

Counterexamples over $\overline{\mathbb{F}}_p$ to a Generic Equivalence Problem of Kraft and Russell

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Abstract

In Oberwolfach Report 01/2007, Kraft and Russell stated that two morphisms of varieties over an algebraically closed field of infinite transcendence degree, whose fibres over all closed points are isomorphic, become isomorphic after a dominant étale base change, and asked whether this holds over every algebraically closed field, or for counterexamples over $\overline{\mathbb{F}}_p$ or $\overline{\mathbb{Q}}$. In 2014 they proved the statement for affine morphisms, with a dominant base change of finite degree. The étale form already fails over every algebraically closed field of positive characteristic, because of Russell’s classical purely inseparable forms of the affine line; we record this, and note that it immediately answers a question on positive characteristic raised in a remark of Kaliman. We show that over $\overline{\mathbb{F}}_p$ even the finite-degree form fails. For every prime p we give a pair of smooth affine families of threefolds over an open subset Y of the affine line, defined over $\mathbb{F}_p$, whose fibres over each closed point of Y are isomorphic as $\overline{\mathbb{F}}_p$ -varieties, but which do not become isomorphic after any base change $U \rightarrow Y$ whose image contains the generic point, whether it is étale, of finite degree, or neither. The fibres are affine modifications of $\mathbb{G}_m^2 \times \mathbb{A}^1$ at two points, and their isomorphism classes are governed by $\mathrm{GL}_2(\mathbb{Z})$ -orbits. Over $\overline{\mathbb{F}}_p$ every point of $\mathbb{G}_m^2$ has finite order, and the Frobenius twist stays in the orbit at every closed point but not at the generic point. For $p \equiv 1 \pmod{4}$ we give smooth projective families with the same properties: blow-ups of $E \times E$ at two points, where E is the curve $y^2 = x^3 - x$ with complex multiplication by $\mathbb{Z}[i]$, parametrised by $E \times E$ minus the origin or by a curve in it. The case of $\overline{\mathbb{Q}}$ remains open. This is an unrefereed note.

1 Introduction

1.1 The problem

Throughout, a *variety* over a field k is an integral separated scheme of finite type over k ; the examples below are smooth and irreducible, so they are varieties under any of the usual conventions. In the Oberwolfach Report on Affine Algebraic Geometry [FRZ07, pp. 69–70], Kraft and Russell stated the following result. Let k be an algebraically closed field of infinite transcendence degree over its prime field, and let $S \rightarrow X$ and $T \rightarrow X$ be morphisms of k -varieties such that the schematic fibres S_x and T_x are isomorphic for all $x \in X$. Then there are a dominant étale morphism $\phi: U \rightarrow X$ and an isomorphism $S \times_X U \cong T \times_X U$ over U . Here x runs over the closed points of X (the abstract of [KR14] speaks of isomorphic “(closed) fibers”); if the generic fibres were assumed to be isomorphic, the conclusion would follow by spreading out. The authors remark that their proof uses the hypothesis on k in an essential way, and their Problem 2 asks to show that the proposition holds over every

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algebraically closed field, “or give counter examples over $\overline{\mathbb{F}}_p$ or $\overline{\mathbb{Q}}$ ”. The problem is record OWR-1452-024 of the UnsolvedMath dataset [ula26]; the record writes $\mathbb{F}_p$ and $\mathbb{Q}$ without the bars that are present in the report.

The result appeared as the Generic Equivalence Theorem of [KR14], for affine morphisms. According to [KR14, §2], a preliminary write-up had appeared in the lecture notes [BNM07] of a school held in Hanoi in 2006; we have not seen them. We could not access the publisher’s online version of [KR14]. In the journal-formatted accepted manuscript deposited in the repository of the University of Basel, the conclusion of the theorem is a dominant morphism $U \rightarrow X$ of *finite degree*, and Remark 2.2 says that the authors do not know whether the theorem holds for all algebraically closed fields, for example for $\overline{\mathbb{Q}}$. The arXiv version [KR14] (v1, 2012) states the theorem with a dominant *étale* morphism and contains the same remark; so the change from an étale to a finite-degree conclusion is due to Kraft and Russell themselves. We use the numbering of the accepted manuscript; in arXiv v1, its Remarks 2.2 and 5.2 and Theorem 5.2 are Remarks 2 and 5 and Theorem 5. Kaliman [Kal20] restated the theorem with the finite-degree conclusion and proved several extensions (numbering as in arXiv:1804.09747v3): the affineness assumption can be dropped in any characteristic [Kal20, Thm. 2.5]; over uncountable fields of characteristic zero it suffices that very general fibres be isomorphic [Kal20, Thm. 3.3]; and over an algebraically closed field of characteristic zero and finite transcendence degree, such as $\overline{\mathbb{Q}}$, the conclusion holds for *proper* morphisms [Kal20, Thms. 4.5 and 4.7]. Kaliman shows that the general case over such fields would follow from his [Kal20, Conj. 4.2]. In characteristic zero a dominant morphism of finite degree can be replaced by a dominant étale one. Kraft and Russell already note this in the accepted manuscript of [KR14], for the base change in Theorem B of their introduction (a consequence of the Generic Equivalence Theorem), right after its statement; Kaliman gives the argument (generic smoothness) in [Kal20, Rem. 3.12]. In the same remark, which is attached to his Theorem 3.3 (stated in characteristic zero), Kaliman says he does not know whether the morphism can be made étale in positive characteristic. The negative answer to this question follows immediately from classical facts (Proposition 1.3 and Remark 6.2).

1.2 Results

For the étale conclusion as stated in [FRZ07], a negative answer over $\overline{\mathbb{F}}_p$ already follows from Russell’s classical forms of the affine line (Proposition 1.3 below). Our main result answers the $\overline{\mathbb{F}}_p$ part of Problem 2 negatively for the finite-degree conclusion of [KR14, Kal20], and indeed for the weakest form of the conclusion: no dominant base change of any kind makes the families isomorphic.

Theorem 1.1 (affine families). *Let p be a prime, $k = \overline{\mathbb{F}}_p$, and $Y = \mathbb{A}_k^1 \setminus \{0, 1, -1\}$ with coordinate y . Let $S \subset \mathbb{G}_m^2 \times \mathbb{A}^3 \times Y$, with coordinates $(u, v; w, z_1, z_2; y)$, be defined by*

$$wz_1 = (u - 1)(u - y), \quad wz_2 = (y - 1)(v - 1) - y(u - 1). \quad (1)$$

Let $\text{Fr}: Y \rightarrow Y$ be the morphism $y \mapsto y^p$, and let $T = S \times_{Y, \text{Fr}} Y$; it is defined by (1) with y replaced by y^p . Then S and T are smooth affine varieties defined over $\mathbb{F}_p$, the projections $S \rightarrow Y$ and $T \rightarrow Y$ are smooth affine morphisms of relative dimension 3, and:

- (i) $S_y \cong T_y$ as k -schemes for every closed point $y \in Y$;
- (ii) for every morphism of schemes $U \rightarrow Y$ whose image contains the generic point of Y , the U -schemes $S \times_Y U$ and $T \times_Y U$ are not isomorphic.

In particular there is no dominant morphism of varieties $U \rightarrow Y$, étale, of finite degree or arbitrary, with $S \times_Y U \cong T \times_Y U$ over U .

field k (algebraically closed)	étale base change	base change of finite degree
char. 0, tr. deg. $= \infty$	holds [KR14, Kal20]	holds [KR14, Kal20]
char. 0, tr. deg. $< \infty$, proper maps	holds [Kal20]	holds [Kal20, Thm. 4.7]
char. 0, tr. deg. $< \infty$, in general	open (cf. [Kal20, Conj. 4.2])	open
char. p , tr. deg. $= \infty$	fails (Proposition 1.3)	holds [KR14], [Kal20, Thm. 2.5]
char. p , $k = \overline{\mathbb{F}}_p$	fails (Proposition 1.3)	fails (Thms. 1.1, 1.2)
char. p , $0 < \text{tr. deg.} < \infty$	fails (Proposition 1.3)	not settled here

Table 1: The generic equivalence statement for two morphisms with isomorphic fibres over all closed points. The proposition of [FRZ07] is the étale form for tr. deg. $= \infty$, in any characteristic. In characteristic 0 a base change of finite degree can be replaced by an étale one, by generic smoothness ([KR14], after Theorem B; [Kal20, Rem. 3.12]). The failures of the étale form in characteristic p are classical (Remark 6.2).

