$ polynomial in the x_J with $J \subsetneq I$, and likewise for z), so $(z_I)_{\emptyset \neq I \subseteq [5]}$ are coordinates on $\mathbb{A}^{31}$. For $t \in \mathbb{C}$ and $a, b \in \mathbb{C}^5$ let

$$p(t, a, b) = (1 - t) \alpha(a) + t \alpha(b) \quad (x_\emptyset = 1).$$

Theorem 2.1 ([MOZ15, Lemma 3.1 and Theorem 4.5]). *(a) The secant cumulants of $p(t, a, b)$ are $z_i = (1 - t)a_i + tb_i$ for $i \in [5]$ and $z_I = t(1 - t)(1 - 2t)^{|I|-2} \prod_{i \in I} (b_i - a_i)$ for $|I| \geq 2$.*

(b) Let $T \subset \mathbb{A}^{26}$, with coordinates z_I ($|I| \geq 2$), be the closure of the image of the map $(\mathbb{C}^)^6 \rightarrow \mathbb{A}^{26}$, $(t_0, t_1, \dots, t_5) \mapsto (t_0 \prod_{i \in I} t_i)_{|I| \geq 2}$. In the coordinates z_I , the open set $U_0 = Y \cap \{x_\emptyset \neq 0\}$ is the product $\mathbb{A}^5 \times T$, where $\mathbb{A}^5$ carries the coordinates $z_1, \dots, z_5$.*

Part (b) follows from (a) as in [MOZ15]: with

$$t_0 = \frac{t(1 - t)}{(1 - 2t)^2}, \quad t_i = (1 - 2t)(b_i - a_i) \quad (i \in [5]), \quad (1)$$

one has $z_I = t_0 \prod_{i \in I} t_i$ for $|I| \geq 2$, while $z_1, \dots, z_5$ can be prescribed independently by shifting a_i and b_i by the same amount. The identity (a) was also checked symbolically for $n = 5$ by the program of this note (Section 8). Note that the map $(t, a, b) \mapsto p(t, a, b)$ from $\mathbb{A}^{11}$ to U_0 is dominant and, the dimensions being equal (see below), generically finite.

2.3 The toric variety T

Let $M = \mathbb{Z} \times \mathbb{Z}^5$, with coordinates $(h; x_1, \dots, x_5)$. For $W \subseteq [5]$ let $\chi_W \in \{0, 1\}^5$ be its indicator vector, and let $\mathcal{S} \subset M$ be the monoid generated by the 26 vectors $g_W = (1; \chi_W)$ with $|W| \geq 2$. The coordinate z_I restricts on T to the character $t_0 \prod_{i \in I} t_i$ of weight g_I , so $T = \text{Spec } \mathbb{C}[\mathcal{S}]$; this is the variety $T_{2,5}$ of [MOZ15, Definition 4.1]. Let $C = \mathbb{R}_{\geq 0} \mathcal{S} \subset \mathbb{R}^6$.

Proposition 2.2. (a) $C = \{(h; x) : 0 \leq x_i \leq h \ (1 \leq i \leq 5), \ x_1 + \dots + x_5 \geq 2h\}$.

(b) $\mathbb{Z}\mathcal{S} = M$ and $\mathcal{S} = C \cap M$.

(c) Let $m_0 = (2; 1, 1, 1, 1, 1)$. The lattice points in the interior of C are exactly the points of $m_0 + \mathcal{S}$.

Proof. (a) The right-hand side C' is a cone containing all g_W . Let $(h; x) \in C'$. If $h = 0$ then $x = 0$. If $h > 0$, then $y = x/h$ lies in the polytope $Q = \{y \in [0, 1]^5 : \sum_i y_i \geq 2\}$, and it suffices to show that the vertices of Q are among the χ_W , $|W| \geq 2$. At a vertex y five linearly independent defining inequalities hold with equality. If $\sum y_i \geq 2$ is not among them, then $y \in \{0, 1\}^5$. Otherwise at least four coordinates of y are 0 or 1, and the fifth is 2 minus their sum, an integer in $[0, 1]$. In both cases $y = \chi_W$ with $|W| = \sum_i y_i \geq 2$.

(b) For distinct i, j, l we have $g_{\{i,j,l\}} - g_{\{j,l\}} = (0; e_i)$, and $g_{\{1,2\}} - (0; e_1) - (0; e_2) = (1; 0)$; so $\mathbb{Z}\mathcal{S} = M$. Clearly $\mathcal{S} \subseteq C \cap M$. Let $(k; z) \in C \cap M$; we show $(k; z) \in \mathcal{S}$ by induction on k . For $k = 0$, $z = 0$; for $k = 1$, $(k; z) = g_W$ with $W = \{i : z_i = 1\}$. Let $k \geq 2$, $A = \{i : z_i = k\}$ and $B = \{i : z_i \geq 1\}$. Since $z_i \leq k$ for all i and $\sum_i z_i \geq 2k$, we have $|B| \geq 2$, and $A \subseteq B$. Choose W with $A \subseteq W \subseteq B$ and $|W| = \max(2, |A|)$. Then $0 \leq z_i - [i \in W] \leq k - 1$ for all i , and $\sum_i z_i - |W| \geq 2(k - 1)$: if $|W| = 2$ this follows from $\sum_i z_i \geq 2k$, and if $|W| = |A| \geq 3$ from $\sum_i z_i \geq k|A|$. So $(k - 1; z - \chi_W) \in C \cap M$, which lies in $\mathcal{S}$ by induction, and $(k; z) = g_W + (k - 1; z - \chi_W) \in \mathcal{S}$.

(c) The cone C is full-dimensional, and since each of the eleven linear forms in (a) is non-zero, its interior is the set where all eleven inequalities are strict. For $(k; z) \in M$ this means $1 \leq z_i \leq k - 1$ and $\sum_i z_i \geq 2k + 1$, that is, $0 \leq z_i - 1 \leq k - 2$ and $\sum_i (z_i - 1) \geq 2(k - 2)$, that is, $(k; z) - m_0 \in C \cap M = \mathcal{S}$. $\square$

The point m_0 is the point $2e_0^\vee + \sum_{i=1}^5 e_i^\vee$ of [MOZ15, Corollary 5.7], where the sum is printed with upper limit 4; the value 1 on all facet normals requires the upper limit 5.

Corollary 2.3 ([MOZ15, Corollaries 4.10 and 5.7]). *T is a normal affine toric variety of dimension 6 with rational singularities. It is Cohen–Macaulay and Gorenstein, and its canonical module is free: in terms of the coordinates $t_0 = \chi^{(1;0)}$ and $t_i = \chi^{(0;e_i)}$ of its torus, the canonical sheaf ω_T is generated by the 6-form $\chi^{m_0} \frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_5}{t_5} = t_0 dt_0 \wedge dt_1 \wedge \dots \wedge dt_5$.*

Proof. By Proposition 2.2(b) the monoid $\mathcal{S}$ is saturated in $\mathbb{Z}\mathcal{S} = M$, so T is a normal toric variety of dimension $\text{rank } M = 6$ [CLS11, Chapter 1]. Normal toric varieties have rational singularities [CLS11, Theorem 11.4.2] and are Cohen–Macaulay (Hochster’s theorem, [BH98, Chapter 6]). By the theorem of Danilov and Stanley, the canonical module of $\mathbb{C}[\mathcal{S}]$ is spanned by the monomials χ^m with m in the interior of C [BH98, Chapter 6]; in terms of differential forms, the sections of ω_T are the forms $\chi^m dt_0/t_0 \wedge \dots \wedge dt_5/t_5$ with $m \in \text{int}(C) \cap M$ [CLS11, Chapter 8]. By Proposition 2.2(c) this module is generated by the single element χ^{m_0} , and $\chi^{m_0} = t_0^2 t_1 \dots t_5$. $\square$

Corollary 2.4. (a) Y is covered by the 32 open sets U_K , each isomorphic to $\mathbb{A}^5 \times T$. Hence Y is a normal projective variety of dimension 11 with rational singularities, in particular Cohen–Macaulay [MOZ15, Corollary 4.10].

