

The Travelling Polytope Constant of a Product of Trees: An Answer to a Question of Genevois

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Abstract

The travelling polytope constant $\text{TP}(X)$ of a graph X compares the travelling salesman problem through a finite set F of vertices with the shortest way to sweep F by convex hulls which grow or shrink by one adjacent vertex at a time. A. Genevois introduced it in his thesis (2017), as the travelling salesman constant of a $\text{CAT}(0)$ cube complex; he proved that it equals $d/(2d-1)$ for $\mathbb{Z}^d$ and lies in $[1/2, 2/3] \cup \{1\}$ for unbounded median graphs of bounded degree, and asked for its value for a product of two trees. At the Oberwolfach workshop Median Geometry and Applications (2026) he asked again to compute it for products of (regular) trees. We show that $\text{TP}(T_1 \times \cdots \times T_d) = d'/(2d'-1)$, where $d' \geq 1$ is the number of factors of unbounded diameter, without any assumption on the degrees, and treat the remaining cases, in which all factors have finite diameter. So a product of d regular trees of degree at least 2 has constant $d/(2d-1)$, as Genevois's guess for cocompact cube complexes predicts. The upper bound is Genevois's grid example for $\mathbb{Z}^d$, placed on geodesic segments of the factors. The new part is the lower bound: by Genevois's formula, the shortest sweep has twice as many steps as there are hyperplanes separating two of the points, and Euler tours of the projected hulls give a 1-Lipschitz map from a box onto the hull of the points, through which the classical strip bound for the travelling salesman problem in a cube applies. The values obtained were known values of the constant, so the note gives no new value; which values the constant takes on median graphs remains open. This is an unrefereed note.

1 Introduction

1.1 The problem

Let X be a connected graph with vertex set $V(X)$ and graph metric d . A set of vertices is *convex* if it contains every vertex of every geodesic between two of its vertices; $\text{hull}(S)$ denotes the smallest convex set containing S , and a *polytope* is the convex hull of a finite nonempty set of vertices. For a set Q of vertices, ∂Q is the set of the vertices outside Q that are adjacent to a vertex of Q . A *path of polytopes* of length k is a sequence $P_0, \dots, P_k$ of polytopes such that, for every $j < k$, one of P_j, P_{j+1} equals $\text{hull}(Q \cup \{u\})$, where Q is the other one and $u \in \partial Q$. The following definitions are Genevois's [Gen26a].

Definition 1.1. Let $x, y \in V(X)$, let $F \subset V(X)$ be finite and $N = |F|$.

- (a) $\text{TS}(x, F, y)$ is the length of a shortest walk from x to y which passes through every vertex of F ; equivalently, the minimum of $d(x, f_1) + \sum_{j < N} d(f_j, f_{j+1}) + d(f_N, y)$ over all orderings $f_1, \dots, f_N$ of F , and $d(x, y)$ if $F = \emptyset$.

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- (b) $\text{TP}(x, F, y)$ is the least length of a path of polytopes $P_0 = \{x\}, P_1, \dots, P_k = \{y\}$ with $F \subset P_0 \cup \dots \cup P_k$.
- (c) An exponent $\alpha > 0$ is *admissible* for X if there is $C > 0$ such that

$$\text{TP}(x, F, y) + |F| \geq C (\text{TS}(x, F, y) + |F|)^\alpha \quad (1)$$

for all $x, y \in V(X)$ and all finite $F \subset V(X)$. The *travelling polytope constant* $\text{TP}(X)$ is the supremum of the admissible exponents.

By Lemma 2.6(a) below, $\text{TP}(x, F, y)$ is finite. We read the points which [Gen26a] leaves implicit as follows. A path through F in (a) is a walk (a path without repeated vertices through three leaves of a star does not exist). The set ∂Q is the outer vertex boundary: this is Genevois's notation $\partial_X Q$ in [Gen26b, Lemma 3.4 and Claim 3.5], and the steps of a path of polytopes are then the edges of his graph of finite convex subcomplexes [Gen17, Section 9.1, Remark 9.5]. If u were an arbitrary vertex, then $\text{TP}(x, F, y) \leq 2|F| + 2$ always, and $\text{TP}(X)$ would vanish for every unbounded graph, contrary to the values listed in [Gen26a]. A step may be read in either direction, F may be empty and $x = y$ is allowed; we use $0^\alpha = 0$.

Products of graphs are Cartesian products: two tuples of vertices are adjacent if they differ in exactly one coordinate, where they are adjacent. Trees may be infinite and need not be locally finite; a tree has unbounded diameter if and only if it contains geodesic segments of every length. For $k \geq 2$, $\mathbb{T}_k$ is the k -regular tree; $\mathbb{T}_2$ is the two-way infinite path, and a product of d copies of it is the grid $\mathbb{Z}^d$.

1.2 Known results and the questions

Genevois introduced the constant in his thesis, where it enters a lower bound for the ℓ^p -compression of wreath products [Gen17, Section 9.4, Theorem 9.37], as the *travelling salesman constant* $\text{TS}(X)$ of a CAT(0) cube complex X [Gen17, Definition 9.38]. (Median graphs are the 1-skeleta of CAT(0) cube complexes [Che00], and the hyperplanes of the complex [Sag95] correspond to the Djoković–Winkler classes of edges [Djo73, Win84].) There $\text{TC}(x, F, y)$ is twice the number of hyperplanes of $\text{hull}(F \cup \{x, y\})$, and $\text{TS}(X)$ is the supremum of the α with $\text{TC}(x, F, y) + |F| \geq C \cdot \text{TS}(x, F, y)^\alpha$ for some $C > 0$ and all x, y, F . By [Gen17, Remark 9.39], $\text{TC} = \text{TP}$ (we reprove this for products of trees in Proposition 2.4), and by Lemma 2.6(b),(c) the two normalisations give the same supremum; so $\text{TP}(X)$ is Genevois's $\text{TS}(X)$ for every infinite median graph X . It appears again, with $\text{TC}/2$ in place of TC , in [Gen22, Section 4.2].

