

On a Question of Freslon, Gerontogiannis and Skalski about Quantum Isometries of Cuntz–Krieger Algebras

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Abstract

For a primitive 0–1 matrix A , Freslon, Gerontogiannis and Skalski computed the quantum isometry group $\mathbb{G}_A^\infty$ of the log-Laplacian spectral triple on the Cuntz–Krieger algebra $\mathcal{O}_A$. They showed that the quantum automorphism group $\text{QAut}(A)$ is a quantum subgroup of a larger quantum group $\mathbb{G}_A^1$ and asked if it is one of $\mathbb{G}_A^\infty$ when $\text{QAut}(A) \neq \text{Aut}(A)$. We show that the answer is negative in general. For the Cuntz algebras $\mathcal{O}_N$ and for $A = J_N - I_N$, with $N \geq 4$, the quantum permutation group $S_N^+ = \text{QAut}(A)$ is not a quantum subgroup of $\mathbb{G}_A^\infty$. For every primitive A with $\text{QAut}(A) \neq \text{Aut}(A)$, the canonical inclusion of $\text{QAut}(A)$ in $\mathbb{G}_A^1$ does not extend to $\mathbb{G}_A^\infty$, and the natural action of $\text{QAut}(A)$ on $\mathcal{O}_A$ is not isometric for this spectral triple. Quantum groups such as S_M^+ , $M \geq 4$, cannot sit in $\mathbb{G}_A^\infty$ through an embedding that fixes a vertex. On the other hand, for two explicit primitive 4×4 matrices, the non-classical $\text{QAut}(A)$ (the dual of the infinite dihedral group, and the hyperoctahedral quantum group H_2^+) is a quantum subgroup of $\mathbb{G}_A^\infty$. For general A it remains open whether S_M^+ , $M \geq 4$, can sit in $\mathbb{G}_A^\infty$ through an embedding that fixes no vertex. This is an unrefereed note.

1 Introduction

Let $N \geq 2$ and let $A \in M_N(\{0, 1\})$ be *primitive*: some power of A has only positive entries. This is the standing assumption of Freslon, Gerontogiannis and Skalski [FGS26]. The Cuntz–Krieger algebra $\mathcal{O}_A$ [CK80] carries a spectral triple built from the log-Laplacian of the underlying topological Markov chain, with classical isometry group $\mathbb{T} \wr \text{Aut}(A)$ [GGM25]. In [FGS26] its quantum isometry group is computed: it is the compact matrix quantum group $\mathbb{G}_A^\infty$, the last member of a family $\mathbb{G}_A^\ell$, $\ell \in \mathbb{N} \cup \{\infty\}$ (Definition 2.1), in which $\mathbb{G}_A^{\ell'}$ is a quantum subgroup of $\mathbb{G}_A^\ell$ for $\ell \leq \ell'$. By [FGS26, Proposition 5.11], the quantum automorphism group $\text{QAut}(A)$ of the graph of A [Ban05] is a quantum subgroup of $\mathbb{G}_A^1$, and $\text{Aut}(A)$ is one of $\mathbb{G}_A^\infty$; just before that proposition the authors write that it is unclear whether $\mathbb{G}_A^\infty$ contains $\text{QAut}(A)$. In Skalski’s abstract in Oberwolfach Report 4/2026 [Ska26, p. 230] this is Question 1: “does the same hold for $\mathbb{G}_A^\infty$, when $\text{QAut}(A) \neq \text{Aut}(A)$?” It is record OWR-14299906-003 of the UnsolvedMath dataset [ula26].

Quantum subgroups are defined by surjective quantum group homomorphisms (Section 2). The question has an abstract and a canonical reading: (Q_{abs}) is there a surjective quantum group homomorphism $C(\mathbb{G}_A^\infty) \rightarrow C(\text{QAut}(A))$? (Q_{can}) does the homomorphism $\kappa: C(\mathbb{G}_A^1) \rightarrow C(\text{QAut}(A))$, $u_{ij} \mapsto p_{ij}$, of [FGS26, Proposition 5.11] factor through the quotient $C(\mathbb{G}_A^\infty)$? A positive answer to (Q_{can}) gives one to (Q_{abs}) .

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Let J_N be the all-ones $N \times N$ matrix (written $\mathbf{1}_N$ in [FGS26]); $\mathcal{O}_{J_N}$ is the Cuntz algebra $\mathcal{O}_N$ [Cun77]. Let $d_A(a, c)$ be the length of a shortest directed path from a to c , and $L_A = 1 + \max_{a \neq c} d_A(a, c) \leq N$. Thus $L_A = 2$ for J_N and $J_N - I_N$.

Theorem 1.1. *Let A be primitive with $\text{QAut}(A) \neq \text{Aut}(A)$. There is no $*$ -homomorphism $C(\mathbb{G}_A^{L_A}) \rightarrow C(\text{QAut}(A))$ with $u_{ij} \mapsto p_{ij}$. Hence κ does not factor through $C(\mathbb{G}_A^\ell)$ for any $\ell \geq L_A$, in particular not through $C(\mathbb{G}_A^\infty)$, and the action $S_i \mapsto \sum_j p_{ij} \otimes S_j$ of $\text{QAut}(A)$ on $\mathcal{O}_A$ is not D -isometric.*

The abstract reading needs a property of the target. A compact quantum group $\mathbb{H}$ has *property (G)* if its polynomial algebra $\text{Pol}(\mathbb{H})$ is not commutative and the coefficients of every finite-dimensional unitary representation of $\mathbb{H}$ other than the identity matrix generate $\text{Pol}(\mathbb{H})$ as a $*$ -algebra. By Banica's fusion rules [Ban99], S_M^+ has property (G) for $M \geq 4$ (Lemma 2.5).