Theorem 1.2 (projective families). *Let $p \equiv 1 \pmod{4}$ be a prime and $k = \overline{\mathbb{F}}_p$. Let E be the elliptic curve $y^2 = x^3 - x$ over $\mathbb{F}_p$, $A = E \times E$, $\pi: A \rightarrow A$ the p -power Frobenius endomorphism, and $Y = A \setminus \{0\}$. Let $\Delta, \Gamma \subset A \times Y$ be the graphs of the inclusion $Y \rightarrow A$ and of $\pi|_Y$, and put*

$$S = \text{Bl}_{(\{0\} \times Y) \sqcup \Delta}(A \times Y), \quad T = \text{Bl}_{(\{0\} \times Y) \sqcup \Gamma}(A \times Y).$$

Then $S \rightarrow Y$ and $T \rightarrow Y$ are smooth projective morphisms of relative dimension 2 between smooth varieties defined over $\mathbb{F}_p$, with fibres $S_a \cong \text{Bl}_{\{0,a\}}(A)$ and $T_a \cong \text{Bl}_{\{0,\pi(a)\}}(A)$ over the closed points $a \in Y$, and statements (i) and (ii) of Theorem 1.1 hold for them; in particular $S_a \cong T_a$ as k -schemes for every closed point $a \in Y$.

In both theorems T is the pull-back of S along an endomorphism of Y of Frobenius type (Fr, resp. $\pi|_Y$; see Remark 5.5(3)). Since S is defined over $\mathbb{F}_p$, each closed fibre T_y is the Frobenius twist of S_y , that is, its base change along the automorphism $x \mapsto x^p$ of k , and so it is isomorphic to S_y as an abstract scheme; the content of (i) is that there is a k -linear isomorphism. Accordingly, every isomorphism between fibres in this note is an isomorphism of schemes over the base field (k , or the field Ω of Sections 3 and 5).

The mechanism is the same in both theorems. The isomorphism class of the fibre over y is determined by the orbit of a point under an arithmetic group ($\text{GL}_2(\mathbb{Z})$ acting on $\mathbb{G}_m^2$ by monomial maps, or $\text{GL}_2(\mathbb{Z}[i])$ acting on A). Over $\overline{\mathbb{F}}_p$ every point in question has finite order, and the Frobenius twist of such a point lies in its orbit, by an explicit matrix (Lemma 2.1). At the generic point the twist lies in no orbit, because the matrix would have to be p times the identity (resp. π times the identity). The matrices needed at the closed points have unbounded entries, and each fixed matrix serves only finitely many closed points (Remarks 4.1 and 5.5). This uses in an essential way that every element of $\overline{\mathbb{F}}_p^*$ is a root of unity; over number fields the same mechanism fails (Proposition 7.1).

For completeness we also record the following classical fact, which concerns only the étale form of the conclusion. It is not new in substance (Remark 6.2), and it already refutes the étale form of the proposition of [FRZ07] over $\overline{\mathbb{F}}_p$, and indeed over every algebraically closed field of positive characteristic. It also answers negatively, in the natural reading, the positive-characteristic question of [Kal20, Rem. 3.12]; this answer follows immediately from classical facts.

Proposition 1.3 (étale form, positive characteristic). *Let k be an algebraically closed field of characteristic $p > 0$, of any transcendence degree. The families $S = \mathbb{A}^1 \times \mathbb{A}_t^1$ and $T = \{y^p = x + tx^p\} \subset \mathbb{A}^2 \times \mathbb{A}_t^1$ over $\mathbb{A}_t^1$ are smooth and affine, all their closed fibres are isomorphic to $\mathbb{A}^1$, and they become isomorphic after the purely inseparable base change $t = s^p$; but there is no étale morphism $U \rightarrow \mathbb{A}_t^1$ whose image contains the generic point with $S \times_{\mathbb{A}^1} U \cong T \times_{\mathbb{A}^1} U$ over U .*

Table 1 summarizes what is known. The finite-degree form over algebraically closed fields of characteristic p and finite positive transcendence degree is not settled by our methods (see Section 7).

2 A matrix lemma

Lemma 2.1. *Let $\mathcal{O}$ be a principal ideal domain, let $\mu \in \mathcal{O} \setminus \{0\}$ and $\lambda \in \mathcal{O}$ with $\lambda\mathcal{O} + \mu\mathcal{O} = \mathcal{O}$, and let $r \in \mathcal{O}^2$. Then there is $M \in \mathrm{SL}_2(\mathcal{O})$ with $Mr \equiv \lambda r \pmod{\mu\mathcal{O}^2}$.*

Proof. If $r = 0$ take $M = I$. Otherwise let d generate the ideal generated by the two entries of r , and write $r = dr_0$ with $r_0 = (a, b)^\top$ and $a\mathcal{O} + b\mathcal{O} = \mathcal{O}$. Choose $\xi, \zeta \in \mathcal{O}$ with $a\xi + b\zeta = 1$; then $B = \begin{pmatrix} a & -\zeta \\ b & \xi \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ and $Be_1 = r_0$. Let e generate $d\mathcal{O} + \mu\mathcal{O}$ and put $\mu' = \mu/e$. Since μ' divides μ , we have $\lambda\mathcal{O} + \mu'\mathcal{O} = \mathcal{O}$; choose β, δ with $\lambda\delta - \beta\mu' = 1$ and put $D = \begin{pmatrix} \lambda & \beta \\ \mu' & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$. For $M = BDB^{-1} \in \mathrm{SL}_2(\mathcal{O})$ we get

$$Mr = dBDe_1 = dB(\lambda e_1 + \mu' e_2) = \lambda r + d\mu' Be_2,$$

and $d\mu' = (d/e)\mu \in \mu\mathcal{O}$. □

The auxiliary element e is not needed: since $\lambda\mathcal{O} + \mu\mathcal{O} = \mathcal{O}$, one may take $\mu' = \mu$, and then $Mr = \lambda r + d\mu Be_2$ directly. We keep the version above because it is the one implemented in the programs of items 1 and 2 of the Verification paragraph at the end of the paper; the programs of item 3 there test both versions.

We use the lemma for $\mathcal{O} = \mathbb{Z}$, $\lambda = p$, $\mu = p^n - 1$, and for $\mathcal{O} = \mathbb{Z}[i]$, $\lambda = \pi$, $\mu = \pi^n - 1$, where π is a Gaussian prime of norm p .

3 Affine modifications of the torus at two points

Let Ω be an algebraically closed field of any characteristic and $\mathbb{T} = \mathbb{G}_m^2 = \mathrm{Spec} \Omega[u^{\pm 1}, v^{\pm 1}]$, with unit element $\mathbf{1} = (1, 1)$. For $M = (m_{ij}) \in \mathrm{GL}_2(\mathbb{Z})$ and $x = (x_1, x_2) \in \mathbb{T}(\Omega)$ put

$$M \cdot x = (x_1^{m_{11}} x_2^{m_{12}}, x_1^{m_{21}} x_2^{m_{22}}).$$

This is a left action by group automorphisms, and $(-M) \cdot x = (M \cdot x)^{-1}$. Let μ_M denote the corresponding automorphism of $\mathbb{T}$, so $\mu_M^* u = u^{m_{11}} v^{m_{12}}$ and $\mu_M^* v = u^{m_{21}} v^{m_{22}}$. If $x_1 = g^a$ and $x_2 = g^b$ for some $g \in \Omega^*$, then $M \cdot x = (g^{c_1}, g^{c_2})$ with $(c_1, c_2)^\top = M(a, b)^\top$.

For $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}(\Omega)$ with $\varphi_1 \neq 1$ put

$$f_1^\varphi = (u - 1)(u - \varphi_1), \quad f_2^\varphi = (\varphi_1 - 1)(v - 1) - (\varphi_2 - 1)(u - 1), \quad I_\varphi = (f_1^\varphi, f_2^\varphi) \subset \mathcal{O}(\mathbb{T}),$$

and let $X_\varphi \subset \mathbb{T} \times \mathbb{A}^3$ (coordinates $u, v; w, z_1, z_2$) be defined by $wz_1 = f_1^\varphi$ and $wz_2 = f_2^\varphi$, with projection $\mathrm{pr}_\varphi: X_\varphi \rightarrow \mathbb{T}$. In the language of [KZ99], X_φ is the affine modification of $\mathbb{T} \times \mathbb{A}_w^1$ along the divisor $\{w = 0\}$ with centre $\{\mathbf{1}, \varphi\} \times \{0\}$.