The known values are Genevois's: $\text{TS}(X) \in [1/2, 1]$ for every unbounded CAT(0) cube complex [Gen17, Lemma 9.44] (the bound $1/2$ holds in all median spaces [Gen22, Lemma 4.7]); for uniformly locally finite X , $\text{TS}(X) = 1$ if and only if X is hyperbolic [Gen17, Proposition 9.45 and Lemma 9.46]; $\text{TS}(\mathbb{R}^d) = d/(2d - 1)$ for the standard cubulation of $\mathbb{R}^d$, whose 1-skeleton is $\mathbb{Z}^d$ [Gen17, Lemma 9.50]; $\text{TS}(X) \leq 2/3$ if X is not hyperbolic, and $\text{TS}(X) = 1/2$ if X is infinite-dimensional [Gen17, Corollaries 9.52 and 9.53]. They are listed with these references in [Gen22, Section 4.2], and without references in [Gen26a].

On p. 248 of [Gen17], Genevois asks, as a guess, whether $\text{TS}(X) = d/(2d - 1)$ for every cocompact CAT(0) cube complex X , where d is the largest integer such that X contains an isometrically embedded subcomplex isomorphic to $\mathbb{R}^d$ (Question 12.16). As easier test cases he asks “What is $\text{TS}(\mathbb{T}_1 \times \mathbb{T}_2)$ where $\mathbb{T}_1, \mathbb{T}_2$ are two trees?” (Question 12.17), and whether $\text{TS}(X) > 1/2$ for every finite-dimensional X (Question 12.18). In the problem session of the Oberwolfach workshop *Median Geometry and Applications* (February 2026) he asked which values TP takes on median graphs, and suggested, as an instance, to compute the travelling polytope constant of a product of (regular) trees [Gen26a]. This is record OWR-14299911-020 of the corpus [ula26].

1.3 Result

Theorem 1.2. *Let $d \geq 1$, let $T_1, \dots, T_d$ be trees, let $X = T_1 \times \dots \times T_d$, and let d' be the number of factors of unbounded diameter.*

- (a) *If $d' \geq 1$, then $\text{TP}(X) = d'/(2d' - 1)$. In particular $\text{TP}(X) = d/(2d - 1)$ if every factor has unbounded diameter, and $\text{TP}(X) = 1$ if exactly one factor has unbounded diameter.*
- (b) *If $d' = 0$ and X is infinite, then $\text{TP}(X) = 1$.*
- (c) *If X is finite, then every $\alpha > 0$ is admissible, so $\text{TP}(X) = +\infty$.*

The proof gives explicit constants: if $T_1, \dots, T_{d'}$ are the factors of unbounded diameter and D is the sum of the diameters of the others, then $\text{TP} + |F| \geq (\text{TS} + |F|)^{d'/(2d'-1)} / (4c_{d'} + 2D + 1)$ for all x, y, F , where $c_k = (k+1)2^{k-1} + k$ (Proposition 3.3). Part (c) is a degenerate case; the known values listed in [Gen26a] concern unbounded graphs.

Corollary 1.3. (a) *For all integers $k_1, \dots, k_d \geq 2$, $\text{TP}(\mathbb{T}_{k_1} \times \dots \times \mathbb{T}_{k_d}) = d/(2d - 1)$, and the guess of [Gen17, Question 12.16] holds for these cocompact $\text{CAT}(0)$ cube complexes.*

- (b) ([Gen17, Question 12.17].) *Let T_1, T_2 be trees. Then $\text{TS}(T_1 \times T_2) = 2/3$ if both have unbounded diameter, and $\text{TS}(T_1 \times T_2) = 1$ if exactly one of them has unbounded diameter, or if both have finite diameter and one of them is infinite. If both are finite, every exponent is admissible.*
- (c) *Every product X of finitely many trees satisfies $\text{TP}(X) > 1/2$; so the answer to [Gen17, Question 12.18] is yes for these complexes.*

1.4 What is new, what is not, and what stays open

The constant, the formula $\text{TP}(x, F, y) = 2 \text{hyp}(F \cup \{x, y\})$ of Proposition 2.4, the inequality $\text{TP} \leq 2 \text{TS}$ of Lemma 2.6(a), the values and bounds quoted above, and the sets F which give the upper bound in Proposition 4.1 are Genevois's [Gen17, Gen22]; the questions are his [Gen17, Gen26a]. We reprove what we use, for products of trees. Lemma 3.2 is a version of the strip method of Few [Few55]; in the proof of [Gen17, Lemma 9.50] a decomposition into cuboids plays the same role. For $d = 1$, Theorem 1.2(a) also follows from the argument of [Gen17, Proposition 9.45], as a finite tree has as many hyperplanes as edges. What the note adds is the lower bound of Proposition 3.3 for products of trees: Euler tours of the projected hulls give a 1-Lipschitz map from a box with sides at most $2 \text{hyp}(F \cup \{x, y\})$ onto the hull of $F \cup \{x, y\}$ (Lemma 3.1), which reduces a product of trees to the grid; and with it the values of Theorem 1.2 and Corollary 1.3. We did not find these statements in print (see ‘‘Scope and priority’’ in Section 7). The values $d/(2d - 1)$ and 1 were known values of the constant, for $\mathbb{Z}^d$ and for hyperbolic graphs; so the note gives no new value in answer to the main question of [Gen26a], which values TP takes on median graphs. This question is open, and so is the behaviour of the constant under products of other median graphs (Section 6).

2 Products of trees, polytopes and hyperplanes

In this section and the next two, $T_1, \dots, T_d$ are trees and $X = T_1 \times \dots \times T_d$. We write $v = (v_1, \dots, v_d)$ for the vertices of X , and $\pi_i(v) = v_i$. In a tree T , $[a, b]$ denotes the vertex set of the unique geodesic from a to b ; for an edge e of T , the graph $T - e$ has two components, the two *sides* of e . A *subtree* is a set of vertices which spans a connected subgraph.