Theorem 1.2. *Let A be primitive, let $\mathbb{H}$ have property (G), and let $\ell \in \mathbb{N} \cup \{\infty\}$. There is no surjective quantum group homomorphism $\pi: C(\mathbb{G}_A^\ell) \rightarrow C(\mathbb{H})$ in each of the following cases:*

- (a) $\ell \geq 2$ and $A \in \{J_N, J_N - I_N\}$;
- (b) $\ell \geq L_A + 1$ and π fixes a vertex: $\pi(p_{zz}) = 1$ for some z ;
- (c) $\ell \geq L_A$ and each $\pi(u_{ij})$ is a scalar multiple of $\pi(p_{ij})$.

Corollary 1.3. *Let $N \geq 4$ and $A \in \{J_N, J_N - I_N\}$. Then $\text{QAut}(A) = S_N^+ \neq S_N = \text{Aut}(A)$, and no S_M^+ with $M \geq 4$ is a quantum subgroup of $\mathbb{G}_A^\ell$ for any $\ell \geq 2$. In particular, S_N^+ is a quantum subgroup of $\mathbb{G}_A^1$ but not of the quantum isometry group $\mathbb{G}_A^\infty$.*

By contrast, $S_N^+ \times S_N^+$ is a subquotient of $\mathbb{G}_{J_N}^\infty$ [FGS26, Proposition 5.12]. Level two does not suffice in general: for the primitive 5×5 matrix A_{hub} of Example 4.2, $\text{QAut}(A_{\text{hub}}) = S_4^+$ is a quantum subgroup of $\mathbb{G}_{A_{\text{hub}}}^2$ but not of $\mathbb{G}_{A_{\text{hub}}}^3$.

Proposition 1.4. *The matrices $A_D = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ and $A_E = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ are primitive. $\text{QAut}(A_D)$ is the dual of the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$, and $\text{QAut}(A_E)$ is the quantum automorphism group of the 4-cycle, which is the hyperoctahedral quantum group H_2^+ of [BBC07]. Neither is classical, and each is a quantum subgroup of $\mathbb{G}_{A_D}^\infty$, respectively $\mathbb{G}_{A_E}^\infty$.*

So (Q_{can}) always has a negative answer, while the answer to (Q_{abs}) depends on A . For general A it remains open whether a quantum group with property (G), such as S_M^+ , can be a quantum subgroup of $\mathbb{G}_A^\infty$ through an embedding that fixes no vertex (Problem 4.3). The arguments are short: Theorem 1.1 rests on writing $p_{ab}p_{cd}$ as a sum of partial isometries with orthogonal ranges and sources (Theorem 3.1), and Theorem 1.2 combines the commutation relations forced by the partial isometry condition with property (G).

2 Preliminaries

Conventions. We follow [FGS26, Wor87]; references to [FGS26] use the numbering of its arXiv version 3. A compact quantum group $\mathbb{G}$ is a unital C^* -algebra $C(\mathbb{G})$ with a coassociative unital $*$ -homomorphism $\Phi: C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ satisfying the usual density conditions. A matrix v over $C(\mathbb{G})$ is a representation if $\Phi(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$. Coefficients of finite-dimensional unitary representations span the dense Hopf $*$ -algebra $\text{Pol}(\mathbb{G})$, with counit $\varepsilon(v_{ij}) = \delta_{ij}$ and invertible antipode $\mathcal{S}(v_{ij}) = v_{ji}^*$; such v decompose into irreducible ones. A group-like unitary is a unitary g with $\Phi(g) = g \otimes g$. A quantum group homomorphism is a unital $*$ -homomorphism $\pi: C(\mathbb{G}) \rightarrow C(\mathbb{H})$ with

$\Phi_{\mathbb{H}}\pi = (\pi \otimes \pi)\Phi_{\mathbb{G}}$; if π is surjective, $\mathbb{H}$ is a quantum subgroup of $\mathbb{G}$ [FGS26, Section 5.3]. The paper [FGS26] also uses surjective Hopf $*$ -homomorphisms $\text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{H})$; our non-existence proofs only use relations between images of generators, so they apply to these verbatim, and the maps of Section 5 restrict to such homomorphisms. A *magic unitary* is a matrix of projections whose rows and columns sum to 1. $C(\text{QAut}(A))$ is the universal C^* -algebra generated by the entries of a magic unitary p with $Ap = pA$, with $\Delta(p_{ij}) = \sum_k p_{ik} \otimes p_{kj}$ [Ban05], [FGS26, Definition 5.9]. Its abelianization is $C(\text{Aut}(A))$, where $\text{Aut}(A) = \{\sigma \in S_N : A_{\sigma(i)\sigma(j)} = A_{ij}\}$; we write $\text{QAut}(A) \neq \text{Aut}(A)$ if $C(\text{QAut}(A))$ is not commutative. Magic unitaries commute with J_N and I_N , so $\text{QAut}(J_N) = \text{QAut}(J_N - I_N) = S_N^+$, Wang's quantum permutation group, which is non-commutative for $N \geq 4$ [Wan98].

A word $\alpha \in [N]^n$ is admissible if $A_{\alpha_i \alpha_{i+1}} = 1$ for $i < n$; V_A^n is the set of admissible words of length n , and $\vec{u}$ the right Perron–Frobenius eigenvector of A .