(b) The canonical sheaf ω_Y , which is the dualizing sheaf of the normal Cohen–Macaulay variety Y , is invertible. On $U_0 = \mathbb{A}^5 \times T$ it is generated by

$$s_0 = dz_1 \wedge \dots \wedge dz_5 \wedge t_0 dt_0 \wedge dt_1 \wedge \dots \wedge dt_5,$$

and on U_K by $s_K = \sigma_K^* s_0$.

Proof. (a) follows from Theorem 2.1(b), Corollary 2.3 and the isomorphisms $\sigma_K: U_K \rightarrow U_0$; the dimension is $5 + 6$. (b) The variety $\mathbb{A}^5 \times T = \text{Spec } \mathbb{C}[\mathbb{N}^5 \times \mathcal{S}]$ is the normal affine toric variety of the cone $\mathbb{R}_{\geq 0}^5 \times C$, whose interior lattice points form the set $(1, \dots, 1) + \mathbb{N}^5$ times $m_0 + \mathcal{S}$. By the same description of the canonical sheaf, ω_{U_0} is generated by $(z_1 \cdots z_5) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_5}{z_5} \wedge \chi^{m_0} \frac{dt_0}{t_0} \wedge \cdots \wedge \frac{dt_5}{t_5} = s_0$. The statement on U_K follows by transport of structure. $\square$

Thus Y is Gorenstein at every point; this is the local result of [MOZ15] (their Corollary 5.7 and Theorem 7.18). The rest of the proof of Theorem A is global.

3 The Hilbert series

Raicu determined the decomposition of the homogeneous coordinate ring of the secant line variety of every Segre–Veronese variety, over a field of characteristic zero, into irreducible representations [Rai12, Theorem 4.1]. For Y it reads as follows. For $r \geq 0$ and a 5-tuple $\lambda = (\lambda^1, \dots, \lambda^5)$ of partitions of r , the Schur module $S_{\lambda^1} \mathbb{C}^2 \otimes \cdots \otimes S_{\lambda^5} \mathbb{C}^2$ occurs in R_r with multiplicity m_λ , where, with $f = \max_i \lambda_i^2$ and $e = \sum_i \lambda_i^2$: $m_\lambda = 0$ if some λ^i has more than two parts or $e < 2f$; if $e \geq r - 1$, then $m_\lambda = \lfloor r/2 \rfloor - f + 1$, except that $m_\lambda = \lfloor r/2 \rfloor - f$ when e is odd and r is even; if $2f \leq e < r - 1$, then $m_\lambda = \lfloor (e + 1)/2 \rfloor - f + 1$, except that $m_\lambda = \lfloor (e + 1)/2 \rfloor - f$ when e is odd. The partitions with at most two parts are $\lambda^i = (r - j_i, j_i)$, $0 \leq j_i \leq q := \lfloor r/2 \rfloor$, and $\dim S_{(r-j,j)} \mathbb{C}^2 = r - 2j + 1$. Writing $m(r, j)$ for m_λ and $\Pi_j = \prod_{i=1}^5 (r - 2j_i + 1)$, we obtain

$$\text{HF}(r) := \dim R_r = \sum_{j \in \{0, \dots, q\}^5} m(r, j) \Pi_j. \quad (2)$$

Proposition 3.1. *Let*

$$h = (h_0, \dots, h_8) = (1, 20, 210, 772, 1250, 772, 210, 20, 1), \quad P(r) = \sum_{i=0}^8 h_i \binom{r-i+11}{11},$$

where $\binom{N}{11} = N(N-1) \cdots (N-10)/11!$ for all $N \in \mathbb{Z}$. Then $\text{HF}(r) = P(r)$ for every $r \geq 0$. Consequently

$$\sum_{r \geq 0} \dim R_r t^r = \frac{h_0 + h_1 t + \cdots + h_8 t^8}{(1-t)^{12}},$$

$\dim R = 12$, $\deg Y = h_0 + \cdots + h_8 = 3256$, and

$$P(-4-k) = -P(k) \quad \text{for all } k \in \mathbb{Z}. \quad (3)$$

The proof has four steps: a closed form for $m(r, j)$, a telescoping that removes the maximum f , a lemma on sums of polynomials over simplices, which shows that HF is a polynomial of degree at most 11 on each of three residue classes of r , and a finite computation.

Step 1: a closed form. For $e \geq 0$ let $\delta_e \in \{0, 1\}$ with $\delta_e \equiv e \pmod{2}$, and $c(r, e) = \lfloor \min(r - \delta_e, e)/2 \rfloor + 1$. Then, with $e = |j| = \sum_i j_i$ and $f = \max_i j_i$,

$$m(r, j) = [e \geq 2f] (c(r, e) - f) \quad (j \in \{0, \dots, q\}^5). \quad (4)$$

Indeed, if $e \leq r - 2$ then $\min(r - \delta_e, e) = e$, and $\lfloor e/2 \rfloor + 1$ equals Raicu's $\lfloor (e + 1)/2 \rfloor + 1$ for even e and $\lfloor (e + 1)/2 \rfloor$ for odd e . If $e \geq r - 1$: for r even and e odd, $\min(r - 1, e) = r - 1$ and $c = r/2$; for r

and e even, $e \geq r$ and $c = r/2 + 1$; for r odd and e even, $\min(r, e) \in \{r-1, r\}$ and $c = (r-1)/2 + 1$; for r and e odd, $e \geq r$, $\min(r-1, e) = r-1$ and $c = (r-1)/2 + 1$. These are Raicu's values plus f .

Step 2: telescoping. For $g \geq 0$ put $w_g(e) = [e \geq 2g](c(r, e) - g)$, so that $m(r, j) = w_f(e)$. Then

$$w_g(e) - w_{g+1}(e) = [e \geq 2g+2] + [2g \leq e \leq 2g+1](c(r, e) - g) \quad (e \geq 0),$$

since for $e \geq 2g+2$ the two values differ by 1, for $e \in \{2g, 2g+1\}$ we have $w_{g+1}(e) = 0$, and for $e < 2g$ both vanish. If $g \leq q-1$ and $e \in \{2g, 2g+1\}$, then $e \leq r-1$, so $c(r, e) = g+1$ and the difference is $[e \geq 2g]$. For $g = q$ and $e \geq 2q$ one finds $c(r, e) - q = 1$, except that $c(r, e) - q = 0$ when r is even and e is odd. Writing $w_f = \sum_{g=f}^{q-1} (w_g - w_{g+1}) + w_q$ and exchanging the summations in (2), we get

$$\text{HF}(r) = \sum_{g=0}^q S(r, g) - [r \text{ even}] S^{\text{odd}}(r, q), \quad S(r, g) = \sum_{\substack{j \in \{0, \dots, g\}^5 \\ |j| \geq 2g}} \Pi_j, \quad (5)$$

where $S^{\text{odd}}(r, g)$ is the same sum restricted to odd $|j|$.

Lemma 3.2. *Let $\varphi(u, j)$ be a polynomial in variables u and $j = (j_1, \dots, j_5)$. There are polynomials F, A, B in (u, N) such that for all integers $N \geq -1$*

$$\sum_{\substack{j \in \mathbb{N}^5 \\ |j| \leq N}} \varphi(u, j) = F(u, N), \quad \sum_{\substack{j \in \mathbb{N}^5 \\ |j| \leq N}} (-1)^{|j|} \varphi(u, j) = A(u, N) + (-1)^N B(u, N),$$

and F, A, B have degree at most $\deg \varphi + 5$.