Every step of a walk in X changes one coordinate, along an edge of the corresponding factor. So the i -th coordinates of a walk form, after deleting repetitions, a walk in T_i , while changing the coordinates one at a time along geodesics of the factors gives a walk of length $\sum_i d_{T_i}(v_i, w_i)$. Hence

$$d_X(v, w) = \sum_{i=1}^d d_{T_i}(v_i, w_i), \quad (2)$$

and a walk in X is a geodesic if and only if each of its projections, after deleting repetitions, is a geodesic of its factor.

Lemma 2.1. (a) *In a tree T , a set of vertices is convex if and only if it is a subtree. If S is finite and nonempty and $s_0 \in S$, then $\text{hull}_T(S) = \bigcup_{t \in S} [s_0, t]$; this set is finite, and both endpoints of an edge e lie in $\text{hull}_T(S)$ if and only if both sides of e meet S .*

(b) *A nonempty set $K \subset V(X)$ is convex if and only if $K = A_1 \times \cdots \times A_d$ with subtrees A_i of T_i . For every nonempty $S \subset V(X)$, $\text{hull}_X(S) = \prod_i \text{hull}_{T_i}(\pi_i(S))$. In particular, the polytopes of X are the products of finite subtrees of the factors.*

Proof. (a) A path without repeated vertices between two vertices of a tree is the geodesic between them. So a subtree contains the geodesic between any two of its vertices and is convex; conversely, a convex set contains a geodesic between any two of its vertices, so it spans a connected subgraph. The set $U = \bigcup_{t \in S} [s_0, t]$ is a finite subtree containing S , and every subtree containing S contains each $[s_0, t]$; so $U = \text{hull}_T(S)$. If S lies in one side of e , this side is a subtree containing S , hence contains $\text{hull}_T(S)$, and the other endpoint of e is not in $\text{hull}_T(S)$. If $s, t \in S$ lie in different sides, then $[s, t]$ contains both endpoints of e .

(b) Let K be convex, $v, w \in K$ and $1 \leq i \leq d$. The walk from v to w which first changes the i -th coordinate along $[v_i, w_i]$, and then the other coordinates one at a time along geodesics, is a geodesic by (2); so it lies in K , and the vertex obtained from v by replacing v_i with w_i belongs to K . Starting from a vertex of K and replacing its coordinates one at a time, we obtain every tuple $(u_1, \dots, u_d)$ with $u_i \in \pi_i(K)$; hence $K = \prod_i \pi_i(K)$. Each $\pi_i(K)$ is convex: if $a, a' \in \pi_i(K)$ and $v \in K$, the vertices obtained from v by replacing v_i with a and with a' lie in K , and the geodesic of T_i from a to a' , placed in the i -th coordinate, is a geodesic of X between them. Conversely, if the A_i are subtrees, every geodesic of X between two vertices of $\prod_i A_i$ has geodesic projections, which lie in the A_i ; so $\prod_i A_i$ is convex. Finally, $\prod_i \text{hull}_{T_i}(\pi_i(S))$ is convex and contains S , and every convex $K = \prod_i K_i$ containing S satisfies $K_i \supset \pi_i(S)$, hence $K_i \supset \text{hull}_{T_i}(\pi_i(S))$. The last statement follows from (a), since a product of finite subtrees is its own hull. $\square$

Definition 2.2. A *hyperplane* of X is a pair (i, e) , where $1 \leq i \leq d$ and e is an edge of T_i . It *separates* two vertices v, w of X if v_i and w_i lie in different sides of e . For a finite set $S \subset V(X)$, $\text{hyp}(S)$ is the number of hyperplanes which separate two points of S .

In the median graph X , the edges parallel to the i -th factor whose projection is e form a Djoković–Winkler class [Djo73, Win84], and (i, e) corresponds to a hyperplane of the cube complex; we shall not use this. By Lemma 2.1(a), the hyperplanes separating two points of S are the pairs (i, e) such that e is an edge of the finite tree $H_i = \text{hull}_{T_i}(\pi_i(S))$; therefore

$$\text{hyp}(S) = \sum_{i=1}^d (|H_i| - 1). \quad (3)$$

Lemma 2.3. *Let $Q = A_1 \times \cdots \times A_d$ be a polytope of X , and let $u \in \partial Q$ be adjacent to $q \in Q$. Then u and q differ in exactly one coordinate i , $u_i \notin A_i$, and*

$$\text{hull}_X(Q \cup \{u\}) = A_1 \times \cdots \times A_{i-1} \times (A_i \cup \{u_i\}) \times A_{i+1} \times \cdots \times A_d,$$

where u_i is a leaf of the subtree $A_i \cup \{u_i\}$. So each step of a path of polytopes adds a leaf to one factor of the smaller of the two polytopes.

Proof. Adjacent vertices differ in exactly one coordinate i , and $u_j = q_j \in A_j$ for $j \neq i$; since $u \notin Q$, we have $u_i \notin A_i$. By Lemma 2.1(b), $\text{hull}_X(Q \cup \{u\})$ is the product of the A_j , $j \neq i$, and of $\text{hull}_{T_i}(A_i \cup \{u_i\})$; and $A_i \cup \{u_i\}$ is a subtree, as u_i is adjacent to $q_i \in A_i$, so it is its own hull. If u_i had a second neighbour in A_i , a path in A_i between its two neighbours would close a cycle in T_i . $\square$

Proposition 2.4 (Genevois [Gen17, Remark 9.39]). *For all $x, y \in V(X)$ and every finite $F \subset V(X)$, $\text{TP}(x, F, y) = 2 \text{hyp}(F \cup \{x, y\})$.*

Proof. Let $S = F \cup \{x, y\}$ and $H_i = \text{hull}_{T_i}(\pi_i(S))$; by Lemma 2.1, $\text{hull}_X(S) = \prod_i H_i$.