Lemma 3.1. *Let $\varphi \in \mathbb{T}(\Omega)$ with $\varphi_1 \neq 1$.*

- (a) I_φ is the ideal of the reduced subscheme $\{\mathbf{1}, \varphi\}$ of $\mathbb{T}$.
- (b) X_φ is a smooth irreducible affine threefold, and $\mathcal{O}(X_\varphi) = \mathcal{O}(\mathbb{T})[w, I_\varphi w^{-1}] \subset \mathcal{O}(\mathbb{T})[w^{\pm 1}]$, where $z_i = f_i^\varphi / w$.
- (c) $\mathcal{O}(X_\varphi)^* = \Omega^* \cdot u^{\mathbb{Z}} \cdot v^{\mathbb{Z}}$.

(d) For $x \in \mathbb{T}(\Omega)$ the fibre $\text{pr}_\varphi^{-1}(x)$ is 2-dimensional if $x \in \{\mathbf{1}, \varphi\}$, and isomorphic to $\mathbb{G}_m$ otherwise.

Proof. (a) Since $\varphi_1 \neq 1$ we have $f_2^\varphi = (\varphi_1 - 1)(v - l(u))$ with $l(u) = 1 + \rho(u - 1)$ and $\rho = (\varphi_2 - 1)/(\varphi_1 - 1)$. Hence $\mathcal{O}(\mathbb{T})/(f_2^\varphi) \cong \Omega[u^{\pm 1}, l(u)^{-1}]$, and dividing further by $(u - 1)(u - \varphi_1)$ gives $\Omega \times \Omega$ by the Chinese remainder theorem, the factors corresponding to $u = 1$ and $u = \varphi_1$. (Both u and $l(u)$ are invertible there: $l(1) = 1$ and $l(\varphi_1) = \varphi_2 \neq 0$.) So $\mathcal{O}(\mathbb{T})/I_\varphi \cong \Omega^2$, with the two maximal ideals of $\mathbf{1}$ and $(\varphi_1, \varphi_2) = \varphi$.

(b) Put $g_i = wz_i - f_i^\varphi$. At a point with $w \neq 0$ the Jacobian minor $\partial(g_1, g_2)/\partial(z_1, z_2)$ equals $w^2 \neq 0$. At a point with $w = 0$ we have $f_1^\varphi = f_2^\varphi = 0$, so the point lies over $\mathbf{1}$ or φ by (a), and there $\partial(g_1, g_2)/\partial(u, v) = \partial_u f_1^\varphi \cdot \partial_v f_2^\varphi = (2u - 1 - \varphi_1)(\varphi_1 - 1) = \mp(\varphi_1 - 1)^2 \neq 0$. Hence X_φ is smooth of pure dimension 3. The open subset $\{w \neq 0\}$ is isomorphic to $\mathbb{T} \times \mathbb{G}_m$ (there $z_i = f_i^\varphi/w$), and its complement is $\{\mathbf{1}, \varphi\} \times \{0\} \times \mathbb{A}^2$, of dimension 2; so no component of X_φ lies in $\{w = 0\}$, and X_φ is irreducible. As X_φ is integral and $w \neq 0$, restriction embeds $\mathcal{O}(X_\varphi)$ into $\mathcal{O}(X_\varphi)[w^{-1}] = \mathcal{O}(\mathbb{T})[w^{\pm 1}]$, with image generated by $\mathcal{O}(\mathbb{T})$, w and $z_i = f_i^\varphi/w$.

(c) A unit h of $\mathcal{O}(X_\varphi)$ is a unit of $\Omega[u^{\pm 1}, v^{\pm 1}, w^{\pm 1}]$, so $h = cu^a v^b w^n$ with $c \in \Omega^*$. Since u, v are units of $\mathcal{O}(X_\varphi)$, so is w^n . But w vanishes on the nonempty set $\text{pr}_\varphi^{-1}(\mathbf{1}) \cap \{w = 0\}$, so $n = 0$.

(d) Over $x \notin \{\mathbf{1}, \varphi\}$ some $f_i^\varphi(x)$ is nonzero, so $w \neq 0$ and z is determined by w . Over $x \in \{\mathbf{1}, \varphi\}$ the fibre is $\{wz_1 = wz_2 = 0\} = \{w = 0\} \cup \{z_1 = z_2 = 0\}$. $\square$

Proposition 3.2. *Let $\varphi, \psi \in \mathbb{T}(\Omega)$ with $\varphi_1 \neq 1 \neq \psi_1$. Then $X_\varphi \cong X_\psi$ as Ω -schemes if and only if $\psi \in \text{GL}_2(\mathbb{Z}) \cdot \varphi$. More precisely, for every isomorphism of Ω -schemes $F: X_\varphi \rightarrow X_\psi$ there are $M \in \text{GL}_2(\mathbb{Z})$ and $c \in \mathbb{T}(\Omega)$ with $\text{pr}_\psi \circ F = \tau_c \circ \mu_M \circ \text{pr}_\varphi$, where τ_c is translation by c , and either $c = \mathbf{1}$ and $M \cdot \varphi = \psi$, or $c = \psi$ and $M \cdot \varphi = \psi^{-1}$ (that is, $(-M) \cdot \varphi = \psi$).*

Proof. If $M \cdot \varphi = \psi$, then μ_M is an automorphism of $\mathbb{T}$ with $\mu_M^{-1}(\{\mathbf{1}, \psi\}) = \{\mathbf{1}, \varphi\}$, so $\mu_M^*(I_\psi) = I_\varphi$ by Lemma 3.1(a). Extending μ_M^* to $\mathcal{O}(\mathbb{T})[w^{\pm 1}]$ by $w \mapsto w$, it maps $\mathcal{O}(\mathbb{T})[w, I_\psi w^{-1}]$ onto $\mathcal{O}(\mathbb{T})[w, I_\varphi w^{-1}]$, and Lemma 3.1(b) gives an isomorphism of Ω -schemes $X_\varphi \cong X_\psi$.

Conversely, let F be an isomorphism of Ω -schemes. Then F^* is Ω -linear, so $F^*(\Omega^*) = \Omega^*$, and F^* induces an isomorphism $\mathcal{O}(X_\psi)^*/\Omega^* \rightarrow \mathcal{O}(X_\varphi)^*/\Omega^*$, and both groups are free with basis u, v by Lemma 3.1(c). Hence $F^*u = c_1 u^{m_{11}} v^{m_{12}}$ and $F^*v = c_2 u^{m_{21}} v^{m_{22}}$ with $c = (c_1, c_2) \in \mathbb{T}(\Omega)$ and $M = (m_{ij}) \in \text{GL}_2(\mathbb{Z})$, that is, $\text{pr}_\psi \circ F = \gamma \circ \text{pr}_\varphi$ with $\gamma = \tau_c \circ \mu_M \in \text{Aut}(\mathbb{T})$. Consequently F maps $\text{pr}_\varphi^{-1}(x)$ isomorphically onto $\text{pr}_\psi^{-1}(\gamma(x))$ for every x , and Lemma 3.1(d) gives $\gamma(\{\mathbf{1}, \varphi\}) = \{\mathbf{1}, \psi\}$. If $\gamma(\mathbf{1}) = \mathbf{1}$, then $c = \mathbf{1}$ and $M \cdot \varphi = \gamma(\varphi) = \psi$. Otherwise $\gamma(\mathbf{1}) = \psi$ and $\gamma(\varphi) = \mathbf{1}$, so $c = \psi$ and $\psi \cdot (M \cdot \varphi) = \mathbf{1}$. $\square$

4 Proof of Theorem 1.1

For a point y of Y with values in a field put $c(y) = (y, y + 1)$; the first coordinate is $\neq 1$. The fibre of S over y is $X_{c(y)}$ in the notation of Section 3 (note $c_1 - 1 = y - 1$ and $c_2 - 1 = y$), and the fibre of T over y is $X_{c(y^p)} = X_{c(y)^p}$, because $(y + 1)^p = y^p + 1$. The morphism Fr maps Y into Y since $0, 1, -1$ are fixed by $y \mapsto y^p$ and this map is injective on k .