Definition 2.1 ([FGS26, Definition 3.6]). For $\ell \in \mathbb{N} \cup \{\infty\}$, $C(\mathbb{G}_A^\ell)$ is the universal C^* -algebra generated by partial isometries u_{ij} , $i, j \in [N]$, with range projections $p_{ij} = u_{ij}u_{ij}^*$ and source projections $q_{ij} = u_{ij}^*u_{ij}$, such that (I) p and q are magic unitaries with $p\vec{u} = \vec{u}1 = q\vec{u}$; (II) $Ap = qA$; and (III) for $n \leq \ell$ and $\alpha, \beta \in V_A^n$, $u_{\alpha, \beta} = u_{\alpha_1 \beta_1} \cdots u_{\alpha_n \beta_n}$ is a partial isometry. The coproduct is $\Phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ [FGS26, Proposition 3.12].

For $\ell \leq \ell'$ the identity on generators is a surjective quantum group homomorphism $C(\mathbb{G}_A^\ell) \rightarrow C(\mathbb{G}_A^{\ell'})$, so quantum subgroups of $\mathbb{G}_A^\ell$ are quantum subgroups of $\mathbb{G}_A^{\ell'}$. The algebra $\mathcal{O}_A$ is generated by $S_1, \dots, S_N$ with $\sum_j S_j S_j^* = 1$ and $S_i^* S_k = \delta_{ik} \sum_j A_{ij} S_j S_j^*$ [CK80], and $\varphi_A(S_i) = \sum_j u_{ij} \otimes S_j$ is an action of $\mathbb{G}_A^\ell$ on $\mathcal{O}_A$ [FGS26, Proposition 4.2]. For $\ell = \infty$ it is D -isometric in the sense of [FGS26, Definition 3.4] and final [FGS26, Theorem 4.5]: every D -isometric action φ of a compact quantum group $\mathbb{G}$ on $\mathcal{O}_A$ is $(\rho \otimes \text{id}) \circ \varphi_A$ for a unique quantum group homomorphism $\rho: C(\mathbb{G}_A^\infty) \rightarrow C(\mathbb{G})$. Since the p_{ij} are projections, $u \mapsto p$ gives the surjective quantum group homomorphism $\kappa: C(\mathbb{G}_A^1) \rightarrow C(\text{QAut}(A))$ [FGS26, Proposition 5.11].

Lemma 2.2. *In a C^* -algebra: (a) for partial isometries V, W , the product VW is a partial isometry if and only if V^*V commutes with WW^* ; (b) for projections P, Q , the product PQ is a partial isometry if and only if $PQ = QP$; (c) a sum of partial isometries with pairwise orthogonal ranges and pairwise orthogonal sources is a partial isometry; (d) projections with sum 1 are pairwise orthogonal.*

Proof. (b) If PQ is a partial isometry, QPQ is a projection, so $X = (1 - Q)PQ$ has $X^*X = QPQ - (QPQ)^2 = 0$; thus $PQ = QPQ = QP$. (a) Put $E = V^*V$, $F = WW^*$. If $EF = FE$, then $(W^*EW)^2 = W^*EFEW = W^*EW$, so $(VW)^*VW$ is a projection. Conversely, if W^*EW is a projection, so is $W(W^*EW)W^* = FEF = (EF)^*EF$; then EF is a partial isometry and $EF = FE$ by (b). (This criterion is also used in [FGS26].) (c) $V_i^*V_j = V_i^*(V_iV_i^*)(V_jV_j^*)V_j = 0$ for $i \neq j$, so $(\sum V_i)^* \sum V_j = \sum V_i^*V_i$ is a projection. (d) $E_i = E_i(\sum_j E_j)E_i$ gives $\sum_{j \neq i} (E_j E_i)^* E_j E_i = 0$. $\square$

Lemma 2.3. (a) *In $C(\mathbb{G}_A^1)$, $u_{ik}u_{il}^* = u_{ik}^*u_{il} = u_{ki}u_{li}^* = u_{ki}^*u_{li} = 0$ for $k \neq l$; hence $\Phi(p_{ij}) = \sum_k p_{ik} \otimes p_{kj}$ and $\Phi(q_{ij}) = \sum_k q_{ik} \otimes q_{kj}$. Moreover $(A_{ac} - A_{bd})q_{ab}p_{cd} = 0$ for all a, b, c, d . (b) In $C(\mathbb{G}_A^\ell)$ with $\ell \geq 2$, q_{ab} commutes with p_{cd} if $A_{ac} = 1$ or $A_{bd} = 1$. If $A \in \{J_N, J_N - I_N\}$, every q_{ab} commutes with every p_{cd} . (c) If p is a magic unitary with $Ap = pA$, then $(A_{ik} - A_{jl})p_{ij}p_{kl} = 0$ for all i, j, k, l .*

Proof. (a) See [FGS26, Remark 3.8 and the proof of Lemma 3.10]: for instance $u_{ik}u_{il}^* = u_{ik}q_{ik}q_{il}u_{il}^* = 0$; expand $\Phi(u_{ij})\Phi(u_{ij})^*$ and $\Phi(u_{ij})^*\Phi(u_{ij})$; and multiply the (a, d) entry of $Ap = pA$ by q_{ab} on the left and by p_{cd} on the right. (b) If $A_{ac} = A_{bd} = 1$, then $u_{ab}u_{cd}$ is a partial isometry by (III), and Lemma 2.2(a) applies. If exactly one of them is 1, then $q_{ab}p_{cd} = 0 = p_{cd}q_{ab}$ by (a). For $J_N - I_N$ the only remaining case is $a = c$, $b = d$, and there $p_{ab} = 1 - \sum_{y \neq b} p_{ay}$, where each p_{ay} commutes with q_{ab} as $A_{by} = 1$. (c) Multiply the (i, l) entry of $Ap = pA$ by p_{ij} on the left and by p_{kl} on the right. $\square$