Upper bound. Put $Q_0 = \{x\}$. As long as $Q_j \neq \text{hull}_X(S)$, choose a vertex $u \in \text{hull}_X(S) \setminus Q_j$ adjacent to Q_j , which exists because $\prod_i H_i$ spans a connected subgraph, and put $Q_{j+1} = \text{hull}_X(Q_j \cup \{u\})$; it is contained in the convex set $\text{hull}_X(S)$. By Lemma 2.3, $\sum_i |\pi_i(Q_j)|$ increases by one at each step, from d to $\sum_i |H_i|$. So this path of polytopes reaches $\text{hull}_X(S)$ after exactly $\text{hyp}(S)$ steps, by (3). The same construction from $\{y\}$, read backwards, leads from $\text{hull}_X(S)$ to $\{y\}$ in $\text{hyp}(S)$ steps. Together they form a path of polytopes of length $2 \text{hyp}(S)$ which contains $\text{hull}_X(S) \supset F$.

Lower bound. Let $P_0 = \{x\}, \dots, P_k = \{y\}$ be a path of polytopes with $F \subset \bigcup_j P_j$, and write $P_j = A_1^{(j)} \times \cdots \times A_d^{(j)}$ (Lemma 2.1). For a hyperplane (i, e) and $0 \leq j \leq k$, the *status* of (i, e) at time j is “crossing” if both endpoints of e lie in $A_i^{(j)}$; otherwise the subtree $A_i^{(j)}$ lies in one side of e , and the status is this side. We claim that from time j to time $j+1$ exactly one hyperplane changes its status, from a side to crossing or from crossing to a side. By Lemma 2.3, for some i , the larger of P_j, P_{j+1} is obtained from the smaller one by adding to its i -th factor A a vertex $w \notin A$ adjacent to some $w' \in A$. The status of $(i, \{w, w'\})$ changes between the side containing w' and crossing. The status of (i', e) with $i' \neq i$ does not change. If $e \neq \{w, w'\}$ is an edge of T_i , then either both endpoints of e lie in A , hence in $A \cup \{w\}$, or A lies in one side of e ; in the latter case w , which is joined to w' by an edge other than e , lies in the same side, and so does $A \cup \{w\}$. This proves the claim.

Now let (i, e) separate two points $p, p' \in S$, and let $\Sigma \neq \Sigma'$ be the sides of e which contain p_i and p'_i . Every point of S lies in some P_j ; say $p \in P_s$ and $p' \in P_t$. The status of (i, e) is a side at the times 0 and k , because singletons cross no hyperplane; it is Σ or crossing at time s , and Σ' or crossing at time t . If it never changed, it would be one side at all times, equal to Σ and to Σ' . If it changed exactly once, the change would lead from a side to crossing, and the status at time k would be crossing. Both are impossible, so the status of (i, e) changes at least twice. As each step changes exactly one status, $k \geq 2 \text{hyp}(S)$. $\square$

Remark 2.5. Genevois states Proposition 2.4 for all CAT(0) cube complexes [Gen17, Remark 9.39]; see also [Gen26b, Lemma 3.4, Lemma 3.9, Proposition 3.8], and [Gen22, Section 3.2] for median spaces. The proof above extends to median graphs: if Q is convex and $u \in \partial Q$ is adjacent to $q \in Q$, then Q lies in the halfspace of the hyperplane J dual to qu which contains q (otherwise every halfspace containing Q would contain u , and u would lie in Q , the intersection of these halfspaces), so only the status of J changes.

Lemma 2.6. *Let X be a connected graph.*

(a) $\text{TP}(x, F, y) \leq 2 \text{TS}(x, F, y)$ for all x, y, F .

- (b) If X is infinite, then for every $\alpha > 1$ and every $C > 0$ there are x, y, F with $\text{TP}(x, F, y) + |F| < C \cdot \text{TS}(x, F, y)^\alpha$. In particular, no $\alpha > 1$ is admissible, and $\text{TP}(X) \leq 1$.
- (c) Let $0 < \alpha \leq 1$. There is $C > 0$ such that $\text{TP}(x, F, y) + |F| \geq C(\text{TS}(x, F, y) + |F|)^\alpha$ for all x, y, F if and only if there is $C' > 0$ such that $\text{TP}(x, F, y) + |F| \geq C' \cdot \text{TS}(x, F, y)^\alpha$ for all x, y, F .

Inequality (a) is stated as clear in [Gen26a]; for TC in place of TP it is [Gen17, Claim 9.43].

Proof. (a) Let $x = w_0, w_1, \dots, w_m = y$ be a walk through F with $m = \text{TS}(x, F, y)$, and put $Q_j = \text{hull}(\{w_0, \dots, w_j\})$. Then $Q_j = \text{hull}(Q_{j-1} \cup \{w_j\})$; either $w_j \in Q_{j-1}$ and $Q_j = Q_{j-1}$, or $w_j \in \partial Q_{j-1}$, because w_j is adjacent to w_{j-1} . Deleting repetitions, we obtain a path of polytopes of length at most m from $\{x\}$ to $W = \text{hull}(\{w_0, \dots, w_m\})$, which contains F . The same construction along the reversed walk gives a path of length at most m from $\{y\}$ to W ; reversing it and concatenating, we obtain a path of polytopes of length at most $2m$ from $\{x\}$ to $\{y\}$ which contains W .

(b) Let $N \geq 1$, choose a vertex x and a set F of N vertices other than x , and put $y = x$. A walk from x through F meets $N+1$ distinct vertices, so $\text{TS}(x, F, x) \geq N$, and $\text{TP}(x, F, x) + N \leq 3 \text{TS}(x, F, x)$ by (a). Since $C \cdot \text{TS}^\alpha > 3 \text{TS}$ as soon as $\text{TS}^{\alpha-1} > 3/C$, it suffices to take N large. As $\text{TS}^\alpha \leq (\text{TS} + |F|)^\alpha$, no $\alpha > 1$ is admissible.