Smoothness. Let g_1, g_2 be the two equations (1). At every point $s \in S$ (closed or not) the relative Jacobian matrix of (g_1, g_2) with respect to (u, v, w, z_1, z_2) has rank 2: if $w(s) \neq 0$ use the columns of z_1, z_2 ; if $w(s) = 0$ then $u(s) \in \{1, y(s)\}$ and the columns of u, v give the minor $(2u - 1 - y)(y - 1) = \mp(y - 1)^2$, which is nonzero because $y - 1$ is a unit on Y . By the Jacobian criterion [BLR90, §2.2, Prop. 7], $S \rightarrow Y$ is smooth of relative dimension 3; so is its base change $T \rightarrow Y$. Both are affine morphisms, and S and T are smooth over k of pure dimension 4. The open subset $\{w \neq 0\}$ of S is isomorphic to $\mathbb{G}_m^3 \times Y$, and its complement is the union of two copies of $\mathbb{A}^2 \times Y$, of dimension 3; hence S is irreducible, and likewise T . All equations have coefficients in $\mathbb{F}_p$.

(i) *Closed fibres.* Let $y \in Y(k)$ and choose $n \geq 1$ with $y \in \mathbb{F}_q$, $q = p^n$. Let g generate the cyclic group $\mathbb{F}_q^*$, put $N = q - 1$, and write $y = g^a$ and $y + 1 = g^b$ with $a, b \in \mathbb{Z}$. Since $\gcd(p, N) = 1$, Lemma 2.1 with $\mathcal{O} = \mathbb{Z}$, $\lambda = p$, $\mu = N$ and $r = (a, b)^\top$ gives $M \in \mathrm{SL}_2(\mathbb{Z})$ with $M(a, b)^\top \equiv p(a, b)^\top \pmod{N}$. Hence $M \cdot c(y) = (g^{pa}, g^{pb}) = c(y)^p$, and Proposition 3.2 (with $\Omega = k$) gives an isomorphism of k -schemes $S_y = X_{c(y)} \cong X_{c(y)^p} = T_y$.

(ii) *The generic point.* Let $U \rightarrow Y$ be a morphism whose image contains the generic point of Y , and suppose that $S \times_Y U \cong T \times_Y U$ over U . Choose $u_0 \in U$ over the generic point and an algebraic closure Ω of the residue field $\kappa(u_0)$. Base change along $\mathrm{Spec} \Omega \rightarrow U$ gives an isomorphism of Ω -schemes $X_\varphi \cong X_{\varphi^p}$, where $\varphi = (t, t + 1)$ and $t \in \Omega$ is the image of the coordinate y ; here $k(t) \subset \Omega$ is a purely transcendental extension of k . By Proposition 3.2 there is $M \in \mathrm{GL}_2(\mathbb{Z})$ with $M \cdot \varphi = \varphi^p$, that is,

$$t^{m_{11}}(t + 1)^{m_{12}} = t^p, \quad t^{m_{21}}(t + 1)^{m_{22}} = (t + 1)^p$$

in $k(t)^*$. Since t and $t + 1$ are non-associate prime elements of $k[t]$, unique factorisation gives $M = pI$, which is impossible because $\det(pI) = p^2 \neq \pm 1$.

For the final assertion, if $U \rightarrow Y$ is a dominant morphism of varieties then its image contains a dense open subset of Y , hence the generic point. $\square$

Remark 4.1 (no bounded family of isomorphisms). Every isomorphism of k -schemes $S_y \cong T_y$ at a closed point y determines a matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ as in Proposition 3.2, with $M \cdot c(y) = c(y)^p$ or, in the swapped case, $M \cdot c(y) = c(y)^{-p}$. For fixed M and $\epsilon = \pm 1$ let $Y_{M, \epsilon} \subset Y(k)$ be the set of y with $M \cdot c(y) = c(y)^{\epsilon p}$. Each row i of this equation reads $y^\alpha(y + 1)^\beta = 1$ with $(\alpha, \beta) = (m_{i1} - \epsilon p \delta_{i1}, m_{i2} - \epsilon p \delta_{i2})$. Not both rows can have $(\alpha, \beta) = (0, 0)$, since $\epsilon p I \notin \mathrm{GL}_2(\mathbb{Z})$. For a row with $(\alpha, \beta) \neq (0, 0)$, the equation $y^{\alpha^+}(y + 1)^{\beta^+} = y^{\alpha^-}(y + 1)^{\beta^-}$ (where $\alpha = \alpha^+ - \alpha^-$ with $\alpha^\pm \geq 0$, and similarly for β) is a nonzero polynomial equation of degree at most $|\alpha| + |\beta|$, by unique factorisation. Hence $\#Y_{M, \epsilon}$ is at most the minimum of $|\alpha| + |\beta|$ over the rows with $(\alpha, \beta) \neq (0, 0)$. Thus $Y(k)$ is a countable union of the finite sets $Y_{M, \epsilon}$, and the matrices needed at the closed points have unbounded entries. This is the countability phenomenon that the hypothesis on the transcendence degree excludes. The programs described at the end of the paper confirm this bound for all $M \in \mathrm{GL}_2(\mathbb{Z})$ with entries in $[-3, 3]$ and both signs, in all fields $\mathbb{F}_q$ with $q \leq 256$ that they treat. For example, if $p = 2$ and $y \in \mathbb{F}_{2^7} \setminus \mathbb{F}_2$, then $c(y)$ has prime order 127, so $M \mp 2I$ is singular modulo 127. Since $\det(M \mp 2I) = \det M \mp 2 \mathrm{tr} M + 4$ and $\det M = \pm 1$, this forces $|\mathrm{tr} M| \geq 61$, and some entry of M has absolute value at least 31.

Remark 4.2 (quasi-affine variant). Let $S^\circ \subset \mathbb{G}_m^2 \times Y$ be the complement of the images of the two sections $y \mapsto \mathbf{1}$ and $y \mapsto c(y)$, and $T^\circ = S^\circ \times_{Y, \mathrm{Fr}} Y$. Every isomorphism of Ω -schemes $\mathbb{T} \setminus \{\mathbf{1}, \varphi\} \cong \mathbb{T} \setminus \{\mathbf{1}, \psi\}$ extends to an automorphism of the Ω -scheme $\mathbb{T}$, because $\mathbb{T}$ is normal and affine and the removed sets have codimension 2. Such automorphisms have the form $\tau_c \circ \mu_M$ (compare the proof of Proposition 3.2). The argument above therefore applies verbatim and shows that the smooth quasi-affine families $S^\circ \rightarrow Y$ and $T^\circ \rightarrow Y$ of surfaces also satisfy (i) and (ii).

5 Proof of Theorem 1.2

In this section $p \equiv 1 \pmod{4}$, $\iota \in \mathbb{F}_p$ satisfies $\iota^2 = -1$, and E, A, π are as in Theorem 1.2. The automorphism $[i](x, y) = (-x, \iota y)$ of E is defined over $\mathbb{F}_p$, fixes the origin and satisfies $[i]^2 = [-1]$. So $a + bi \mapsto [a] + [b][i]$ is a ring homomorphism from $\mathbb{Z}[i]$ to the endomorphism ring of E over any field containing $\mathbb{F}_p$, and it is injective: if $[a] + [b][i] = 0$, composing with $[a] - [b][i]$ gives $[a^2 + b^2] = 0$, so $a = b = 0$ [Sil09, Prop. III.4.2(a)]. We let $\mathrm{GL}_2(\mathbb{Z}[i])$ act on $A = E \times E$ by matrices.