Lemma 2.4. (a) Let P be a magic unitary representation of a compact quantum group $\mathbb{H}$. Then $i \sim j \iff P_{ij} \neq 0$ is an equivalence relation on $[N]$, $P_{ij} = 0$ if exactly one of i, j lies in a class O , and $P_O = (P_{ij})_{i,j \in O}$ is a magic unitary representation, not the identity matrix if $|O| \geq 2$. (b) If P, Q are magic unitaries with $AP = QA$, and r_i, c_i are the out- and in-degrees of i , then $(r_i - r_j)Q_{ij} = 0$ and $(c_i - c_j)P_{ij} = 0$.

Proof. (a) This is the orbit relation of quantum permutation groups [LMR20]: $\varepsilon(P_{ii}) = 1$; $\mathcal{S}(P_{ij}) = P_{ji}$ with $\mathcal{S}$ injective; and if $P_{ik} \neq 0 \neq P_{kj}$, then $(P_{ik} \otimes P_{kj})\Phi(P_{ij}) = P_{ik} \otimes P_{kj} \neq 0$. The rest follows as rows and columns sum to 1. (b) Applying $AP = QA$ to the all-ones vector gives $\sum_k Q_{ik}r_k = r_i$; multiply by Q_{ij} on the left. Multiplying $AP = QA$ on the left by the all-ones row gives $\sum_k c_k P_{kj} = c_j$; multiply by P_{ij} on the left. $\square$

Lemma 2.5. (a) If $\mathbb{H}$ has property (G), its only group-like unitary is 1. (b) For $N \geq 4$, S_N^+ has property (G).

Proof. (a) A group-like unitary $g \neq 1$ would generate $\text{Pol}(\mathbb{H})$, which would be commutative. (b) $\text{Pol}(S_N^+)$ is non-commutative [Wan98] and is generated by the entries of the fundamental magic unitary p [Wor87]. By Banica's theorem [Ban99, Theorem 4.1] for $B = \mathbb{C}^N$, the irreducible representations of S_N^+ can be labelled $r_0 = 1, r_1, r_2, \dots$ with $r_k \otimes r_l \cong r_{|k-l|} \oplus r_{|k-l|+1} \oplus \dots \oplus r_{k+l}$, and $p \cong r_0 \oplus r_1$ (in [Ban99], r_1 is the complement of the trivial subrepresentation of p). A finite-dimensional unitary representation v other than the identity matrix contains some r_k , $k \geq 1$, so the coefficients of r_k are linear combinations of those of v ; products of two of them span the coefficients of $r_k \otimes r_k \supseteq r_1$. Hence all p_{ij} lie in the $*$ -algebra generated by the coefficients of v , which contains $1 = \sum_j v_{ij}v_{ij}^*$. $\square$

3 The canonical inclusion does not extend

Theorem 3.1. Let $A \in M_N(\{0, 1\})$ be irreducible and p a magic unitary in a unital C^* -algebra with $Ap = pA$. If $T_{\alpha, \beta} = p_{\alpha_1 \beta_1} \cdots p_{\alpha_n \beta_n}$ is a partial isometry for all $n \leq L_A$ and $\alpha, \beta \in V_A^n$, then the p_{ij} commute pairwise.

Proof. If $a = c$, then $p_{ab}p_{cd} = \delta_{bd}p_{ab} = p_{cd}p_{ab}$. Let $a \neq c$, $m = d_A(a, c) \leq L_A - 1$, and fix $\alpha = (a, x_1, \dots, x_{m-1}, c) \in V_A^{m+1}$. Inserting $1 = \sum_y p_{x_i y}$ for each intermediate letter gives

$$p_{ab}p_{cd} = \sum_{y \in [N]^{m-1}} T_{\alpha, (b, y, d)}. \quad (1)$$

If $\beta \in [N]^{m+1}$ is not admissible, then $T_{\alpha, \beta} = 0$ by Lemma 2.3(c), since α is admissible; for admissible β it is a partial isometry. Summing over all β and using that the rows of p sum to 1 (starting from β_{m+1} , respectively β_1), we get $\sum_{\beta} T_{\alpha, \beta} T_{\alpha, \beta}^* = 1 = \sum_{\beta} T_{\alpha, \beta}^* T_{\alpha, \beta}$. By Lemma 2.2(d) the ranges of the $T_{\alpha, \beta}$, $\beta \in V_A^{m+1}$, are pairwise orthogonal, and so are their sources. By (1) and Lemma 2.2(c), $p_{ab}p_{cd}$ is a partial isometry, and Lemma 2.2(b) gives $p_{ab}p_{cd} = p_{cd}p_{ab}$. $\square$