(c) The first inequality implies the second with $C' = C$, as $\text{TS}^\alpha \leq (\text{TS} + |F|)^\alpha$. Conversely, $t \mapsto t^\alpha$ is concave and vanishes at 0, so $(\text{TS} + N)^\alpha \leq \text{TS}^\alpha + N^\alpha \leq \text{TS}^\alpha + N$ for $N = |F|$; the second inequality then gives $(\text{TS} + N)^\alpha \leq (1/C' + 1)(\text{TP} + N)$. $\square$

Since $\text{TP} = \text{TC}$ (Proposition 2.4; [Gen17, Remark 9.39] for all median graphs), Lemma 2.6(b),(c) shows that $\text{TP}(X)$ is the constant $\text{TS}(X)$ of [Gen17, Definition 9.38] for every infinite median graph X , and also the constant of [Gen22, Section 4.2], where $\text{TC}/2$ replaces TC , which only changes C .

3 The lower bound

Lemma 3.1. *Let H be a finite tree with $h \geq 1$ vertices. There is a surjective map $\varphi: \{0, 1, \dots, 2h-2\} \rightarrow V(H)$ such that $\varphi(s)$ and $\varphi(s+1)$ are equal or adjacent for $0 \leq s < 2h-2$. Consequently, $d_H(\varphi(s), \varphi(t)) \leq |s-t|$ for all s, t .*

Proof. Induction on h . For $h = 1$, φ maps 0 to the vertex of H . For $h \geq 2$, let ℓ be a leaf of H with neighbour w , let $\varphi': \{0, \dots, 2h-4\} \rightarrow V(H) \setminus \{\ell\}$ be a map as in the lemma for the tree $H - \ell$, and choose s_0 with $\varphi'(s_0) = w$. Put $\varphi(s) = \varphi'(s)$ for $s \leq s_0$, $\varphi(s_0 + 1) = \ell$, $\varphi(s_0 + 2) = w$, and $\varphi(s) = \varphi'(s-2)$ for $s \geq s_0 + 3$. The last statement follows from the triangle inequality. $\square$

The map φ is an Euler tour of H ; the corresponding walk is also used in the proof of [Gen17, Proposition 9.45].

Lemma 3.2 (strip method). *Let $k \geq 1$ and $L \geq 0$, and let $a, b, p_1, \dots, p_n$ ($n \geq 1$) be points of the cube $[0, L]^k \subset \mathbb{R}^k$, not necessarily distinct. There is a permutation π of $\{1, \dots, n\}$ such that, with $q_0 = a$, $q_j = p_{\pi(j)}$ for $1 \leq j \leq n$ and $q_{n+1} = b$,*

$$\sum_{j=0}^n \|q_{j+1} - q_j\|_1 \leq c_k (n^{1-1/k} + 1)L, \quad c_k = (k+1)2^{k-1} + k.$$

This is a version of the strip method of Few [Few55]; we include the proof, as we need the ℓ^1 -metric and an explicit constant ($c_1 = 3$, $c_2 = 8$, $c_3 = 19$).

Proof. For $k = 1$, visit the points in increasing order; the length is $|a - p_{\min}| + (p_{\max} - p_{\min}) + |p_{\max} - b| \leq 3L$. Let $k \geq 2$ and $m = \lceil n^{1/k} \rceil$. Divide $[0, L]$ into the intervals $I_\kappa = [\kappa L/m, (\kappa + 1)L/m]$ for $0 \leq \kappa \leq m - 2$ and $I_{m-1} = [(m - 1)L/m, L]$, and the cube $[0, L]^{k-1}$ of the last $k - 1$ coordinates into the $M = m^{k-1}$ products of such intervals, the *cells*. Assign each point p_j to the cell which contains its last $k - 1$ coordinates, and order the points assigned to each cell by their first coordinate. The walk starts at a , visits the points of the first nonempty cell in this order, then those of the next nonempty cell, and so on, and ends at b . Two consecutive points p, p' of the same cell satisfy $\|p' - p\|_1 \leq (p'_1 - p_1) + (k - 1)L/m$, and the first terms telescope; so a cell with n_κ points contributes at most $L + n_\kappa(k - 1)L/m$. The at most $M + 1$ remaining steps (from a , between cells, and to b) have length at most kL each. The total is at most

$$ML + \frac{(k - 1)nL}{m} + (M + 1)kL = (k + 1)ML + kL + \frac{(k - 1)nL}{m}.$$

Since $n^{1/k} \leq m \leq n^{1/k} + 1 \leq 2n^{1/k}$, we have $M \leq 2^{k-1}n^{1-1/k}$ and $n/m \leq n^{1-1/k}$, so the total is at most $((k + 1)2^{k-1} + k - 1)n^{1-1/k}L + kL \leq c_k(n^{1-1/k} + 1)L$. $\square$

Proposition 3.3. *Let $1 \leq k \leq d$, suppose that $T_{k+1}, \dots, T_d$ have finite diameter, and let $D = \sum_{j>k} \text{diam}(T_j)$, with $D = 0$ if $k = d$. Then for all $x, y \in V(X)$ and every finite $F \subset V(X)$,*

$$\text{TP}(x, F, y) + |F| \geq \frac{1}{4c_k + 2D + 1} (\text{TS}(x, F, y) + |F|)^{k/(2k-1)}.$$

In particular, $k/(2k - 1)$ is admissible. For every product of d trees, $d/(2d - 1)$ is admissible with $C = 1/(4c_d + 1)$; for instance $C = 1/13$ for $d = 1$ and $C = 1/33$ for $d = 2$.

Proof. Let $N = |F|$, $S = F \cup \{x, y\}$, $H_i = \text{hull}_{T_i}(\pi_i(S))$, $h_i = |H_i|$, $\sigma = \sum_{i \leq k} (h_i - 1)$, and $\alpha = k/(2k - 1) \in (1/2, 1]$. By Proposition 2.4 and (3), $\text{TP}(x, F, y) = 2 \sum_{i \leq d} (h_i - 1) \geq 2\sigma$.