Lemma 5.1. (a) *E is ordinary.*

(b) *For every algebraically closed field $\Omega \supseteq \mathbb{F}_p$ we have $\mathrm{End}(E_\Omega) = \mathbb{Z}[i]$, $\mathrm{End}(A_\Omega) = M_2(\mathbb{Z}[i])$ and $\mathrm{Aut}_{\mathrm{gp}}(A_\Omega) = \mathrm{GL}_2(\mathbb{Z}[i])$. In particular all endomorphisms of A_Ω are defined over $\mathbb{F}_p$.*

(c) The Frobenius endomorphism of E is an element $\pi \in \mathbb{Z}[i]$ of norm p ; in particular it is not a unit. It acts bijectively on $A(\Omega)$ for every algebraically closed field $\Omega \supseteq \mathbb{F}_p$.

Proof. (a) Put $m = (p-1)/2$, which is even. The coefficient of x^{p-1} in $(x^3 - x)^m = x^m(x^2 - 1)^m$ is $(-1)^{m/2} \binom{m}{m/2}$, which is nonzero modulo p . By [Sil09, Thm. V.4.1(a)], E is not supersingular. (This is Deuring's criterion [Deu41]: E has complex multiplication by $\mathbb{Z}[i]$, and p splits in $\mathbb{Q}(i)$.)

(b) The p -torsion points of E_Ω are the Ω -points of the finite $\mathbb{F}_p$ -group scheme $E[p]$, so they are defined over $\overline{\mathbb{F}}_p$, and by (a) there are p of them. Hence E_Ω is ordinary, and by Deuring's theorem [Deu41], [Sil09, Thm. V.3.1(a)], its endomorphism ring is not an order in a quaternion algebra. By [Sil09, §III.9] it is then $\mathbb{Z}$ or an order in an imaginary quadratic field. It contains $\mathbb{Z}[i]$, which is the maximal order of $\mathbb{Q}(i)$, so $\text{End}(E_\Omega) = \mathbb{Z}[i]$. Hence $\text{End}(A_\Omega) = M_2(\mathbb{Z}[i])$ and $\text{Aut}_{\text{gp}}(A_\Omega) = \text{GL}_2(\mathbb{Z}[i])$; the generator $[i]$ is defined over $\mathbb{F}_p$.

(c) The p -power Frobenius $\pi(x, y) = (x^p, y^p)$ is an endomorphism of degree p [Sil09, Prop. II.2.11]. The dual isogeny is additive, fixes $\mathbb{Z}$ and sends the automorphism $[i]$ to $[i]^{-1} = -[i]$ [Sil09, Thm. III.6.2], so on $\mathbb{Z}[i]$ it is complex conjugation. Therefore $\pi\bar{\pi} = \hat{\pi}\pi = [\deg \pi] = p$. Finally $x \mapsto x^p$ is bijective on Ω , so π is bijective on $A(\Omega)$. $\square$

Lemma 5.2. *Every finitely generated $\mathbb{Z}[i]$ -submodule of $E(\overline{\mathbb{F}}_p)$ is finite and cyclic. Moreover $E(\mathbb{F}_{p^n}) \cong \mathbb{Z}[i]/(\pi^n - 1)$ as $\mathbb{Z}[i]$ -modules for every $n \geq 1$.*

Proof. The second statement is Lenstra's theorem [Len96, Thm. 1(a)], since $\text{End}_{\mathbb{F}_p}(E) = \mathbb{Z}[i]$ by Lemma 5.1(b) and $\pi \notin \mathbb{Z}$. Lenstra also shows [Len96, Thm. 3] that $E(\overline{\mathbb{F}}_p) \cong \mathbb{Z}[i]_{(\pi)}/\mathbb{Z}[i]$, where $\mathbb{Z}[i]_{(\pi)}$ is the localisation at the prime ideal $\pi\mathbb{Z}[i]$ (the kernel of the action on the tangent space); this implies the first statement. For completeness we give a direct argument for the first statement.

A finitely generated submodule N is finitely generated over $\mathbb{Z}$ and torsion, hence finite, and $N = \bigoplus_\ell N_\ell$ with N_ℓ its ℓ -primary part. For $\ell \neq p$, the Tate module $T_\ell E$ is free of rank 2 over $\mathbb{Z}_\ell$ [Sil09, Prop. III.7.1] and is a faithful module over $\mathcal{O}_\ell = \mathbb{Z}[i] \otimes \mathbb{Z}_\ell$ [Sil09, Thm. III.7.4]. The ring $\mathcal{O}_\ell$ is a product of complete discrete valuation rings $\mathcal{O}_\mathfrak{l}$, one for each prime $\mathfrak{l} \mid \ell$ of $\mathbb{Z}[i]$, with $\sum_\mathfrak{l} \text{rank}_{\mathbb{Z}_\ell} \mathcal{O}_\mathfrak{l} = 2$. The corresponding factors of $T_\ell E$ are torsion-free, hence free, and nonzero by faithfulness; counting $\mathbb{Z}_\ell$ -ranks shows that each is free of rank 1. So $T_\ell E \cong \mathcal{O}_\ell$, and $E[\ell^m] \cong T_\ell E / \ell^m T_\ell E \cong \mathbb{Z}[i] / \ell^m \mathbb{Z}[i]$ is cyclic. A submodule of a cyclic module over the principal ideal ring $\mathbb{Z}[i] / \ell^m \mathbb{Z}[i]$ is cyclic, so N_ℓ is cyclic. For $\ell = p$, $E[p^m](\overline{\mathbb{F}}_p) \cong \mathbb{Z}/p^m \mathbb{Z}$ because E is ordinary [Sil09, Thm. V.3.1(b)], so N_p is already cyclic as a group. Finally, if $N_\ell = \mathbb{Z}[i]G_\ell$ for each ℓ , then $G = \sum_\ell G_\ell$ generates N : by the Chinese remainder theorem some integer multiple of G equals G_ℓ , for each ℓ . $\square$

Lemma 5.3. *Let B be an abelian surface over an algebraically closed field Ω , and let $a, b \in B(\Omega) \setminus \{0\}$. Then $\text{Bl}_{\{0,a\}}(B) \cong \text{Bl}_{\{0,b\}}(B)$ as Ω -schemes if and only if $b = \alpha(a)$ for some $\alpha \in \text{Aut}_{\text{gp}}(B)$.*

Proof. If $b = \alpha(a)$, then α is an automorphism of B mapping $\{0, a\}$ onto $\{0, b\}$, and it lifts to an isomorphism of the blow-ups by functoriality of blowing up.

Conversely, let $F: X_a = \text{Bl}_{\{0,a\}}(B) \rightarrow X_b = \text{Bl}_{\{0,b\}}(B)$ be an isomorphism of Ω -schemes, and let $\beta_a: X_a \rightarrow B$, $\beta_b: X_b \rightarrow B$ be the blow-downs. By [Har77, Prop. V.3.1], X_a is a smooth projective surface, β_a is an isomorphism outside the two exceptional curves $D_0, D_a \cong \mathbb{P}^1$, and so the irreducible curves contracted by β_a are exactly D_0 and D_a ; similarly for X_b . The curve $F(D_0)$ is rational. Every rational map from $\mathbb{P}^1$ to an abelian variety is constant [Mil08, Prop. I.3.9], so β_b contracts $F(D_0)$, and $F(D_0)$ is one of the exceptional curves of X_b ; the same holds for $F(D_a)$, and $F(D_0) \neq F(D_a)$. The rational map $h = \beta_b \circ F \circ \beta_a^{-1}: B \dashrightarrow B$ extends to a morphism, because B is smooth and the target is an abelian variety [Mil08, Thm. I.3.2]. Then $h \circ \beta_a = \beta_b \circ F$, as both sides agree on a dense open subset. The same construction for F^{-1} gives an inverse of h , so h is an automorphism of the Ω -variety B , and

$$h(\{0, a\}) = h(\beta_a(D_0 \cup D_a)) = \beta_b(F(D_0) \cup F(D_a)) = \{0, b\}.$$

By [Mil08, Cor. I.1.2], $h(x) = \alpha(x) + h(0)$ with $\alpha \in \text{Aut}_{\text{gp}}(B)$. If $h(0) = 0$ then $b = h(a) = \alpha(a)$. Otherwise $h(0) = b$ and $0 = h(a) = \alpha(a) + b$, so $b = (-\alpha)(a)$. $\square$

Lemma 5.4. *Let K be an algebraically closed field, B an abelian surface over K , V a quasi-projective K -variety, and $\sigma_1, \sigma_2: V \rightarrow B$ morphisms whose graphs $\Gamma_1, \Gamma_2 \subset B \times V$ are disjoint. Let $X = \text{Bl}_{\Gamma_1 \sqcup \Gamma_2}(B \times V)$. Then $X \rightarrow V$ is smooth and projective of relative dimension 2, and for every field $L \supseteq K$ and every K -morphism $v: \text{Spec } L \rightarrow V$ we have $X \times_V \text{Spec } L \cong \text{Bl}_{\{\sigma_1(v), \sigma_2(v)\}}(B_L)$ over L .*

Proof. We use three standard facts: blowing up commutes with restriction to open subsets and with isomorphisms; for a closed subscheme $C \subset W$ and any morphism $W' \rightarrow W$ the blow-up of W' along $C \times_W W'$ is a closed subscheme of $\text{Bl}_C(W) \times_W W'$, because the natural map of Rees algebras $\bigoplus_n I^n \otimes_R R' \rightarrow \bigoplus_n (IR')^n$ is surjective; and this closed immersion is an isomorphism if $W' \rightarrow W$ is flat, because then $I^n \otimes_R R' = I^n R'$.