Proof of Theorem 1.1. A $*$ -homomorphism $\sigma: C(\mathbb{G}_A^{L_A}) \rightarrow C(\text{QAut}(A))$ with $\sigma(u_{ij}) = p_{ij}$ maps $u_{\alpha, \beta}$ to $T_{\alpha, \beta}$, so by Theorem 3.1 (primitive matrices are irreducible) the generators of $C(\text{QAut}(A))$ would commute. For $\ell \geq L_A$, $C(\mathbb{G}_A^\ell)$ is a quotient of $C(\mathbb{G}_A^{L_A})$. Next, $\psi(S_i) = \sum_j p_{ij} \otimes S_j$ equals $(\kappa \otimes \text{id}) \circ \varphi_A$ for the action φ_A of $\mathbb{G}_A^1$; as κ is a surjective quantum group homomorphism, ψ is an action in the sense of [FGS26, Definition 3.2]. If ψ were D -isometric, [FGS26, Theorem 4.5] would give $\rho: C(\mathbb{G}_A^\infty) \rightarrow C(\text{QAut}(A))$ with $\sum_j (\rho(u_{ij}) - p_{ij}) \otimes S_j = 0$ for all i . Multiplying by $1 \otimes S_k^*$ on the left gives $(\rho(u_{ik}) - p_{ik}) \otimes S_k^* S_k = 0$, where $S_k^* S_k = \sum_m A_{km} S_m S_m^* \neq 0$: A has no zero row, and $S_m S_m^*$ corresponds to the non-empty cylinder set $C(m)$ under $C(\Sigma_A) \subset \mathcal{O}_A$ [FGS26, Section 2]. A slice map gives $\rho(u_{ik}) = p_{ik}$, so ρ would factor κ through $C(\mathbb{G}_A^\infty)$, which is excluded. $\square$

Remark 3.2. The bound L_A in Theorem 3.1 cannot be lowered in general. Let A_7 be the matrix of the graph on $[7]$ with edges $t \rightarrow 5$, $7 \rightarrow t$ for $t \in T = \{1, 2, 3, 4\}$, and $5 \rightarrow 5$, $5 \rightarrow 6$, $6 \rightarrow 7$; it is primitive with $L_{A_7} = 5$, and $p = q \oplus I_3$ commutes with A_7 for the fundamental magic unitary q of S_4^+ . Letters from T in an admissible word are at least four positions apart, and $p_{\alpha_i \beta_i} = 0$ unless $\alpha_i, \beta_i \in T$ or $\alpha_i = \beta_i \notin T$. So for $n \leq 4$ each $T_{\alpha, \beta}$ is 0, 1 or some q_{tb} , while $T_{(t, 5, 6, 7, t'), (b, 5, 6, 7, d)} = q_{tb} q_{t'd}$ is not a partial isometry if $q_{tb} q_{t'd} \neq q_{t'd} q_{tb}$.

4 Quantum subgroups with property (G)

Lemma 4.1. *Let A be primitive, let $\mathbb{H}$ have property (G), and let $\pi: C(\mathbb{G}_A^\ell) \rightarrow C(\mathbb{H})$ be a surjective quantum group homomorphism. Put $w = \pi(u)$, $P' = \pi(p)$ and $Q' = \pi(q)$ entrywise. (a) P' and Q' are magic unitary representations, and $P'_{ij} \neq 0 \iff Q'_{ij} \neq 0 \iff w_{ij} \neq 0$; we call the classes of the resulting equivalence relation (Lemma 2.4) the orbits of π . If $\{x\}$ is an orbit, then $w_{xx} = 1$. (b) Some orbit has at least two elements. (c) If O_1, O_2 are orbits with at least two elements (possibly equal), then $Q'_{ab} P'_{cd} \neq P'_{cd} Q'_{ab}$ for some $a, b \in O_1$ and $c, d \in O_2$.*

Proof. (a) Use Lemma 2.3(a) and $\|w_{ij}\|^2 = \|P'_{ij}\| = \|Q'_{ij}\|$. If $\{x\}$ is an orbit, then $P'_{xx} = Q'_{xx} = 1$ and $w_{xj} = w_{jx} = 0$ for $j \neq x$, so w_{xx} is a group-like unitary and equals 1 by Lemma 2.5(a). (b) Otherwise w is the identity matrix and $C(\mathbb{H}) = \pi(C(\mathbb{G}_A^\ell)) = \mathbb{C}1$. (c) By Lemma 2.4(a) and property (G), the coefficients of Q'_{O_1} generate $\text{Pol}(\mathbb{H})$, and so do those of P'_{O_2} ; if they commuted, $\text{Pol}(\mathbb{H})$ would be commutative. $\square$

Proof of Theorem 1.2. Case (a): by Lemma 2.3(b), every Q'_{ab} commutes with every P'_{cd} , which contradicts Lemma 4.1(b),(c).

Case (b): let F be the set of vertices forming one-element orbits and U its complement; $F \neq \emptyset$ by assumption and $U \neq \emptyset$ by Lemma 4.1(b). As A is irreducible, there are an edge $x \rightarrow f_1$ with $x \in U$, $f_1 \in F$, and a shortest path $f_1 \rightarrow \dots \rightarrow f_k \rightarrow y$ from f_1 to U ; so $f_1, \dots, f_k \in F$, $y \in U$ and $1 \leq k \leq L_A - 1$. Let $[x], [y]$ be the orbits of x, y . Since $P'_{mf_1} = \delta_{mf_1}$, the (x, f_1) entry of $AP' = Q'A$ reads $1 = A_{xf_1} = \sum_{m \in [x]} Q'_{xm} A_{mf_1}$, and as the Q'_{xm} , $m \in [x]$, are non-zero orthogonal projections with sum 1, $A_{mf_1} = 1$ for all $m \in [x]$. Likewise the (f_k, y) entry gives $\sum_{m \in [y]} A_{f_k m} P'_{my} = 1$, so $A_{f_k m} = 1$ for $m \in [y]$. Hence for $a, b \in [x]$ and $c, d \in [y]$ the words $(a, f_1, \dots, f_k, c)$ and $(b, f_1, \dots, f_k, d)$ are admissible, of length $k + 2 \leq \ell$, and since $w_{ff} = 1$ for $f \in F$, the image of the corresponding $u_{\alpha, \beta}$ is $w_{ab} w_{cd}$, a partial isometry. By Lemma 2.2(a), Q'_{ab} commutes with P'_{cd} , contradicting Lemma 4.1(c) for $O_1 = [x]$, $O_2 = [y]$.