Proof. For one variable: if $\psi(k)$ is a polynomial (with coefficients polynomial in parameters), let Ψ be the polynomial with $\Psi(k) - \Psi(k-1) = \psi(k)$ and $\Psi(-1) = 0$, and let β be the polynomial with $\beta(k) + \beta(k-1) = \psi(k)$; it exists and is unique because $\beta \mapsto \beta(k) + \beta(k-1)$ is $2 - \nabla$ with $\nabla \beta(k) = \beta(k) - \beta(k-1)$ nilpotent on polynomials of bounded degree. Telescoping gives $\sum_{k=0}^M \psi(k) = \Psi(M)$ and $\sum_{k=0}^M (-1)^k \psi(k) = \beta(-1) + (-1)^M \beta(M)$ for all $M \geq -1$ (both sides vanish for $M = -1$), with $\deg \Psi \leq \deg \psi + 1$ and $\deg \beta = \deg \psi$. Now write the sum over the simplex as $\sum_{j_1=0}^N \sum_{j_2=0}^{N-j_1} \dots \sum_{j_5=0}^{N-j_1-\dots-j_4}$ and sum from the inside out; every inner upper limit is non-negative and the outer one is $N \geq -1$. In the alternating case, after the sums over $j_5, \dots, j_{k+1}$ the result is a sum of terms $(-1)^{j_1+\dots+j_k} G_1$ and $(-1)^N G_2$ with polynomials G_1, G_2 in $(u, j_1, \dots, j_k, N)$, because $(-1)^{j_1+\dots+j_k} (-1)^{N-j_1-\dots-j_k} = (-1)^N$; summing over j_k preserves this form. Each summation raises the degree by at most one. $\square$

Step 3: quasi-polynomiality. Let $g \geq 1$. If $j \in \mathbb{N}^5$ and $|j| \leq 2g-1$, at most one coordinate of j exceeds g . Hence, by the symmetry of Π_j in the coordinates,

$$S(r, g) = ((g+1)(r-g+1))^5 - \sum_{\substack{j \in \mathbb{N}^5 \\ |j| \leq 2g-1}} \Pi_j + 5 \sum_{\substack{j' \in \mathbb{N}^5 \\ |j'| \leq g-2}} \Pi_{(g+1+j'_1, j'_2, \dots, j'_5)},$$

where the first term is $\sum_{j \in \{0, \dots, g\}^5} \Pi_j = (\sum_{i=0}^g (r-2i+1))^5$. By Lemma 3.2 (with $N = 2g-1 \geq 1$ and $N = g-2 \geq -1$), $S(r, g) = \Sigma(r, g)$ for all $g \geq 1$, with a polynomial Σ of total degree at most 10. Since $S(r, 0) = (r+1)^5$, the sum $\sum_{g=0}^q S(r, g)$ is a polynomial in (r, q) of degree at most 11 for all $q \geq 0$. For even $r = 2q$ we write $S^{\text{odd}}(r, q) = \frac{1}{2}(S(r, q) - S^\pm(r, q))$, where $S^\pm$ carries the extra factor $(-1)^{|j|}$, and decompose $S^\pm$ in the same way; the alternating sums of Lemma 3.2 with $N = 2q-1$ and $N = q-2$, the factor $(-1)^{q+1}$ of the third term, and the one-variable alternating sum in the first term

show that $S^\pm(2q, q)$ is a polynomial in q on each parity class of $q \geq 1$. Consequently $\text{HF}(r)$ coincides with a polynomial in r of degree at most 11 on each of the classes

$$\{r \text{ odd}, r \geq 1\}, \quad \{r \equiv 2 \pmod{4}, r \geq 2\}, \quad \{r \equiv 0 \pmod{4}, r \geq 4\}.$$

Step 4: a finite computation. An exact evaluation of (2) gives $\text{HF}(r) = P(r)$ for $0 \leq r \leq 120$ (Section 8). Each of the three classes contains at least 12 of these values of r (for instance $1, 3, \dots, 23$; $2, 6, \dots, 46$; $4, 8, \dots, 48$), and two polynomials of degree at most 11 that agree at 12 points are equal; with $\text{HF}(0) = 1 = P(0)$ this proves $\text{HF}(r) = P(r)$ for all $r \geq 0$. Since $\binom{r-i+11}{11} = 0$ for $0 \leq r < i \leq 8$, we get $\sum_{r \geq 0} P(r)t^r = \sum_i h_i t^i / (1-t)^{12}$, hence the Hilbert series, $\dim R = 12$ and $\deg Y = 3256$. Finally $\binom{-m}{11} = -\binom{m+10}{11}$ and $h_i = h_{8-i}$ give

$$P(-4-k) = \sum_i h_i \binom{7-k-i}{11} = -\sum_i h_i \binom{k+i+3}{11} = -\sum_i h_{8-i} \binom{k-i+11}{11} = -P(k). \quad \square$$

Remark 3.3. The proof of Proposition 3.1 uses Raicu's theorem in all degrees. The values of HF in low degrees were also obtained without it, by linear algebra modulo a prime in each weight space of the grading of R by $\mathbb{Z}^5$ (Section 8): ranks of matrices of monomials evaluated at integer points of $\widehat{Y}$ give lower bounds, and ranks of products of the 3×3 minors of flattenings with monomials give upper bounds. The bounds agree and give $\text{HF}(2), \dots, \text{HF}(5) = 528, 5216, 35539, 184096$, so $I(Y)$ contains no quadrics and a 768-dimensional space of cubics; the second and the third verification runs (Section 8) obtained $\text{HF}(6) = 774368$ in the same way. With $\dim R = 12$, the values up to degree 5 determine $h_0, \dots, h_5$; they agree with Proposition 3.1. In particular $h_1 = 20$ is the codimension of Y , $h_2 = \binom{21}{2}$ and $h_3 = \binom{22}{3} - 768$.

4 The canonical sheaf

We prove $\omega_Y \cong \mathcal{O}_Y(-4)$ twice: first through the hyperplane section $H_0 = Y \setminus U_0$ and the Hilbert polynomial (Sections 4.1 and 4.2), then directly by gluing (Section 4.3). The second proof uses neither Proposition 3.1 nor Lemma 4.2.

4.1 The hyperplane section H_0

For a smooth projective variety $Z \subset \mathbb{P}^N$ and $p \in Z$, the embedded tangent space $T_p Z \subset \mathbb{P}^N$ is the projective linear subspace whose affine cone $\widehat{T}_p Z \subset \mathbb{C}^{N+1}$ is the tangent space of the affine cone of Z at any non-zero point over p .