Let $Q_i = (B \times V) \setminus \Gamma_{3-i}$ for $i = 1, 2$; these open sets cover $B \times V$, and $X|_{Q_i} = \text{Bl}_{\Gamma_i \cap Q_i}(Q_i)$. The V -automorphism $\tau_i(x, v) = (x - \sigma_i(v), v)$ of $B \times V$ maps Γ_i onto $\{0\} \times V$, and $\text{Bl}_{\{0\} \times V}(B \times V) = \text{Bl}_0(B) \times V$ by flat base change along $V \rightarrow \text{Spec } K$. Hence $X|_{Q_i}$ is V -isomorphic to an open subscheme of $\text{Bl}_0(B) \times V$. Since $\text{Bl}_0(B)$ is a smooth projective surface [Har77, Prop. V.3.1], $X \rightarrow V$ is smooth of relative dimension 2. It is projective, as a composite of the projective morphisms $X \rightarrow B \times V$ [Har77, Prop. II.7.16] and $B \times V \rightarrow V$.

Let $v: \text{Spec } L \rightarrow V$ and let $j: \text{Bl}_{\{\sigma_1(v), \sigma_2(v)\}}(B_L) \rightarrow X \times_V \text{Spec } L$ be the closed immersion from the second fact. Over $Q_i \times_V \text{Spec } L$, transport by τ_i (whose base change is translation by $-\sigma_i(v)$) identifies j with the corresponding closed immersion for the blow-up of $B \times V$ along $\{0\} \times V$, which is the natural map $\text{Bl}_0(B_L) \rightarrow \text{Bl}_0(B) \times_K L$; this is an isomorphism by flat base change along $\text{Spec } L \rightarrow \text{Spec } K$. So j is an isomorphism. $\square$

Proof of Theorem 1.2. The graphs of the inclusion $Y \rightarrow A$ and of the zero map are disjoint, and so are the graphs of $\pi|_Y$ and of the zero map, because π is injective on points (Lemma 5.1(c)). By Lemma 5.4 with $K = k$, $S \rightarrow Y$ and $T \rightarrow Y$ are smooth and projective of relative dimension 2, $S_a = \text{Bl}_{\{0, a\}}(A)$ and $T_a = \text{Bl}_{\{0, \pi(a)\}}(A)$ for closed points a , and the same formulas hold for points with values in any field containing k . By [Har77, Prop. II.7.16], S and T are varieties, birational to $A \times Y$. They are defined over $\mathbb{F}_p$, since A, Y and the centres are, and blowing up commutes with the flat base change from $\mathbb{F}_p$ to k .

(i) *Closed fibres.* Let $a = (a_1, a_2) \in Y(k)$ and choose $n \geq 1$ with $a_1, a_2 \in E(\mathbb{F}_q)$, $q = p^n$. Since $[i]$ is defined over $\mathbb{F}_p$, $E(\mathbb{F}_q)$ is a $\mathbb{Z}[i]$ -submodule of $E(\overline{\mathbb{F}_p})$, so Lemma 5.2 gives $E(\mathbb{F}_q) = \mathbb{Z}[i]G$ for some G , and $\mu = \pi^n - 1$ annihilates $E(\mathbb{F}_q)$ since π^n fixes $\mathbb{F}_q$ -points. We have $\mu \neq 0$ and $\pi\mathbb{Z}[i] + \mu\mathbb{Z}[i] = \mathbb{Z}[i]$, because π is not a unit and any common divisor of π and μ divides $\pi^n - \mu = 1$. Write $a_j = r_j G$ with $r_j \in \mathbb{Z}[i]$. Lemma 2.1 with $\mathcal{O} = \mathbb{Z}[i]$, $\lambda = \pi$ and $r = (r_1, r_2)^T$ gives $M \in \text{SL}_2(\mathbb{Z}[i])$ with $Mr \equiv \pi r \pmod{\mu}$, and hence

$$M(a) = ((Mr)_1 G, (Mr)_2 G) = (\pi r_1 G, \pi r_2 G) = \pi(a).$$

Since $M \in \text{Aut}_{\text{gp}}(A)$, Lemma 5.3 gives an isomorphism of k -schemes $S_a \cong T_a$.

(ii) *The generic point.* Let $U \rightarrow Y$ be a morphism whose image contains the generic point ξ of Y , and suppose $S \times_Y U \cong T \times_Y U$ over U . Choose $u_0 \in U$ over ξ , an algebraic closure Ω of $\kappa(u_0)$, and let $\eta \in A(\Omega)$ be the composite $\text{Spec } \Omega \rightarrow U \rightarrow Y \subset A$; its image in A is the generic point of A . Base change gives an isomorphism of Ω -schemes $\text{Bl}_{\{0, \eta\}}(A_\Omega) \cong \text{Bl}_{\{0, \pi(\eta)\}}(A_\Omega)$, so by Lemma 5.3 and Lemma 5.1(b) there is $\alpha \in \text{GL}_2(\mathbb{Z}[i])$ with $\alpha(\eta) = \pi(\eta)$. The endomorphism $\beta = \alpha - \pi I$ of A is defined over $\mathbb{F}_p$ and satisfies $\beta \circ \eta = 0$. Its kernel is therefore a closed subscheme of A containing the generic point, so β maps every point of A to 0, and $\beta = 0$ because A is reduced. Thus $\pi I = \alpha \in \text{GL}_2(\mathbb{Z}[i])$, so $\pi^2 = \det \alpha$ is a unit of $\mathbb{Z}[i]$, contradicting $\pi \bar{\pi} = p$. $\square$

Remark 5.5. (1) A fixed $\alpha \in \mathrm{GL}_2(\mathbb{Z}[i])$ serves only finitely many closed points, as in Remark 4.1: $\det(\alpha - \pi I) = \det \alpha - \pi \operatorname{tr} \alpha + \pi^2$ is nonzero, since otherwise π would divide the unit $\det \alpha$, so $\alpha - \pi I$ is an isogeny and $\{a : \alpha(a) = \pi(a)\}$ lies in its finite kernel.

(2) The base can be replaced by the curve $C = (E \setminus \{0\}) \times \{0\} \subset Y$. The restricted families $S|_C \rightarrow C$ and $T|_C \rightarrow C$ still satisfy (i), and they satisfy (ii) as well. Indeed, at the generic point $\eta = (\eta_E, 0)$ of C the equation $\alpha(\eta) = \pi(\eta)$ gives $(\alpha_{11} - \pi)(\eta_E) = 0$ and $\alpha_{21}(\eta_E) = 0$, hence $\alpha_{11} = \pi$ and $\alpha_{21} = 0$ as in the proof above. Then π divides the unit $\det \alpha = \pi \alpha_{22}$, a contradiction.

(3) As in Theorem 1.1, T is the pull-back of S along the endomorphism $\pi|_Y$ of Y . Indeed $\pi|_Y$ is flat, being a finite surjective morphism between smooth varieties of the same dimension, blowing up commutes with the flat base change $\operatorname{id}_A \times \pi|_Y$, and the preimage of $(\{0\} \times Y) \sqcup \Delta$ under this map is $(\{0\} \times Y) \sqcup \Gamma$. We have not tried to extend Theorem 1.2 to primes $p \not\equiv 1 \pmod{4}$; the affine Theorem 1.1 covers every p .