Case (c): write $w_{ij} = \lambda_{ij} P'_{ij}$; then $P'_{ij} = |\lambda_{ij}|^2 P'_{ij}$, so we may take $|\lambda_{ij}| = 1$, and $Q' = P'$. Thus $AP' = P'A$, and $P'_{\alpha_1 \beta_1} \dots P'_{\alpha_n \beta_n}$ is a unimodular multiple of $\pi(u_{\alpha, \beta})$, hence a partial isometry, for $\alpha, \beta \in V_A^n$, $n \leq L_A$. By Theorem 3.1 the P'_{ij} commute, contradicting Lemma 4.1(b),(c). $\square$

Proof of Corollary 1.3. $\text{QAut}(A) = S_N^+$ and $\text{Aut}(A) = S_N$ (Section 2); S_M^+ has property (G) by Lemma 2.5(b), so Theorem 1.2(a) applies; and $S_N^+ \subset \mathbb{G}_A^1$ by [FGS26, Proposition 5.11]. $\square$

Example 4.2 (Level two does not suffice). Let $A_{\text{hub}} = \begin{pmatrix} I_4 & \mathbf{1} \\ \mathbf{1}^\top & 1 \end{pmatrix}$, $\mathbf{1}$ the all-ones column: four looped vertices, each joined in both directions to a looped hub 5. It is primitive, $L_{A_{\text{hub}}} = 3$, and its Perron–Frobenius eigenvector is $(1, 1, 1, 1, 2)/6$. (i) The hub is the only vertex of out-degree 5, so by Lemma 2.4(b) the magic unitaries commuting with A_{hub} are the $q \oplus 1$ with q any 4×4 magic unitary (these commute with A_{hub} as $q\mathbf{1} = \mathbf{1}$, $\mathbf{1}^\top q = \mathbf{1}^\top$). So $\text{QAut}(A_{\text{hub}}) = S_4^+ \neq S_4 = \text{Aut}(A_{\text{hub}})$. (ii) For the fundamental magic unitary q of S_4^+ , $u \mapsto w = q \oplus 1$ is a surjective quantum group homomorphism $C(\mathbb{G}_{A_{\text{hub}}}^2) \rightarrow C(S_4^+)$: $P' = Q' = w$ satisfies (I) and (II), and for $\alpha, \beta \in V_{A_{\text{hub}}}^n$ with $n \leq 2$ (words $tt, t5, 5t, 55$) each $w_{\alpha, \beta}$ is 0, 1 or some q_{tb} . (iii) For $\ell \geq 3$ no $\mathbb{H}$ with property (G) is a

quantum subgroup of $\mathbb{G}_{A_{\text{hub}}}^\ell$; in particular $\text{QAut}(A_{\text{hub}})$ is not one of $\mathbb{G}_{A_{\text{hub}}}^\infty$. Indeed, for a surjection π , Lemma 2.4(b) for $AP' = Q'A$ makes the hub a one-element orbit, so $w_{55} = 1$; for $a, b, c, d \leq 4$ the words $(a, 5, c)$, $(b, 5, d)$ are admissible, so $w_{ab}w_{cd}$ is a partial isometry and Q'_{ab} commutes with P'_{cd} , contradicting Lemma 4.1(c). So the level-two argument of case (a) fails for general A ; here level 3 suffices.

Problem 4.3. *Let A be primitive and $\mathbb{H}$ with property (G), for example S_M^+ with $M \geq 4$. Is there a surjective quantum group homomorphism $C(\mathbb{G}_A^\infty) \rightarrow C(\mathbb{H})$? By Theorem 1.2 the answer is negative for $A \in \{J_N, J_N - I_N\}$, and in general such a homomorphism would fix no vertex and not be as in Theorem 1.2(c). A negative answer for S_M^+ would imply that $\text{QAut}(A)$ is not a quantum subgroup of $\mathbb{G}_A^\infty$ whenever S_M^+ is one of $\text{QAut}(A)$, for instance if every permutation of some M vertices, fixing the others, is an automorphism of A (then $q \oplus I$ commutes with A for every $M \times M$ magic unitary q).*

5 Two positive examples

Proposition 5.1. *A_D is primitive, $C(\text{QAut}(A_D)) \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ with free generators s, t satisfying $\Delta(s) = s \otimes s$, $\Delta(t) = t \otimes t$, and $\text{Aut}(A_D) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The assignment $u \mapsto \text{diag}(s, t, 1, 1)$ is a surjective quantum group homomorphism $C(\mathbb{G}_{A_D}^\infty) \rightarrow C(\text{QAut}(A_D))$.*