Lemma 4.1. *Let $Z \subset \mathbb{P}^N$ be a smooth projective variety. Every point of $\sigma_2(Z)$ lies on a line $\langle p, q \rangle$ with $p \neq q \in Z$, or on a line through a point $p \in Z$ contained in $T_p Z$.*

Proof. The union of the lines $\langle p, q \rangle$, $p \neq q$, is constructible, so its Zariski closure $\sigma_2(Z)$ is also its closure in the Euclidean topology [Mum99, Chapter I, §10]. Let $x \in \sigma_2(Z)$, and let $x_k \rightarrow x$ with $x_k \in \langle p_k, q_k \rangle$, $p_k \neq q_k \in Z$. Passing to a subsequence, $p_k \rightarrow p$ and $q_k \rightarrow q$ in Z , and the lines $\langle p_k, q_k \rangle$ converge in the Grassmannian of lines to a line ℓ , which contains p, q and x . If $p \neq q$, then $\ell = \langle p, q \rangle$. If $p = q$, choose affine coordinates $(u, v) \in \mathbb{C}^n \times \mathbb{C}^{N-n}$ around $p = 0$, $n = \dim Z$, in which Z is near p the graph $v = g(u)$ of a holomorphic map with $g(0) = 0$ and $dg(0) = 0$; then $T_p Z$ is the closure of $\{v = 0\}$. With $p_k = (u_k, g(u_k))$ and $q_k = (u'_k, g(u'_k))$ we have $u_k \neq u'_k$ and $|g(u'_k) - g(u_k)| \leq \sup \|dg\| \cdot |u'_k - u_k|$, the supremum over the segment $[u_k, u'_k]$, which tends to 0. Hence the limits of the directions of $q_k - p_k$ lie in $\{v = 0\}$, and ℓ , the limit of the lines through p_k in these directions, is a line through p contained in $T_p Z$. $\square$

For $p \in X$ the affine tangent space $\hat{T}_p X \subset V$ has dimension 6; for $p = [a^1 \otimes \cdots \otimes a^5]$ it is $\sum_{i=1}^5 a^1 \otimes \cdots \otimes \mathbb{C}^2 \otimes \cdots \otimes a^5$ ($\mathbb{C}^2$ in the i -th factor). Since X is smooth, these spaces form a vector subbundle of rank 6 of $X \times V$. Its total space is an irreducible closed subvariety of $X \times V$ of dimension 11, so its image

$$\hat{\tau} = \bigcup_{p \in X} \hat{T}_p X \subset V$$

is closed (X is complete), irreducible, and of dimension at most 11. Let $\varphi: (\mathbb{C}^2)^{10} \rightarrow V$, $\varphi(a^1, \dots, a^5, b^1, \dots, b^5) = a^1 \otimes \cdots \otimes a^5 + b^1 \otimes \cdots \otimes b^5$. By Lemma 4.1, every non-zero $u \in \hat{Y}$ is of the form $\lambda v + \mu w$ with $v, w \in \hat{X}$, hence lies in the image of φ , or lies in $\hat{\tau}$:

$$\hat{Y} \subseteq \varphi((\mathbb{C}^2)^{10}) \cup \hat{\tau}. \quad (6)$$

Lemma 4.2. *The hyperplane section $H_0 = Y \cap \{x_\emptyset = 0\} = Y \setminus U_0$ is irreducible of dimension 10, and the divisor of the section $x_\emptyset$ of $\mathcal{O}_Y(1)$ is H_0 , with multiplicity one. In particular $\mathcal{O}_Y(H_0) \cong \mathcal{O}_Y(1)$.*

Proof. Write $a^i = (a_0^i, a_1^i)$ and $b^i = (b_0^i, b_1^i)$. Then $\varphi^* x_\emptyset = \mathfrak{p} := a_0^1 a_0^2 a_0^3 a_0^4 a_0^5 + b_0^1 b_0^2 b_0^3 b_0^4 b_0^5$. This polynomial is irreducible, being linear in a_0^1 with coprime coefficient and constant term; so $V(\mathfrak{p}) \subset (\mathbb{C}^2)^{10}$ is an irreducible hypersurface of dimension 19 whose ideal is generated by $\mathfrak{p}$. Let Z be the closure of $\varphi(V(\mathfrak{p}))$; it is irreducible and contained in $\hat{Y} \cap \{x_\emptyset = 0\}$.

At the point $P_0 \in V(\mathfrak{p})$ given by $(a^1, \dots, a^5) = ((1, 1), (1, 2), (1, 3), (1, 4), (1, 5))$ and $(b^1, \dots, b^5) = ((-1, 1), (1, 1), (1, 2), (1, 7), (1, 11))$ the 32×20 Jacobian matrix of φ has rank 12 (an exact computation, Section 8). The Zariski tangent space at P_0 of the fibre of φ through P_0 lies in the kernel of $d\varphi_{P_0}$, so the fibre of $\varphi|_{V(\mathfrak{p})}: V(\mathfrak{p}) \rightarrow Z$ through P_0 has dimension at most 8 at P_0 . Every fibre of this dominant morphism has dimension at least $\dim V(\mathfrak{p}) - \dim Z$ at each of its points [Har77, Exercise II.3.22]; hence $\dim Z \geq 19 - 8 = 11$.

Since $\hat{Y}$ is irreducible of dimension 12 and $x_\emptyset$ is a non-zero non-unit on $\hat{Y}$ (it takes the value 1 at $e_\emptyset \in \hat{X}$ and 0 at 0), every irreducible component W of $\hat{Y} \cap \{x_\emptyset = 0\}$ has dimension 11. If $u = \varphi(P)$ with $x_\emptyset(u) = 0$, then $P \in V(\mathfrak{p})$; so by (6), $W \subseteq Z \cup (\hat{\tau} \cap \{x_\emptyset = 0\})$. As $\hat{\tau}$ is irreducible of dimension at most 11 and contains $e_\emptyset$, where $x_\emptyset = 1$, the second set has dimension at most 10. Hence $W \subseteq Z$, and $W = Z$ because Z is irreducible of dimension at most 11. So $\hat{Y} \cap \{x_\emptyset = 0\} = Z$ is irreducible of dimension 11, and H_0 is irreducible of dimension 10.

Since Y is normal, the local ring $\mathcal{O}_{Y, H_0}$ is a discrete valuation ring; let v be its valuation. Let $\ell = x_{[5]}$, a linear form with $\ell(\varphi(P_0)) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 + 1 \cdot 1 \cdot 2 \cdot 7 \cdot 11 = 274 \neq 0$; it does not vanish identically on H_0 , and the multiplicity of H_0 in the divisor of $x_\emptyset$ is $m = v(x_\emptyset/\ell) \geq 1$. The rational map $\Phi = [\varphi]: (\mathbb{C}^2)^{10} \dashrightarrow Y$ is dominant and maps the generic point of $V(\mathfrak{p})$ to the generic point of H_0 . So it induces an injective local homomorphism $\Phi^*: \mathcal{O}_{Y, H_0} \rightarrow \mathcal{O}_{(\mathbb{C}^2)^{10}, V(\mathfrak{p})}$ into the discrete valuation ring with uniformizer $\mathfrak{p}$. Here $\Phi^*(x_\emptyset/\ell) = \mathfrak{p}/\varphi^*\ell$, and $\mathfrak{p}$ does not divide $\varphi^*\ell$ because $\varphi^*\ell(P_0) \neq 0 = \mathfrak{p}(P_0)$. Writing $x_\emptyset/\ell = \epsilon\pi^m$ with a unit ϵ and a uniformizer π of $\mathcal{O}_{Y, H_0}$, we get $1 = v_{\mathfrak{p}}(\mathfrak{p}/\varphi^*\ell) = v_{\mathfrak{p}}(\Phi^*\epsilon) + m v_{\mathfrak{p}}(\Phi^*\pi) \geq m$. Hence $m = 1$. $\square$

4.2 First proof of $\omega_Y \cong \mathcal{O}_Y(-4)$

Proposition 4.3. $\omega_Y \cong \mathcal{O}_Y(-4)$.

Proof. By Corollary 2.4(b), ω_Y is invertible and its section s_0 over $U_0 = Y \setminus H_0$ has no zeros there. As a rational section of ω_Y on Y , s_0 has a divisor supported on the prime divisor H_0 (Lemma 4.2); so $\text{div}(s_0) = aH_0$ for some $a \in \mathbb{Z}$, and $\omega_Y \cong \mathcal{O}_Y(aH_0) \cong \mathcal{O}_Y(a)$ by Lemma 4.2.