If $N = 0$, then $\text{TS}(x, \emptyset, y) = d_X(x, y)$ and $H_i = [x_i, y_i]$, so $\text{TP}(x, \emptyset, y) = 2d_X(x, y)$ by (2). As $t \geq t^\alpha$ for $t \in \{0\} \cup [1, \infty)$, the inequality holds, even with the constant 1.

Let $N \geq 1$. For $i \leq k$ let $\varphi_i: \{0, \dots, 2h_i - 2\} \rightarrow H_i$ be a map as in Lemma 3.1, let $\ell = \max_{i \leq k} (2h_i - 2) \leq 2\sigma$, and let $\Phi = (\varphi_1, \dots, \varphi_k)$ on the box $B = \prod_{i \leq k} \{0, \dots, 2h_i - 2\} \subset [0, \ell]^k$. For every $p \in S$ choose $p' \in B$ with $\Phi(p') = (p_1, \dots, p_k)$; this is possible because $p_i \in H_i$. By (2) and Lemma 3.1, for $p, q \in S$,

$$d_X(p, q) = \sum_{i \leq k} d_{T_i}(p_i, q_i) + \sum_{j>k} d_{T_j}(p_j, q_j) \leq \|p' - q'\|_1 + D. \quad (4)$$

Lemma 3.2, applied to $a = x'$, $b = y'$ and the N points f' ($f \in F$), gives an ordering $f_1, \dots, f_N$ of F for which the sum of the ℓ^1 -distances between consecutive points of $x', f'_1, \dots, f'_N, y'$ is at most $c_k(N^{1-1/k} + 1)\ell$. We concatenate geodesics of X from x to f_1 , from f_1 to f_2 , $\dots$, from f_N to y , and use (4) for each of the $N + 1$ pieces. As $1 \leq N^{1-1/k}$ and $\ell \leq 2\sigma$, we obtain

$$\text{TS}(x, F, y) \leq c_k(N^{1-1/k} + 1)\ell + (N + 1)D \leq 4c_k N^{1-1/k} \sigma + 2ND.$$

The function $t \mapsto t^\alpha$ is subadditive on $[0, \infty)$; moreover $(4c_k)^\alpha \leq 4c_k$, $(1 - 1/k)\alpha = (k - 1)/(2k - 1)$, and $t^\alpha \leq t$ for $t \geq 1$. Hence

$$(\text{TS} + N)^\alpha \leq (4c_k N^{1-1/k} \sigma)^\alpha + ((2D + 1)N)^\alpha \leq 4c_k N^{\frac{k-1}{2k-1}} \sigma^{\frac{k}{2k-1}} + (2D + 1)N.$$

The exponents $\frac{k-1}{2k-1}$ and $\frac{k}{2k-1}$ are nonnegative with sum 1, so the weighted inequality between the arithmetic and the geometric mean gives $N^{\frac{k-1}{2k-1}} \sigma^{\frac{k}{2k-1}} \leq N + \sigma$; for $k = 1$ this reads $\sigma \leq N + \sigma$. Therefore

$$(\text{TS} + N)^\alpha \leq 4c_k(N + \sigma) + (2D + 1)N \leq (4c_k + 2D + 1)(N + \sigma) \leq (4c_k + 2D + 1)(\text{TP} + N),$$

because $\sigma \leq \text{TP}/2$. $\square$

For $T_i = \mathbb{Z}$ the hulls H_i are intervals and Φ can be replaced by a translation, as in [Gen17, Lemma 9.50]; the Euler tours deal with branching factors.

4 The upper bound

Proposition 4.1. *Let $1 \leq k \leq d$, and suppose that $T_1, \dots, T_k$ have unbounded diameter. For every $n \geq 2$ there are $x \in V(X)$ and $F \subset V(X)$ with $|F| = n^k$ such that*

$$\text{TP}(x, F, x) + |F| \leq (2k + 1)n^k \quad \text{and} \quad \text{TS}(x, F, x) + |F| \geq n^{2k-1}.$$

Consequently, no $\alpha > k/(2k - 1)$ is admissible.

The sets F are, up to the range of the indices, those of the proof of [Gen17, Lemma 9.50], placed on geodesic segments of the factors.

Proof. Let $L = n^{k-1}$. For $i \leq k$, the tree T_i contains a geodesic segment $\gamma_i: \{0, 1, \dots, (n-1)L\} \rightarrow V(T_i)$, that is, $d_{T_i}(\gamma_i(s), \gamma_i(t)) = |s - t|$; for $j > k$, fix a vertex b_j of T_j . Let

$$\begin{aligned} F &= \{(\gamma_1(m_1L), \dots, \gamma_k(m_kL), b_{k+1}, \dots, b_d) : 0 \leq m_1, \dots, m_k \leq n-1\}, \\ x &= (\gamma_1(0), \dots, \gamma_k(0), b_{k+1}, \dots, b_d). \end{aligned}$$

Then $x \in F$ and $|F| = n^k$. Two distinct points of F differ in some coordinate $i \leq k$, where their distance is $|m_i - m'_i|L \geq L$. If we list the points of F in the order of their first visits by a walk from x through F , consecutive points are at distance at least L ; so the walk has length at least $(n^k - 1)L$, and

$$\text{TS}(x, F, x) + |F| \geq (n^k - 1)n^{k-1} + n^k \geq n^{2k-1}.$$

For $i \leq k$, the hull of $\pi_i(F)$ in T_i is the geodesic segment $\gamma_i(\{0, \dots, (n-1)L\})$, which has $(n-1)L$ edges; for $j > k$ it is $\{b_j\}$. By Proposition 2.4 and (3), $\text{TP}(x, F, x) = 2k(n-1)L \leq 2kn^k$, so $\text{TP}(x, F, x) + |F| \leq (2k + 1)n^k$. (Explicitly, a shortest path of polytopes grows the box $\prod_{i \leq k} \gamma_i(\{0, \dots, a_i\}) \times \prod_{j > k} \{b_j\}$ one unit at a time up to $a_1 = \dots = a_k = (n-1)L$ and shrinks it back.)