Proof. All entries of A_D^2 are positive. The out-degrees are 3, 3, 2, 2, so by Lemma 2.4(b) a magic unitary commuting with A_D has the form $\begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix} \oplus \begin{pmatrix} f & 1-f \\ 1-f & f \end{pmatrix}$ with projections e, f , and every such matrix commutes with $A_D = \begin{pmatrix} I_2 & J_2 \\ J_2 & 0 \end{pmatrix}$. So $C(\text{QAut}(A_D))$ is the universal C^* -algebra generated by two projections e, f , that is, $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ with $s = 2e - 1$, $t = 2f - 1$, and $\Delta(e) = e \otimes e + (1-e) \otimes (1-e)$ gives $\Delta(s) = s \otimes s$, likewise for t . The entries of $w = \text{diag}(s, t, 1, 1)$ are unitaries or 0, so $P' = Q' = I_4$ and (I)–(II) hold, and each $w_{\alpha, \beta}$ is 0 or unitary, so (III) holds for all n . The map intertwines the coproducts and is onto. (It is the quantum gauge subgroup $\widehat{\mathbb{F}}_4$ of [FGS26, Proposition 5.10] followed by $C^*(\mathbb{F}_4) \rightarrow C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$.) $\square$

Write $A_E = I_4 + B$, with B the adjacency matrix of the 4-cycle $1 - 2 - 3 - 4 - 1$; indices are taken modulo 4.

Lemma 5.2. *Let p be the fundamental magic unitary of $C(\text{QAut}(A_E)) = C(\text{QAut}(B))$, and $a = p_{11} - p_{13}$, $b = p_{12} - p_{14}$, $c = p_{21} - p_{23}$, $d = p_{22} - p_{24}$, $e = a^2$. (a) $p_{i+2, j+2} = p_{ij}$ and $p_{i+2, j} = p_{i, j+2}$. (b) a, b, c, d are self-adjoint, $ab = cd = ac = bd = 0$, $a^2 = d^2 = e$, $b^2 = c^2 = 1 - e$, and $x^3 = x$ for $x \in \{a, b, c, d\}$. (c) Each p_{ij} is $(x^2 \pm x)/2$ for some $x \in \{a, b, c, d\}$; so a, b, c, d generate, and e is a central projection. (d) $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$ and $\Delta(d) = c \otimes b + d \otimes d$. (e) $g = 2e - 1$ is a group-like unitary, $g \neq 1$, and $C(\text{QAut}(A_E))$ is not commutative.*

Proof. (a) $B^2 = 2I + 2R$ with R the permutation matrix of $i \mapsto i + 2$, so p commutes with R . (b) $ab = cd = 0$ by row orthogonality. In ac , the terms $p_{11}p_{21}$, $p_{13}p_{23}$ vanish by column orthogonality and $p_{11}p_{23}$, $p_{13}p_{21}$ by Lemma 2.3(c) for B ($B_{12} = 1$, $B_{13} = B_{31} = 0$); $bd = 0$ likewise. By row orthogonality $a^2 = p_{11} + p_{13}$, $b^2 = p_{12} + p_{14}$, $c^2 = p_{21} + p_{23}$, $d^2 = p_{22} + p_{24}$, so rows 1, 2 and, by (a), columns 1, 2 give $a^2 + b^2 = c^2 + d^2 = a^2 + c^2 = b^2 + d^2 = 1$; hence $a^2 = d^2 = e$ and $b^2 = c^2 = 1 - e$. Then $a^3 = a(1 - b^2) = a$, and similarly for b, c, d . (c) $p_{11} = (a^2 + a)/2$, $p_{13} = (a^2 - a)/2$, similarly in rows 1, 2, and (a) gives rows 3, 4. By (b), $eb = a(ab) = 0 = be$, $ec = 0 = ce$, $ea = a^3 = ae$, $ed = d^3 = de$. (d) By (a), $p_{k1} - p_{k3} = a, c, -a, -c$ for $k = 1, 2, 3, 4$, so $\Delta(a) = \sum_k p_{1k} \otimes (p_{k1} - p_{k3}) = a \otimes a + b \otimes c$; the others follow in the same way from $p_{k2} - p_{k4} = b, d, -b, -d$ and from row 2. (e) $\Delta(e) = \Delta(a)^2 = e \otimes e + (1-e) \otimes (1-e)$

by (d) and (b), so $\Delta(g) = g \otimes g$. On $\mathbb{C}^2 \oplus \mathbb{C}^2$ let $s_1 = \text{diag}(1, -1)$, $s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $a = s_1 \oplus 0$, $d = s_2 \oplus 0$, $b = 0 \oplus s_1$, $c = 0 \oplus s_2$. These satisfy (b), and the matrix given by (c) and (a) is a magic unitary commuting with B (a direct computation, also done by the program of the Verification paragraph). There $e = 1 \oplus 0 \neq 1$, and $p_{11} = \frac{1}{2}(1 + s_1) \oplus 0$ does not commute with $p_{22} = \frac{1}{2}(1 + s_2) \oplus 0$. $\square$

Here $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the cubic matrix of [BBC07, Section 6], and $\text{QAut}(B)$ equals the quantum automorphism group H_2^+ of the complementary graph $J_4 - I_4 - B$ (two disjoint edges) studied there.