The variety Y is projective and Cohen–Macaulay of dimension 11, with dualizing sheaf ω_Y , so Serre duality gives $H^i(Y, \mathcal{O}_Y(k)) \cong H^{11-i}(Y, \omega_Y(-k))^\vee$ for all i and k [Har77, Section III.7]. Hence $\chi(\mathcal{O}_Y(k)) = -\chi(\mathcal{O}_Y(a-k))$ for all $k \in \mathbb{Z}$. The function $k \mapsto \chi(\mathcal{O}_Y(k))$ is the Hilbert polynomial of

Y , which is the Hilbert polynomial of R [Har77, Exercise III.5.2], that is, P by Proposition 3.1. So $P(k) = -P(a-k)$ for all k , and with (3) $P(a-k) = P(-4-k)$ for all k : the non-constant polynomial P is invariant under translation by $a+4$. Therefore $a = -4$. $\square$

4.3 Second proof: gluing the charts

Let $\Phi: \mathbb{A}^{11} \dashrightarrow U_0$, $(t, a, b) \mapsto [p(t, a, b)]$, be the parametrization of Theorem 2.1. It is dominant, and generically finite since $\dim U_0 = 11$; so pull-back of rational 11-forms along Φ is injective. Write $d\mathbf{v} = dt \wedge da_1 \wedge db_1 \wedge \cdots \wedge da_5 \wedge db_5$.

Lemma 4.4. (a) $\Phi^* s_0 = \varepsilon t(1-t) d\mathbf{v}$ for a sign $\varepsilon \in \{\pm 1\}$.

(b) Let $K \subseteq [5]$, $A_K = \prod_{i \in K} a_i$, $B_K = \prod_{i \in K} b_i$ and $N_K = (1-t)A_K + tB_K$, and let $\tau_K(t, a, b) = (\tilde{t}, \tilde{a}, \tilde{b})$ with $\tilde{t} = tB_K/N_K$, $\tilde{a}_i = 1/a_i$, $\tilde{b}_i = 1/b_i$ for $i \in K$ and $\tilde{a}_i = a_i$, $\tilde{b}_i = b_i$ for $i \notin K$. Then $\sigma_K \circ \Phi = \Phi \circ \tau_K$, and $\Phi^*(x_K/x_\emptyset) = N_K$.

(c) $s_K = (x_\emptyset/x_K)^4 s_0$ as rational sections of ω_Y .

Proof. (a) By Theorem 2.1(a), the torus coordinates $t_i = z_{\{i,j,l\}}/z_{\{j,l\}}$ and $t_0 = z_{\{j,l\}}/(t_j t_l)$ of T (i, j, l distinct) pull back to the functions (1). So t_0 depends only on t , with $dt_0/dt = (1-2t)^{-3}$, while z_i and t_i depend only on (t, a_i, b_i) . Modulo dt we have $dz_i \equiv (1-t)da_i + tdb_i$ and $dt_i \equiv (1-2t)(db_i - da_i)$, hence $dz_i \wedge dt_i \equiv (1-2t)da_i \wedge db_i$. Therefore $\Phi^* s_0 = \varepsilon t_0(1-2t)^{-3}(1-2t)^5 d\mathbf{v} = \varepsilon t(1-t) d\mathbf{v}$, the sign coming from reordering.

(b) Since $\sigma_K(e_0 + a_i e_1) = a_i(e_0 + a_i^{-1} e_1)$ for $i \in K$, we have $\sigma_K(\alpha(a)) = A_K \alpha(\tilde{a})$, and so $\sigma_K(p(t, a, b)) = (1-t)A_K \alpha(\tilde{a}) + tB_K \alpha(\tilde{b}) = N_K p(\tilde{t}, \tilde{a}, \tilde{b})$. Moreover $x_K(p(t, a, b)) = N_K$ and $x_\emptyset(p(t, a, b)) = 1$.

(c) By (b) and (a), $\Phi^* s_K = (\sigma_K \circ \Phi)^* s_0 = \tau_K^*(\Phi^* s_0) = \varepsilon \tilde{t}(1-\tilde{t}) \det(D\tau_K) d\mathbf{v}$. Here $\tilde{t}(1-\tilde{t}) = t(1-t)A_K B_K / N_K^2$, and since $\tilde{a}, \tilde{b}$ depend only on (a, b) ,

$$\det(D\tau_K) = \frac{\partial \tilde{t}}{\partial t} \prod_{i \in K} \frac{1}{a_i^2 b_i^2} = \frac{A_K B_K}{N_K^2} \cdot \frac{1}{A_K^2 B_K^2}.$$

Hence $\Phi^* s_K = \varepsilon t(1-t) N_K^{-4} d\mathbf{v} = \Phi^*((x_\emptyset/x_K)^4 s_0)$, and (c) follows from the injectivity of Φ^* . $\square$

Proposition 4.5 (second proof of Proposition 4.3). *The rational section $s_0 \otimes x_\emptyset^4$ of $\omega_Y(4)$ is regular and nowhere zero on Y . Hence $\omega_Y \cong \mathcal{O}_Y(-4)$.*

Proof. On $U_0 \cap U_K$, Lemma 4.4(c) gives $s_0 \otimes x_\emptyset^4 = (x_K/x_\emptyset)^4 s_K \otimes (x_\emptyset/x_K)^4 x_K^4 = s_K \otimes x_K^4$. The right-hand side generates $\omega_Y(4)$ on U_K , because s_K generates $\omega_Y|_{U_K}$ (Corollary 2.4) and x_K^4 generates $\mathcal{O}_Y(4)|_{U_K}$. Since the U_K cover Y , the claim follows. $\square$

Conversely, Proposition 4.5 and Serre duality force $P_Y(-4-k) = -P_Y(k)$ for the Hilbert polynomial P_Y of Y ; that P satisfies (3) is thus a non-trivial check of Proposition 3.1.

5 Proof of Theorem A

Lemma 5.1. $H^i(Y, \mathcal{O}_Y(k)) = 0$ for $1 \leq i \leq 10$ and all $k \in \mathbb{Z}$, and $H^{11}(Y, \mathcal{O}_Y(k)) = 0$ for $k \geq -3$.

Proof. Let $\pi: \tilde{Y} \rightarrow Y$ be a resolution of singularities with $\tilde{Y}$ smooth and projective. Since Y has rational singularities (Corollary 2.4), $\pi_* \mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$ and $R^j \pi_* \mathcal{O}_{\tilde{Y}} = 0$ for $j > 0$. For $m \geq 1$ the projection formula and the Leray spectral sequence give $H^j(Y, \mathcal{O}_Y(-m)) \cong H^j(\tilde{Y}, \pi^* \mathcal{O}_Y(-m))$, which

by Serre duality on $\tilde{Y}$ is dual to $H^{11-j}(\tilde{Y}, \omega_{\tilde{Y}} \otimes \pi^* \mathcal{O}_Y(m))$. The line bundle $\pi^* \mathcal{O}_Y(m)$ is nef and big, so this group vanishes for $11 - j > 0$ by the Kawamata–Viehweg vanishing theorem [Laz04, Theorem 4.3.1]. Thus

$$H^j(Y, \mathcal{O}_Y(-m)) = 0 \quad \text{for } 0 \leq j \leq 10 \text{ and } m \geq 1. \quad (7)$$

Let $1 \leq i \leq 10$. If $k \leq -1$, then $H^i(Y, \mathcal{O}_Y(k)) = 0$ by (7). If $k \geq -3$, Serre duality on Y and $\omega_Y \cong \mathcal{O}_Y(-4)$ (Proposition 4.3 or 4.5) give $H^i(Y, \mathcal{O}_Y(k)) \cong H^{11-i}(Y, \mathcal{O}_Y(-4-k))^\vee$, which vanishes by (7) because $1 \leq 11-i \leq 10$ and $-4-k \leq -1$. Finally $H^{11}(Y, \mathcal{O}_Y(k)) \cong H^0(Y, \mathcal{O}_Y(-4-k))^\vee = 0$ for $k \geq -3$. $\square$

By Serre duality on Y , (7) says that $H^i(Y, \omega_Y \otimes \mathcal{O}_Y(m)) = 0$ for $i > 0$ and $m \geq 1$; this is the form of Kawamata–Viehweg vanishing for varieties with rational singularities, which can also be derived from Grauert–Riemenschneider vanishing [GR70] and Kempf’s description $\pi_* \omega_{\tilde{Y}} = \omega_Y$ of rational singularities [KKMSD73, Chapter I, §3].