6 The étale form in positive characteristic

Proposition 6.1. *Let k be an algebraically closed field of characteristic $p > 0$, $Y = \mathbb{A}_k^1$ with coordinate t , $S = \mathbb{A}^1 \times Y$, and $T = \operatorname{Spec} k[t, x, y]/(y^p - x - tx^p)$ with its projection to Y .*

- (a) *$T \rightarrow Y$ is smooth and affine, and every closed fibre T_c is isomorphic to $\mathbb{A}^1$.*
- (b) *For $U = \operatorname{Spec} k[s] \rightarrow Y$, $t = s^p$, we have $T \times_Y U \cong \mathbb{A}_U^1 = S \times_Y U$.*
- (c) *If $U \rightarrow Y$ is étale and its image contains the generic point of Y , then $S \times_Y U \not\cong T \times_Y U$ over U .*

Proof. (a) and (b). Put $F = y^p - x - tx^p$. Then $\partial F / \partial x = -1$, so $T \rightarrow Y$ is smooth. For $c \in k$ let $s = c^{1/p} \in k$. In $k[x, y]$ we have $(y - sx)^p - x = F(c, x, y)$. Hence $u \mapsto (x, y) = (u^p, u + su^p)$ and $(x, y) \mapsto u = y - sx$ are mutually inverse isomorphisms between $\mathbb{A}^1$ and T_c . The same formulas, with s a coordinate on U , prove (b).

(c) Choose $u_0 \in U$ over the generic point. Its residue field L is a finite separable extension of $K = k(t)$, since $U \rightarrow Y$ is étale. An isomorphism over U induces an L -algebra isomorphism $L[x, y]/(F) \cong L[u]$. Let $f, g \in L[u]$ be the images of x and y , so $g^p = f + tf^p$. If f were constant, then g would be constant too, and the map would not be surjective; so $d = \deg f \geq 1$. Then $\deg(tf^p) = pd > d$, and comparing leading coefficients gives $\operatorname{lc}(g)^p = t \operatorname{lc}(f)^p$. Hence $t^{1/p} \in L$. But $t^{1/p}$ has the inseparable minimal polynomial $X^p - t$ over K , which contradicts the separability of L/K . $\square$

Remark 6.2. Proposition 6.1 is classical in substance. The generic fibre of T is a nontrivial purely inseparable form of the affine line, of the kind classified by Russell [Rus70]; flat families of affine lines were studied by Kambayashi–Miyanishi [KM78] and Kambayashi–Wright [KW85]. In particular $T \rightarrow Y$ is a flat affine morphism all of whose closed fibres are affine lines, but it is not locally trivial in the étale topology. This is consistent with [KW85], where every scheme-theoretic fibre, including the generic one, is assumed to be an affine line over its residue field (see [KR14, Rem. 5.2]), and with [KR14, Thm. 5.2(a)], which is stated over $\mathbb{C}$. A related phenomenon for smooth projective families in characteristic 2, caused by a non-reduced automorphism group scheme, is in Poczobut’s paper [Poc25, §8 of the arXiv version].

That in characteristic zero the base change of finite degree can be chosen étale is noted by Kraft and Russell themselves, in the accepted manuscript of [KR14] right after Theorem B of the introduction, and by Kaliman [Kal20, Rem. 3.12], who gives the argument by generic smoothness; the change from the étale conclusion of arXiv v1 of [KR14] to the finite-degree conclusion is also due to Kraft

and Russell. Kaliman's remark is attached to [Kal20, Thm. 3.3], which is stated in characteristic zero, and says that he does not know whether the morphism of finite degree can be made étale in positive characteristic. In the natural reading, for the finite-degree base changes of [KR14] and [Kal20, Thm. 2.5], the answer is negative, and it follows immediately from the classical facts above: by Proposition 6.1, over every algebraically closed field of positive characteristic, including uncountable ones and fields of infinite transcendence degree, such a base change cannot in general be made étale. In particular the étale conclusion of the proposition in [FRZ07] fails over $\overline{\mathbb{F}}_p$, and in every positive characteristic, for classical reasons, whereas the finite-degree conclusion of the accepted manuscript of [KR14] holds in every characteristic (for fields of infinite transcendence degree). We do not count Proposition 6.1 among the results of this note.

7 Remarks on $\overline{\mathbb{Q}}$

The $\overline{\mathbb{Q}}$ part of Problem 2 remains open: see [KR14, Rem. 2.2] and [Kal20, Conj. 4.2]. Kaliman proved the proper case [Kal20, Thms. 4.5 and 4.7], and by generic smoothness the base change can then be taken étale; so the projective phenomenon of Theorem 1.2 cannot occur over $\overline{\mathbb{Q}}$. The next proposition shows that the mechanism of Theorem 1.1 cannot produce a counterexample over $\overline{\mathbb{Q}}$ either, at least for families of the same shape. Its scope is limited to morphisms from punctured affine lines to tori; it says nothing about other constructions.

For $\gamma \in \mathrm{GL}_r(\mathbb{Z})$ and $x \in \mathbb{G}_m^r$ we write $\gamma \cdot x$ for the monomial action, as in Section 3.

Proposition 7.1. *Let k_0 be a number field, $a_1, \dots, a_s \in k_0$ distinct, $Y_0 = \mathbb{A}_{k_0}^1 \setminus \{a_1, \dots, a_s\}$, and let $c, c': Y_0 \rightarrow \mathbb{G}_m^r$ be morphisms. There is $y_0 \in Y_0(k_0)$ such that, if $c'(y_0) = \gamma \cdot c(y_0)$ for some $\gamma \in \mathrm{GL}_r(\mathbb{Z})$, then $c' = \gamma \cdot c$ as morphisms.*

Proof. The units of $\mathcal{O}(Y_0)$ are the elements $\lambda \prod_k (y - a_k)^{e_k}$ with $\lambda \in k_0^*$, so $c_j = \lambda_j \prod_k (y - a_k)^{A_{jk}}$ and $c'_j = \lambda'_j \prod_k (y - a_k)^{A'_{jk}}$ for integer matrices A, A' and constants $\lambda_j, \lambda'_j \in k_0^*$. Choose distinct primes $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of k_0 at which all λ_j, λ'_j and all $a_k - a_l$ ($k \neq l$) are units and all a_k are integral, and choose uniformisers ϖ_k at $\mathfrak{p}_k$. By the Chinese remainder theorem there is $y_0 \in k_0$ with $y_0 \equiv a_k + \varpi_k \pmod{\mathfrak{p}_k^2}$ for all k . Then $v_{\mathfrak{p}_m}(y_0 - a_l) = \delta_{ml}$ for all m, l ; in particular $y_0 \in Y_0(k_0)$. If $c'(y_0) = \gamma \cdot c(y_0)$, then for each i

$$\prod_k (y_0 - a_k)^{(A' - \gamma A)_{ik}} = \lambda_i'^{-1} \prod_j \lambda_j^{\gamma_{ij}},$$

and the right-hand side is a unit at every $\mathfrak{p}_m$. Taking $v_{\mathfrak{p}_m}$ gives $(A' - \gamma A)_{im} = 0$. Hence $A' = \gamma A$, then $\lambda'_i = \prod_j \lambda_j^{\gamma_{ij}}$, and so $c' = \gamma \cdot c$. $\square$

Corollary 7.2. *Let $c, c': Y \rightarrow \mathbb{G}_m^2$ be morphisms defined over $\overline{\mathbb{Q}}$ on $Y = \mathbb{A}_{\overline{\mathbb{Q}}}^1$ minus finitely many points, whose first coordinates never take the value 1. Let $S_c \subset \mathbb{G}_m^2 \times \mathbb{A}^3 \times Y$ be defined by $wz_i = f_i^{c(y)}$ for $i = 1, 2$, with f_i^φ as in Section 3, and define $S_{c'}$ in the same way, so that their fibres over y are $X_{c(y)}$ and $X_{c'(y)}$. If $S_{c,y} \cong S_{c',y}$ as $\overline{\mathbb{Q}}$ -schemes for all closed points y , then $S_c \cong S_{c'}$ over Y .*

Proof. All data are defined over a number field k_0 ; let y_0 be as in Proposition 7.1. By Proposition 3.2 with $\Omega = \overline{\mathbb{Q}}$, $c'(y_0) = \gamma \cdot c(y_0)$ for some $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, so $c' = \gamma \cdot c$. The Y -automorphism $\mu_\gamma \times \mathrm{id}_Y$ of $\mathbb{G}_m^2 \times Y$ maps the two disjoint sections $\mathbf{1}$ and c to $\mathbf{1}$ and c' . As in Lemma 3.1(a), the ideals generated by the right-hand sides of the equations are the ideals of these pairs of sections (the first coordinates of c, c' minus 1 are units on Y), and as in Lemma 3.1(b) and the first part of the proof of Proposition 3.2, μ_γ^* extended by $w \mapsto w$ induces a Y -isomorphism $S_c \cong S_{c'}$. $\square$

We also do not know whether the finite-degree form of the statement holds over algebraically closed fields of characteristic p and finite positive transcendence degree, such as the algebraic closure of $\mathbb{F}_p(t)$. Over such fields not every point of $\mathbb{G}_m^2$ has finite order, and our constructions do not apply.