If α and C satisfy (1), then $(2k + 1)n^k \geq Cn^{(2k-1)\alpha}$ for all $n \geq 2$, which forces $(2k - 1)\alpha \leq k$. $\square$

5 Proof of the theorem

Proof of Theorem 1.2. Permuting the factors gives an isomorphic graph, so we may assume that $T_1, \dots, T_{d'}$ are the factors of unbounded diameter.

(a) Let $d' \geq 1$. By Proposition 3.3 with $k = d'$, the exponent $d'/(2d' - 1)$ is admissible, and by Proposition 4.1 with $k = d'$ no larger exponent is. So $\text{TP}(X) = d'/(2d' - 1)$, which is 1 for $d' = 1$.

(b) Let $d' = 0$ and let X be infinite; then X has finite diameter $D = \sum_i \text{diam}(T_i) \geq 1$. If $N = |F| \geq 1$, visiting the points of F in any order gives $\text{TS}(x, F, y) \leq (N + 1)D \leq 2ND$, so that $\text{TS} + N \leq (2D + 1)N \leq (2D + 1)(\text{TP} + N)$. If $F = \emptyset$, then $\text{TS}(x, \emptyset, y) = d_X(x, y) \leq \text{TP}(x, \emptyset, y)$ by Proposition 2.4. So $\alpha = 1$ is admissible with $C = 1/(2D + 1)$, and $\text{TP}(X) \leq 1$ by Lemma 2.6(b).

(c) Let X be finite; there are finitely many triples (x, F, y) . If $\text{TS}(x, F, y) + |F| > 0$, then $\text{TP}(x, F, y) + |F| > 0$: either $F \neq \emptyset$, or $F = \emptyset$ and $\text{TP}(x, \emptyset, y) = 2d_X(x, y) = 2\text{TS}(x, \emptyset, y) > 0$. So, for each $\alpha > 0$, the minimum of $(\text{TP} + |F|)/(\text{TS} + |F|)^\alpha$ over the triples with $\text{TS} + |F| > 0$ is a positive constant C for which (1) holds. $\square$

Proof of Corollary 1.3. (a) $\mathbb{T}_k$ has unbounded diameter for $k \geq 2$, so Theorem 1.2(a) applies with $d' = d$. By (2), a product of d two-way infinite geodesics is an isometrically embedded subcomplex isomorphic to $\mathbb{R}^d$, and there is none isomorphic to $\mathbb{R}^{d+1}$, as the complex has no $(d+1)$ -cube. It is cocompact: $\text{Aut}(\mathbb{T}_k)$ is transitive on the vertices and on the edges of $\mathbb{T}_k$, and a cube of the product is a product of vertices and edges of the factors, so the product of the automorphism groups has at most 2^d orbits of cubes. (b) For infinite X , $\text{TS}(X) = \text{TP}(X)$ by Proposition 2.4 and Lemma 2.6(b),(c), and Theorem 1.2(a),(b) applies. For finite X , the argument of Theorem 1.2(c) applies to both normalisations. (c) The values $d'/(2d' - 1)$ with $d' \geq 1$, 1 and $+\infty$ exceed $1/2$. $\square$

Remark 5.1. For $X = T \times K$, with T of unbounded diameter and K of finite diameter D , a direct walk gives a better constant. Let $H_1 = \text{hull}_T(\pi_1(S))$, $S = F \cup \{x, y\}$. Follow a closed walk of length $2(|H_1| - 1)$ from x_1 through all vertices of H_1 , keeping the second coordinate fixed, except that, when the first coordinate reaches f_1 for a point $f \in F$ not yet visited, the second coordinate moves to f_2 ; at the end, go to y . The moves in K cost at most D per point and D at the end, and the last move in T costs at most $|H_1| - 1$. So, by Proposition 2.4, $\text{TS}(x, F, y) \leq 3(|H_1| - 1) + D(|F| + 1) \leq \frac{3}{2} \text{TP}(x, F, y) + D(|F| + 1)$; for $K = K_2$, a single edge, $\text{TS} \leq \frac{3}{2} \text{TP} + |F| + 1$.

6 Products of other median graphs

Let $X = X_1 \times \cdots \times X_d$ be a product of median graphs. By the definition of the Djoković–Winkler relation [Djo73, Win84] and (2), which holds for all products of connected graphs, two edges of X are related if and only if they are parallel to the same factor and their projections to it are related. So the hyperplanes of X are those of the factors, and a hyperplane coming from X_i separates two vertices if and only if it separates their i -th coordinates.

Upper bound. If every X_i has unbounded diameter, then $\text{TP}(X) \leq d/(2d - 1)$. Indeed, define x and F as in the proof of Proposition 4.1, with geodesic segments γ_i of length $(n - 1)L$ in the X_i . In a median graph, the hyperplanes separating two vertices are those crossed by a geodesic between them, each exactly once; so the hyperplanes of X_i separating two of the points $\gamma_i(mL)$ are the $(n - 1)L$ hyperplanes separating $\gamma_i(0)$ and $\gamma_i((n - 1)L)$, and $\text{TP}(x, F, x) = 2d(n - 1)L$ by Genevois’s formula (Remark 2.5). The bound for TS is unchanged. No convexity of the segments γ_i is needed.

Lower bound. Besides the formula, the proof of Proposition 3.3 uses, in each factor, a walk through all vertices of the hull of a finite set whose length is at most a constant times the number of hyperplanes of this hull (Lemma 3.1, with the constant 2). For uniformly locally finite hyperbolic median graphs, such a bound follows from [Gen17, Lemma 9.46], as in the proof of [Gen17, Proposition 9.45]. We have not written out the resulting extension of Theorem 1.2 to products of such graphs, and we do not claim it. For factors which are not hyperbolic this property fails (in $\mathbb{Z}^2$, the hull of $\{(0, 0), (L, L)\}$ has $(L + 1)^2$ vertices and $2L$ hyperplanes).