Proof of Theorem A. Let $R' = \bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(k))$. Since $I(Y)$ is the saturated ideal of all forms vanishing on Y , the natural maps $R_k \rightarrow R'_k$ are injective, and we regard R as a graded subring of R' . For $k \geq 0$, Lemma 5.1 gives $H^i(Y, \mathcal{O}_Y(k)) = 0$ for all $i \geq 1$, so $\dim R'_k = \chi(\mathcal{O}_Y(k)) = P(k)$, the Hilbert polynomial of Y being that of R , namely P (Proposition 3.1). Also $\dim R_k = P(k)$ by Proposition 3.1. Hence $R = R'$. As Y is normal, R' is integrally closed [Har77, Exercise II.5.14]; so Y is projectively normal.

Since $H^0(Y, \mathcal{O}_Y(k)) = 0$ for $k < 0$, the natural map $R \rightarrow \bigoplus_{k \in \mathbb{Z}} H^0(Y, \mathcal{O}_Y(k))$ is an isomorphism. For the finitely generated graded $\mathbb{C}[x]$ -module R and the maximal homogeneous ideal $\mathfrak{m}$ of $\mathbb{C}[x]$ this gives $H_{\mathfrak{m}}^0(R) = H_{\mathfrak{m}}^1(R) = 0$, and $H_{\mathfrak{m}}^{i+1}(R)_k \cong H^i(Y, \mathcal{O}_Y(k))$ for $i \geq 1$ [Eis95, Theorem A4.1]. By Lemma 5.1, $H_{\mathfrak{m}}^j(R) = 0$ for $j \leq 11$, so $\text{depth } R \geq 12 = \dim R$, and R is Cohen–Macaulay [BH98, Section 3.5].

The ring R is a Cohen–Macaulay graded domain whose h -vector is symmetric (Proposition 3.1); by Stanley’s theorem [Sta78, Theorem 4.4] it is Gorenstein. Its only non-zero local cohomology module is $H_{\mathfrak{m}}^{12}(R)$, with $H_{\mathfrak{m}}^{12}(R)_k \cong H^{11}(Y, \mathcal{O}_Y(k)) \cong H^0(Y, \mathcal{O}_Y(-4-k))^\vee$; this is non-zero exactly for $k \leq -4$. So the a -invariant of R is -4 , and by graded local duality [BH98, Section 3.6] the canonical module ω_R , the graded dual of $H_{\mathfrak{m}}^{12}(R)$, has $(\omega_R)_k \neq 0$ exactly for $k \geq 4$. As R is Gorenstein, $\omega_R \cong R(c)$ for some c , and therefore $\omega_R \cong R(-4)$. The regularity of R is $a(R) + \dim R = 8$, the degree of its h -polynomial, and that of $I(Y)$ is 9. Part (c) has been proved along the way (Proposition 4.3, Lemma 5.1). $\square$

Remark 5.2. Given Proposition 4.5, Raicu’s theorem enters the proof of Theorem A only through Proposition 3.1, that is, for the equality $R = R'$ (projective normality of Y) and for the numbers in Theorem A(b). The section ring R' is Gorenstein independently of it. Indeed, Proposition 4.5 and Lemma 5.1, whose proofs do not use Proposition 3.1, show that the finitely generated $\mathbb{C}[x]$ -module R' (it agrees with R in large degrees [Har77, Exercise II.5.9]) is Cohen–Macaulay of dimension 12, with Hilbert function $k \mapsto \chi(\mathcal{O}_Y(k))$ for $k \geq 0$. Serre duality gives $\chi(\mathcal{O}_Y(-4-k)) = -\chi(\mathcal{O}_Y(k))$, and Lemma 5.1 gives $\chi(\mathcal{O}_Y(-1)) = \chi(\mathcal{O}_Y(-2)) = \chi(\mathcal{O}_Y(-3)) = 0$; hence the h -vector of R' is symmetric of degree 8, and R' is Gorenstein by Stanley’s theorem.

6 Lattice hulls of balls

A lattice polytope $P \subset \mathbb{R}^n$ is *normal* if for every $k \geq 1$ every lattice point of kP is a sum of k lattice points of P ; for polytopes whose lattice points affinely generate $\mathbb{Z}^n$, as below, the usual variants of this notion agree. Conjecture 5(1) of [Mic16] asks whether the lattice hull $\text{conv}(B \cap \mathbb{Z}^n)$ of every ball

$B \subset \mathbb{R}^n$ is normal; [BGM16, Question 7.2(b)] asks this for ellipsoids, and Michałek asked it again in 2017 [Mic17, p. 2684, Conjecture 1], which is record OWR-15587-006 of [ula26]. For $n \leq 3$ the answer is positive (for $n = 3$ by [BGM16, Theorem 6.5(c)]), and for ellipsoids Gubeladze constructed non-normal lattice hulls [Gub21]. For balls the answer is negative.

Theorem B ([Fer26, Theorem A and Corollary B]). *Let $P = \text{conv}\{z \in \mathbb{Z}^6 : z_1^2 + \dots + z_6^2 - z_3 - z_5 \leq 158906\}$, the lattice hull of the closed ball with centre $c = (0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$ and squared radius $\varrho^2 = \frac{317813}{2}$, and let $\mathbf{n} = (-36, -46, 50, -57, -41, 1)$.*

(i) *Every lattice point z of P satisfies $\mathbf{n} \cdot z \geq -41504$.*

(ii) *Exactly six lattice points of P lie on the hyperplane $\mathbf{n} \cdot z = -41504$:*

$$\begin{aligned} F_0 &= (136, 177, -192, 218, 157, -3), & F_1 &= (137, 175, -190, 220, 158, -4), \\ F_2 &= (137, 176, -191, 219, 157, -6), & F_3 &= (138, 176, -190, 220, 156, -4), \\ F_4 &= (139, 175, -192, 217, 158, -3), & F_5 &= (139, 177, -189, 218, 158, -4). \end{aligned}$$

They are affinely independent, and $\text{conv}(F_0, \dots, F_5)$ is a facet of P .

(iii) *The point $w = \frac{1}{2}(F_0 + \dots + F_5) = (413, 528, -572, 656, 472, -12)$ is a lattice point of $3P$ but not a sum of three lattice points of P .*

The polytope P is full-dimensional and its lattice points affinely generate $\mathbb{Z}^6$ (0 and $e_1, \dots, e_6$ lie in it), so P is not normal. Padding with zeros gives a ball in $\mathbb{R}^n$ with non-normal lattice hull for every $n \geq 6$.