Verification. The claims were checked as follows. All programs are in the release; they use only the Python standard library.

1. The programs that accompanied the original construction do the following (about seven minutes in total).
 - They build the matrix $M = BDB^{-1}$ of Lemma 2.1 for all 21,926 admissible y in 32 fields $\mathbb{F}_q$ with $q \leq 4096$ and $p \leq 13$, and verify $M \cdot c(y) = c(y)^p$ with schoolbook field arithmetic.
 - For all admissible y in 16 fields with $q \leq 289$ they compute Laurent polynomials a_{ij} with $f_i^\psi \circ \mu_M = \sum_j a_{ij} f_j^\varphi$ exactly, and likewise for M^{-1} . This is an explicit isomorphism $X_{c(y)} \rightarrow X_{c(y)^p}$, $(x, w, z) \mapsto (M \cdot x, w, a(x)z)$. For $q \leq 16$ the map is also checked point by point.
 - They carry out the construction of the proof of Theorem 1.2(i) for every nonzero point of $A(\mathbb{F}_q)$ with $q \in \{25, 125, 625, 169, 289\}$ (549,435 points), and check $M(a) = \pi(a)$ with elliptic-curve arithmetic. They also confirm $\#E(\mathbb{F}_q) = N(\pi^n - 1)$ and that $E(\mathbb{F}_q)$ is a cyclic $\mathbb{Z}[i]$ -module.
 - They verify the identities of Proposition 6.1 for $p \leq 13$.
2. Separate programs, with their own finite-field and elliptic-curve arithmetic and without the matrices of item 1, do the following (about one minute).
 - They test Lemma 2.1 on 20,000 random instances over $\mathbb{Z}$ and over $\mathbb{Z}[i]$.
 - They compute the orbits of $\mathrm{SL}_2(\mathbb{Z})$, generated by the two elementary matrices, on $(\mathbb{F}_q^*)^2$ for 40 fields with $q \leq 1024$ and $p \leq 53$. In each field the number of orbits equals the number of divisors of $q - 1$, as it must, and $c(y)^p$ lies in the orbit of $c(y)$ for all 7,836 admissible y . They also check the bound of Remark 4.1 for both signs.
 - They compute the orbits of $\mathrm{SL}_2(\mathbb{Z}[i])$, generated by four elementary matrices, on $E(\mathbb{F}_q)^2$ for 17 fields with $q \leq 841$. In each field the number of orbits equals the number of ideals of $\mathbb{Z}[i]$ containing $\pi^n - 1$, and $\pi(a)$ lies in the orbit of a for every nonzero point a . These are 1,220,015 points, among them 1,189,434 points over the six fields with $n \geq 2$, where π acts nontrivially.
 - They verify the identities of Proposition 6.1 for $p \leq 23$.
3. A further AI-assisted verification run wrote its own programs, again with separate finite-field, Gaussian-integer and elliptic-curve arithmetic (about four minutes).
 - They test Lemma 2.1, and its simpler version with $\mu' = \mu$, on 10,849 instances over $\mathbb{Z}$ and 15,258 over $\mathbb{Z}[i]$, many of them chosen adversarially.
 - They build M for all 47,698 admissible y in 63 fields with $q \leq 8192$ and $p \leq 97$, and verify $M \cdot c(y) = c(y)^p$ by schoolbook exponentiation, without logarithm tables.
 - They compute the orbits of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z})$ on $(\mathbb{F}_q^*)^2$ for 29 fields with $q \leq 512$ by breadth-first search from the generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$), with the same results as in item 2. For $q = 2^7$ they check that no $M \in \mathrm{GL}_2(\mathbb{Z})$ with entries of absolute value at most 30 satisfies $M \cdot c(y) = c(y)^{\pm 2}$ for any admissible y , as Remark 4.1 predicts.

- For 18 fields with $q \leq 4913$ they check that $E(\mathbb{F}_q) \cong \mathbb{Z}[i]/(\pi^n - 1)$, and they verify $M(a) = \pi(a)$ with elliptic-curve arithmetic for 210,662 points a : all nonzero points of $A(\mathbb{F}_q)$ for $q \leq 289$, and for larger q random samples together with all points of the curve C of Remark 5.5(2).
 - They verify the identities of Proposition 6.1 for $p \leq 31$, and test Proposition 7.1 on an example over $\mathbb{Q}$.
4. Independent verification runs, which were AI-assisted, checked the proofs, and re-ran the programs of items 1 and 2 with identical output.

The computations only illustrate the theorems for finitely many fields; the proofs do not depend on them.

Scope and priority. Theorems 1.1 and 1.2 answer the $\overline{\mathbb{F}}_p$ part of Problem 2 of [FRZ07, p. 69] negatively, for the proposition as stated there (closed fibres, étale base change), for the finite-degree version of [KR14, Kal20], and for affine morphisms (every p) as well as for smooth projective morphisms ($p \equiv 1 \pmod{4}$). For the étale form alone, a negative answer over $\overline{\mathbb{F}}_p$, and in every positive characteristic, already follows from the classical Proposition 6.1; Theorems 1.1 and 1.2 do not rest on it. The $\overline{\mathbb{Q}}$ part remains open [KR14, Rem. 2.2], [Kal20, Conj. 4.2]; Proposition 7.1 only excludes the mechanism of Theorem 1.1 for families over punctured affine lines.

We read the Oberwolfach report [FRZ07], the arXiv version and the accepted manuscript of [KR14], the arXiv versions of [Kal20], [Poc25] and [BBG16], Lenstra’s paper [Len96] and Milne’s notes [Mil08]. We did not see the publisher’s version of [KR14] or the journal versions of [Kal20], [Poc25] and [BBG16], which we quote from the versions just listed. We also did not see the papers of Russell [Rus70], Kambayashi–Miyanishi [KM78], Kambayashi–Wright [KW85], Deuring [Deu41] and Kaliman–Zaidenberg [KZ99], or the lecture notes [BNM07] of the Hanoi school ICPA2006, which according to [KR14, §2] contain a preliminary write-up of the Generic Equivalence Theorem. These works are cited for background and attribution only; the hypothesis of [KW85] is quoted from [KR14, Rem. 5.2], and Deuring’s theorem is used in the form given in [Sil09].

We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, including the works citing [KR14], [Kal20] and [Poc25], and ran three general web searches. Apart from the classical inseparable phenomenon behind Proposition 6.1, which already refutes the étale form of the Oberwolfach proposition over $\overline{\mathbb{F}}_p$ (indeed in every positive characteristic), we found no earlier counterexample over $\overline{\mathbb{F}}_p$ to the finite-degree form of [KR14, Kal20], and none over $\overline{\mathbb{Q}}$. Poczobut [Poc25] proves an étale-local triviality theorem for smooth projective families all of whose fibres are isomorphic to one fixed variety (with reduced automorphism group scheme) over fields of infinite transcendence degree, and says that he does not know how to remove the hypothesis on the transcendence degree; our families are not of that kind, since their fibres vary. Bogomolov, Böhning and Graf von Bothmer ask [BBG16, Question 4.2 of the arXiv version] whether their birational analogue holds over all algebraically closed fields; our families are birationally trivial, so they do not bear on that question. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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