7 Computations and reproducibility

No proof in this note depends on a computation. The package contains the program of this note, `check_note.py` (standard library, about four seconds; all 354,889 checks pass), which checks the finite and exact parts of the proofs and tests the statements on small instances: the constants c_k , the exponents and the integer inequalities $n \leq m^k \leq 2^k n$ used in Lemma 3.2 and Proposition 3.3, and the counting inequalities of Proposition 4.1; Lemmas 2.1 and 2.3, Proposition 2.4 and Lemma 2.6(a) by a literal search from the definitions on eight products of trees (115,394 triples); Lemma 3.1 on all 201 trees with at most 10 vertices; Lemma 3.2 on 1,000 point sets; the walk of the proof of Proposition 3.3 and each of its bounds on 1,200 random triples in products of up to four trees, with and without

factors of finite diameter; and Proposition 4.1 on nine instances, with exact TS. The package also contains the programs with which the result was first obtained (`original/`) and those of the three independent verification runs described below (`verification_run_A/`, `verification_run_B/` and `independent_run_2/`), with their recorded outputs; all programs were run again from the archive of the package, and the outputs agree with the recorded ones up to dates and running times.

Verification. The result was first obtained, for products of trees of unbounded diameter, with the proofs of Propositions 2.4, 3.3 and 4.1. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first re-derived every lemma and inequality, read Problem 7 in the page images of [ACFN26] and the relevant sections of [Gen17, Gen22, Gen26b], showed that the constant of [Gen26a] is Genevois’s $TS(X)$, proved the cases of two trees of which at most one is unbounded, and checked Lemmas 2.1 and 2.3 and Proposition 2.4 by a literal search on six products of trees (17,760 triples). The second checked the proofs line by line, searched the literature, checked the bibliographic data, and tested Proposition 2.4 by a literal search on 875,033 triples in 20 median graphs, with negative controls (a wrong hull, wrong formulas, other readings of ∂Q , graphs which are not median), and every inequality in the proof of Proposition 3.3 on 2,100 random triples. Neither found an error in these proofs. Both required corrections of the account of the literature, which had missed [Gen17, Gen22, Gen26b] and described the known facts as unpublished; the first also found that a side remark on products of other median graphs was wrong (corrected in Section 6) and that a walk in a remark on $T \times K_2$ did not end at y (corrected in Remark 5.1). All required corrections were made. The case of factors of finite diameter (Proposition 3.3 with $D > 0$, Proposition 4.1 with $k < d$) was then written for this note.

A third independent verification run, also AI-assisted, examined the text of this note. It read Problem 7 and the problem session of [ACFN26] and the cited parts of [Gen17, Gen22, Gen26b] again, compared every quoted statement with these texts, re-derived all proofs line by line, the case of factors of finite diameter first, and repeated the literature search. Its two programs, written before it read those of the package, check Lemmas 2.1 and 2.3, Proposition 2.4 and Lemma 2.6(a) by a literal search on 465,402 triples in 15 products of one to three trees, Lemma 2.6(a) also in seven graphs which are not median, the upper bound of Section 6 on geodesic segments which are not convex, and Proposition 4.1 with $k < d$ by a literal search in four products; and, in exact arithmetic, Lemma 3.1 on all 18,249 labelled trees with at most 7 vertices, Lemma 3.2 on 6,000 point sets, every step of the proof of Proposition 3.3 on 6,000 random triples in products of up to five trees, with factors of finite diameter ($D > 0$) in two thirds of them, Proposition 4.1 with $k < d$ in ten cases with exact TS, Theorem 1.2(b) and the walk of Remark 5.1. It found no error. It required that its examination of the case of factors of finite diameter and its searches be recorded here, a clearer definition of d' in the abstract, three more precise statements about the sources, and corrections of the package and of its metadata; these were made. It also ran all programs of the package again from its archive.

Scope and priority. The note answers the question of [Gen26a] for products of trees, and [Gen17, Question 12.17]. The constant, Proposition 2.4, Lemma 2.6(a), the known values and bounds, and the sets F of Proposition 4.1 are Genevois’s [Gen17, Gen22], and the questions are his [Gen17, Gen26a]; Lemma 3.2 is the strip method of [Few55]. The note contributes the lower bound of Proposition 3.3 for products of trees, through Lemma 3.1, and the resulting values. These were known values of the constant, so the note gives no new value in answer to the main question of [Gen26a], which values the constant takes on median graphs; this question, and the behaviour of the constant under products of other median graphs, remain open. We claim no priority.

No printed answer to the question was found in the sources searched: the report [ACFN26] (the problem session, pp. 528–536, and Genevois’s abstract, pp. 507–510, read; the rest searched); [Gen17]

(Sections 9.1, 9.4 and Chapter 12 read, the rest searched), [Gen22] (Sections 3.2, 4.2), [Gen26b] (Section 3.1) and Genevois’s preprints arXiv:2605.20419 and arXiv:2608.24725 (searched); the list of his arXiv preprints (the latest of 25 August 2026), keyword searches in arXiv, zbMATH Open and Crossref, the works citing [ACFN26] in OpenAlex, Semantic Scholar and Crossref (none), the works citing [Gen17] in Semantic Scholar (none on the constant), and web searches, on 17 October 2026; the arXiv interface, OpenAlex and Semantic Scholar were partly unavailable because of rate limits. The closest work found, on a different question, is [EM23], on orderings of metric spaces which are efficient for the travelling salesman problem. A search that finds nothing is not a proof of novelty. We used the open-access file of [ACFN26] and the arXiv versions v1 of [Gen17], v2 of [Gen22] and v1 of [Gen26b], whose numbering we use (the journal version of [Gen22] was not compared); of [EM23] we read the abstract. [Few55, Djo73, Win84, Sag95, Che00] are cited for classical facts and were not read for this note (the publisher’s file of [Few55] could not be accessed). Of the results quoted from [Gen17, Gen22, Gen26b], only those reproved here were checked.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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