The certificate is short. Part (i) is the Cauchy–Schwarz inequality: on the ball, $\mathbf{n} \cdot y$ is at least $\mathbf{n} \cdot c - |\mathbf{n}| \varrho$, and $(\mathbf{n} \cdot c + 41505)^2 - |\mathbf{n}|^2 \varrho^2 = \frac{61643}{4} > 0$, where $\mathbf{n} \cdot c = \frac{9}{2}$ and $|\mathbf{n}|^2 = 10843$. Part (ii) is a finite enumeration of the lattice points of a five-dimensional ball of squared radius $\frac{270429}{43372} \approx 6.24$. For (iii), if $w = p_1 + p_2 + p_3$ with lattice points p_i of P , then $\mathbf{n} \cdot p_i \geq -41504$ and $\sum_i \mathbf{n} \cdot p_i = \mathbf{n} \cdot w = 3 \cdot (-41504)$ force $\mathbf{n} \cdot p_i = -41504$, so each p_i is one of the F_j . But the F_j are linearly independent (they are affinely independent and lie on a hyperplane that does not contain 0), so $w = \frac{1}{2}(F_0 + \dots + F_5)$ is the only way to write w as a linear combination of them, whereas a sum of three of them has integer coefficients; so w is not such a sum. The program of this note repeats these checks (Section 8); the companion note [Fer26] contains the details, further certificates, and the open cases $n = 4$ and $n = 5$.

7 The gradient map on a generic linear subspace

Let k be an algebraically closed field, $n \geq 1$, and $f \in k[x_0, \dots, x_n]$ a non-zero form of degree $d \geq 1$ with $\text{char } k \nmid d$ (for instance $k = \mathbb{C}$). Let $B = V(\partial_0 f, \dots, \partial_n f) \subset \mathbb{P}^n$, let $\nabla f: \mathbb{P}^n \setminus B \rightarrow (\mathbb{P}^n)^\vee$, $x \mapsto [\partial_0 f(x) : \dots : \partial_n f(x)]$, be the gradient map, with values in the space of hyperplanes, and let $\Gamma \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ be the closure of its graph. For a linear subspace $L \subseteq \mathbb{P}^n$ let $L^\perp \subset (\mathbb{P}^n)^\vee$ be the set of hyperplanes containing L . We consider three readings of $\nabla f(L)$: (i) the image $\nabla f(L \setminus B)$; (ii) its Zariski closure, which is the reading of the source (it writes $\overline{\nabla f(L)}$) and of [MSUZ16], where it is the exponential variety; (iii) the total transform $\Gamma_L = \text{pr}_2(\Gamma \cap (L \times (\mathbb{P}^n)^\vee))$. Since Γ_L is closed and contains $\nabla f(L \setminus B)$, (i) $\subseteq$ (ii) $\subseteq$ (iii).

Theorem C. *For every ℓ with $0 \leq \ell \leq n$ there is a non-empty Zariski open subset $\mathcal{U}_\ell$ of the Grassmannian G of ℓ -dimensional linear subspaces of $\mathbb{P}^n$ such that for $L \in \mathcal{U}_\ell$ there is no $(x, H) \in \Gamma$ with $x \in L \subseteq H$. In particular, for $L \in \mathcal{U}_\ell$, $L^\perp \cap \Gamma_L = \emptyset$, hence $L^\perp \cap \overline{\nabla f(L \setminus B)} = \emptyset$ and $L^\perp \cap \nabla f(L \setminus B) = \emptyset$.*

Proof. By Euler's identity $\sum_i x_i \partial_i f = d f$ with $d \neq 0$ in k , the partial derivatives do not all vanish identically; so $B \neq \mathbb{P}^n$, and Γ is irreducible of dimension n . Let $\Gamma_{\text{inc}} = \{(x, H) \in \Gamma : x \in H\}$, a closed subset. Over $\mathbb{P}^n \setminus B$, the condition $x \in \nabla f(x)$ reads $\sum_i x_i \partial_i f(x) = 0$, that is $f(x) = 0$; so this part of Γ_{inc} is isomorphic to $V(f) \setminus B$, of dimension at most $n - 1$. The rest lies in $\Gamma \cap (B \times (\mathbb{P}^n)^\vee)$, a proper closed subset of the irreducible n -dimensional Γ . Hence $\dim \Gamma_{\text{inc}} \leq n - 1$.

For $\ell = n$ the statement is trivial (no hyperplane contains $\mathbb{P}^n$), so let $\ell \leq n - 1$. The set $J = \{(x, H, L) \in \Gamma_{\text{inc}} \times G : x \in L \subseteq H\}$ is closed. For $(x, H) \in \Gamma_{\text{inc}}$ the fibre $\{L : x \in L \subseteq H\}$ is the set of ℓ -dimensional subspaces of $H \cong \mathbb{P}^{n-1}$ through x , a Grassmannian of dimension $\ell(n - 1 - \ell)$. Therefore

$$\dim J \leq n - 1 + \ell(n - 1 - \ell) = (\ell + 1)(n - \ell) - 1 = \dim G - 1.$$

So the image of J in G is a proper closed subset, and its complement $\mathcal{U}_\ell$ has the required property. The last assertions follow from (i) $\subseteq$ (ii) $\subseteq$ (iii). $\square$

Remark 7.1. The hypothesis $\text{char } k \nmid d$ is necessary. If $p = \text{char } k$ divides d , Euler's identity gives $\sum_i x_i \partial_i f = 0$, so $x \in \nabla f(x)$ for every $x \notin B$; for the points $L = \{x\}$, $x \notin B$, we get $\nabla f(x) \in L^\perp$. For example, for $f = x_0^{p-1}x_1$ on $\mathbb{P}^1$ and $L = \{(a : b)\}$ with $a \neq 0$, $\nabla f(L) = \{(-b : a)\} = L^\perp$. Genericity of L is also necessary: if L is a line tangent to a smooth conic $V(f) \subset \mathbb{P}^2$ at x , then $\nabla f(x) = L \in L^\perp$ [MSUZ16, Proposition 5.9].

As explained in Section 1, Theorem C is not new: the source records Teissier's observation that the answer is positive [Mic16, p. 1001], as a special case of [Tei73, Tei82], and refers to [Huh12]. Our proof does not use these works. By the discussion preceding [MSUZ16, Conjecture 5.8] (arXiv numbering; the source cites the journal version, as Conjecture 5.5), the conclusion for the closure is, in the setting of [MSUZ16], equivalent to the equality of the degree and the maximum likelihood degree of the exponential variety defined by L .

8 Computations and reproducibility

The proofs use three finite computations: the exact evaluation of Raicu's formula (2) in Step 4 of the proof of Proposition 3.1 (only $r \leq 48$ is needed), the rank of the 32×20 Jacobian matrix of φ at P_0 in Lemma 4.2, and the enumeration in Theorem B(ii), which belongs to the companion note. Everything else reported in this section is a test: it could have detected an error, but no proof depends on it. The reproducibility package (folder `reproducibility/` of the source archive) contains the following.

The program of this note, `check_note.py` (Python 3, standard library, and numpy for part C; about ten seconds), makes 27 exact checks, all of which pass. Part A evaluates (2) for $r \leq 120$ (term by term for $r \leq 14$) and checks (4), (5), the inclusion-exclusion identity of Step 3, $\text{HF} = P$ for $r \leq 120$, the h -vector and (3). Part B checks the finite statements of Proposition 2.2, the identity of Theorem 2.1(a) by exact polynomial arithmetic, the data of P_0 , and Lemma 4.4 for all 32 sets K at random rational points, computed from the cumulant formulas by exact automatic differentiation. Part C certifies $\text{HF}(2), \dots, \text{HF}(5)$ without Raicu's theorem (Remark 3.3), part D the certificate of Theorem B, part E the example of Remark 7.1. The folder `original/` contains the programs with which the results were first obtained (Raicu's formula, the telescoped form, the certified Hilbert function in degrees 2 to 5, the toric chart and P_0 , and a check of the certificate of Theorem B), the folders `verification_run_A/` and `verification_run_B/` those of the first two verification runs, and `independent_run_2/` those of the third (see the next paragraph), all with recorded outputs; `README.md` lists command lines, running times and what each program checks. On 14 October 2026 all programs were run again from a copy of the package, and the third verification run ran them once more from the source archive; both times the recorded outputs were reproduced up to running times, except two lines of a wrong-input test recorded with an earlier revision of one program of the second

run, and except that one test of the second run does not finish its last example (its partial output was reproduced; see `README.md`).