Proposition 5.3. *A_E is primitive, $\text{Aut}(A_E)$ is dihedral of order 8, and $\text{QAut}(A_E) \neq \text{Aut}(A_E)$. Put $w_{11} = a$, $w_{13} = b$, $w_{31} = c$, $w_{33} = d$, $w_{22} = w_{44} = 1$ and $w_{ij} = 0$ otherwise. Then $u \mapsto w$ is a surjective quantum group homomorphism $C(\mathbb{G}_{A_E}^\infty) \rightarrow C(\text{QAut}(A_E))$.*

Proof. $A_E^2 > 0$, $\text{Aut}(A_E) = \text{Aut}(B)$ is the symmetry group of the square, and $\text{QAut}(A_E) \neq \text{Aut}(A_E)$ by Lemma 5.2(e). The w_{ij} are self-adjoint partial isometries, and $P' = Q' = eI_4 + (1 - e)\Sigma$ with Σ the permutation matrix of $(1\ 3) \in \text{Aut}(A_E)$; so P' , Q' are magic, commute with A_E , and fix the constant eigenvector of the 3-regular A_E . A product of entries of w is 0 or $X = x_1 \cdots x_n$ with x_i self-adjoint and x_i^2 a central projection, so $XX^* = x_n^2 \cdots x_1^2$ is a projection and X a partial isometry. Hence (I)–(III) hold for every ℓ , and the map is onto by Lemma 5.2(c). It intertwines the coproducts: for $i, j \in \{1, 3\}$ by Lemma 5.2(d), as $\sum_k w_{ik} \otimes w_{kj}$ only involves $k \in \{1, 3\}$; for $i = j \in \{2, 4\}$ both sides are $1 \otimes 1$; otherwise both sides vanish. $\square$

Proposition 1.4 follows from Propositions 5.1 and 5.3. The embeddings fix vertices, so by Theorem 1.2(b) neither $\text{QAut}(A_D)$ nor $\text{QAut}(A_E)$ can have property (G); indeed s , t and g are non-trivial group-like unitaries (Lemma 2.5(a)). By Theorem 1.1 the embeddings are necessarily non-canonical.

6 Concluding remarks

Verification. All results are proved in the text; the computations were used only as checks. (i) A standard-library Python program checks every finite claim in exact rational arithmetic: primitivity, L_A , degrees and $|\text{Aut}(A)|$ for J_4 , J_5 , $J_4 - I_4$, $J_5 - I_5$, A_D , A_E , A_{hub} and A_7 ; the model of Lemma 5.2(e), with the relations of Lemma 5.2, the four coproduct formulas and $\Delta(g) = g \otimes g$ in the tensor square of the model; the maps of Propositions 5.1 and 5.3 on all words of length at most 6; and the levels at which the canonical relations fail in Remark 3.2 and Example 4.2. All 36 checks pass. (ii) Numerical programs check the same identities in random finite-dimensional models (Pauli-type models of S_4^+ , magic unitaries from quantum Latin squares), including Lemma 2.2(a), the first failing level of the canonical relations for five graphs (never later than L_A), identity (1), and Propositions 5.1 and 5.3. (iii) An independent verification run, which was AI-assisted, checked the proofs of an earlier draft and its use of [FGS26, Ska26], re-ran the programs of item (ii) (identical outputs), and wrote separate code for A_{hub} , Remark 3.2 and Propositions 5.1 and 5.3 (all words of length at most 7, all four coproduct formulas). It found that the draft's argument for a version of Theorem 1.2(b) without a fixed vertex was incomplete, as Example 4.2 shows, and outlined the corrected statements proved here. (iv) A second independent verification run, also AI-assisted, checked the present text line by line against [FGS26] (versions 1–3), [Ska26], [Ban99, Theorem 4.1] and [BBC07, Section 6], and re-ran all programs from the source archive (identical outputs). Its own exact code checks the finite claims again, in a new rational model of a non-commutative 4×4 magic unitary and in a second model of Lemma 5.2, and tests Theorem 3.1 on all primitive A of size 4 or 5 with two disjoint transpositions in $\text{Aut}(A)$ and on all primitive A of size at most 6 in which every permutation of four vertices, fixing the others, is an automorphism (up to relabelling; 908 cases): the canonical relations always fail at some level at most L_A , with equality in 207 cases. It found no mathematical error and asked only for

changes of presentation, which are made here. The programs and their outputs are included in the source files.

Scope and priority. Theorem 1.1 answers the canonical reading (Q_{can}) of Question 1 of [Ska26] (the question of [FGS26, Section 5.3]) negatively for all primitive A with $Q\text{Aut}(A) \neq \text{Aut}(A)$. Corollary 1.3 and Example 4.2 answer (Q_{abs}) negatively for J_N , $J_N - I_N$ ($N \geq 4$) and A_{hub} , while Proposition 1.4 gives positive cases. Open are a characterization of the A with a positive answer and, in particular, whether for general A a quantum group with property (G) can be a quantum subgroup of $\mathbb{G}_A^\infty$ through an embedding that fixes no vertex (Problem 4.3). We read the question in [Ska26] and in the LaTeX sources of the arXiv versions of [FGS26] (the passage is unchanged from version 1 to version 3). As of 1 October 2026 we found no work addressing it in arXiv (authors Freslon, Gerontogiannis, Skalski, Joardar, Mandal, Banica, Schmidt, Weber, and keyword searches), Crossref, OpenAlex (no recorded citations of [FGS26]), zbMATH and a web search. Related work on quantum symmetries of graph C^* -algebras [SW18, JM18, JS25] does not concern $\mathbb{G}_A^\infty$. For a different spectral triple on graph C^* -algebras, constructed by Farsi et al., Joardar and Sharma [JS25] show that the quantum automorphism group of a strongly connected graph does act isometrically; by Theorem 1.1 this fails for the log-Laplacian triple whenever $Q\text{Aut}(A) \neq \text{Aut}(A)$. The arguments are short and standard; the question was posed as open by the authors of [FGS26]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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