Verification. The results were first obtained, with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first run re-derived every step of the proof of Theorem A (Raicu’s multiplicities and their closed form, the telescoping, the polynomiality on residue classes, the section H_0 , $\omega_Y \cong \mathcal{O}_Y(a)$, the vanishing and the section ring) and the chart of [MOZ15] for $n = 5$; it checked the toric facts with its own programs, found the second proof of $\omega_Y \cong \mathcal{O}_Y(-4)$ (Section 4.3) and checked it symbolically, computed $\text{HF}(r) = P(r)$ for $r \leq 124$, compared Raicu’s multiplicities with ranks of evaluation matrices in 42 weight spaces of degrees 6 to 14 (all equal; a test, as such ranks are lower bounds), and obtained $\deg Y = 3256$ from the double point formula, using that Y is not defective and that a general point of Y lies on a unique secant line. The second run implemented Raicu’s formula independently ($\text{HF}(r) = P(r)$ for $r \leq 300$; deliberately wrong inputs were detected), certified $\text{HF}(r)$ for $r \leq 6$ without Raicu’s theorem, checked the cumulant identity for $n = 4, 5$, the toric chart, Theorem B and the proof of Theorem C (with Gröbner-basis tests on examples not covered by [MSUZ16, Corollary 5.10]), read the sources and checked the bibliography. Both runs confirmed the three answers and found no mathematical error. Their required corrections, all applied here, concern the literature ([DH16] is now discussed; [KMZ20] is quoted as it stands, and that $\sigma_2((\mathbb{P}^1)^5)$ was the only undecided Segre case is stated as our inference from [MOZ15, Oed16]), the multiplicity-one step of Lemma 4.2 (now with $x_\emptyset/\ell$), the reason for $R \subseteq R'$ (saturation of $I(Y)$), the misprint in [MOZ15, Corollary 5.7], the attribution of Theorem 1 of the source, the wording of [BGM16, Question 7.2(b)], and the addition of the gluing proof and of the degree check. When this note was written, the proofs of Proposition 2.2 and Lemma 4.1 (cited facts in the first version), Remark 5.2 and `check_note.py` were added, and the proof of Lemma 5.1 was arranged to use only $R^j \pi_* \mathcal{O}_{\tilde{Y}} = 0$, Serre duality and Kawamata–Viehweg vanishing (the first version also used Grauert–Riemenschneider vanishing and Kempf’s criterion).

A third verification run, also AI-assisted, then examined the text of this note line by line, as if nothing had been checked before. It read the sources again (the whole abstract [Mic16] and the title page of its report; the TeX sources of [MOZ15, Rai12, Oed16, KMZ20, BGM16] and of both arXiv versions of [MSUZ16]; [Mic17], the extended abstract of [DH16] and the published file of [Fer26]), re-derived every proof, and wrote its own exact programs (folder `independent_run_2/`) without using those of the package. They evaluate Raicu’s formula by a different summation ($\text{HF}(r) = P(r)$ for $r \leq 150$) and test Steps 1–3 of the proof of Proposition 3.1 separately; derive Theorem 2.1(a) for $n = 5$ symbolically from the definitions of [MOZ15]; find the facets of C from the generators and test saturation up to height 7; check Lemma 4.4(a), (c) for all 32 sets K by a route that does not use τ_K (the chart coordinates of $\sigma_K(p)/x_K(p)$, with exact derivatives); recompute the data of Lemma 4.2; enumerate the slice of Theorem B with exact bounds; test Theorem C on lines in $\mathbb{P}^2$ by exact gcd computations, also for forms whose base locus meets every line, and with two non-generic controls; and certify $\text{HF}(r)$ for $r \leq 6$ without Raicu’s theorem, weight space by weight space, every dimension being equal to Raicu’s prediction. It also ran the programs of the package from the source archive, as described above, and repeated the literature search. It found no mathematical error and no gap. Its required corrections, all applied here, concerned the dates of the workshop, the wording on the numbering of the conjecture of [MSUZ16], the last step of the sketch of Theorem B(iii), the credit to [Oed16, Conjecture 1.1], and the record of this run.

Scope and priority. The note answers the three questions of [Mic16, p. 1000] recorded as OWR-14601-005: the affine cone over $\sigma_2((\mathbb{P}^1)^5)$ is Gorenstein (Theorem A); the lattice hull of a ball need not be normal (Theorem B, the result of the companion note [Fer26]); and $L^\perp \cap \nabla f(L) = \emptyset$ for generic

L over $\mathbb{C}$ (Theorem C), which the source itself records as Teissier’s observation [Mic16, p. 1001], [Tei73, Tei82]. The questions are Michałek’s. The secant cumulants, the chart $\mathbb{A}^5 \times T$, the toric variety T with its Gorenstein property, and the rational singularities of Y are due to Michałek, Oeding and Zwiernik [MOZ15]; the Hilbert function rests on Raicu’s theorem [Rai12]; Oeding listed the case as unknown and conjectured the answer [Oed16, Table 1 and Remark 3.15], and the Cohen–Macaulay property is a case of his conjecture [Oed16, Conjecture 1.1]; the vanishing and duality theorems are standard. To our knowledge, Theorem A and its proof are new. No proof that this cone is Gorenstein was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and no priority is claimed. Nothing is claimed for Theorem B beyond [Fer26], and Theorem C is not new. Not treated: higher secant varieties, other Segre or Segre–Veronese varieties, positive characteristic in part (a) (Raicu’s theorem and the vanishing theorem are used in characteristic zero), and the open dimensions $n = 4, 5$ of question (1).

What was read: [Mic16], pp. 999–1001, in the publisher’s open file; the arXiv versions of [MOZ15] (v2, Sections 2–5 and 7), [Rai12] (v2, Theorem 4.1), [Oed16] (v2, Sections 1–3 and Table 1), [KMZ20] (v1, Section 4) and [MSUZ16] (v2, Section 5), whose numbering we use (the journal versions were not compared; the source cites the journal version of [MSUZ16], as Conjecture 5.5, which is Conjecture 5.8 in arXiv versions 1 and 2); [DH16] in the authors’ extended abstract; and [Fer26], from which the statements of Section 6 on [BGM16, Gub21, Mic17] were taken. The third verification run read these sources again, and also arXiv version 1 of [MSUZ16], [BGM16] (v2, Theorem 6.5 and Question 7.2), [Mic17] (p. 2684) and the abstract of [Gub21]. The works [Tei73, Tei82, Huh12] are cited only as in the source and were not used; textbook statements are cited without re-checking their proofs.

The searches were made on 14 October 2026, when the results were first obtained and by all three verification runs: arXiv API queries combining secant varieties, Segre products, (arithmetically) Gorenstein, Cohen–Macaulay, Hilbert series, border rank and secant cumulants, and the authors of [MOZ15]; the lists of citing works in OpenAlex for [MOZ15] (23), [Oed16] (4), [KMZ20] (5) and [Rai12] (66), with titles and available abstracts; zbMATH Open, Crossref and Semantic Scholar; and five web searches in all. Apart from [DH16], found among the works citing [MOZ15], nothing on the arithmetic Gorenstein or Cohen–Macaulay property of $\sigma_2((\mathbb{P}^1)^5)$ was found; the latest statements found are [Oed16] (conjectured) and [KMZ20] (open for Segre products). All DOIs in the bibliography were checked with Crossref, and that of [Fer26] with DataCite, when this note was written and again by the third verification run.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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