

Independent-Root Counts Do Not Characterize Local Graph Removal

Alper Ferudun*

27 October 2026

Abstract

We give counterexamples to the fixed-cardinality independent-root formulation of local graph removal in Conjecture 8 of the 2026 Oberwolfach report *Combinatorics*. Every core graph with a maximal independent set smaller than its independence number supplies an obstruction: retain that set as singleton parts and blow up the other vertices. The resulting graphs stay a positive normalized maximum-degree distance from being H -free, while all independent root counts of the prescribed size are asymptotically small. A six-vertex wheel gives one example. For C_9 we obtain graphs on $6m + 3$ vertices with maximum independent-four-root count m^4 and exact removal distance $\lceil m/2 \rceil / (6m + 3)$, tending to $1/12$. More generally, no single root cardinality works for a core that is not well-covered. These conclusions concern the formula as printed; they do not exclude formulations that test several root sizes or otherwise qualify the root patterns.

1 The printed criterion

All graphs are finite, simple, and loopless. Copies are non-induced subgraphs isomorphic to a fixed graph H . Write

$$d_\infty(G, H\text{-free}) = \frac{1}{|V(G)|} \min\{\Delta(D) : D \subseteq G, G - E(D) \text{ is } H\text{-free}\}.$$

Allowing both additions and deletions gives the same minimum. Indeed, if an edited graph G' is H -free, then $G \cap G'$ is H -free, and its difference from G is contained in $E(G) \triangle E(G')$.

For $S \subseteq V(G)$ with $|S| = q$, let $R_q(H, G; S)$ count the H -copies containing S in which S is independent *in the copy*. This need not mean that S is independent in G . Put

$$R_q^{\max}(H, G) = \max_{S \in \binom{V(G)}{q}} R_q(H, G; S).$$

Counting labeled injective embeddings instead multiplies these quantities by $|\text{Aut}(H)|$ and does not change any little- o assertion.

Definition 4 of Wei's contribution to [1], whose graph results are joint work with Jacob Fox, introduces the following parameter. An independent set $A \subseteq V(H)$ is admissible if every endomorphism $\rho : H \rightarrow H$ satisfies

$$(|\rho^{-1}(v)| \leq 1 \text{ for every } v \in A) \implies (|\rho^{-1}(v)| = 1 \text{ for every } v \in A).$$

Let $a(H)$ be the maximum size of an admissible independent set. For $k = |V(H)|$, the fixed-cardinality formulation in Conjecture 8, on pages 60–61 of that report, is

$$d_\infty(G_n, H\text{-free}) = o(1) \iff R_{a(H)}^{\max}(H, G_n) = o(n^{k-a(H)}). \quad (1)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

The independence requirement and the use of *exactly* $a(H)$ roots are essential to the statement being examined.

The examples below disprove the implication from right to left in (1). They leave intact the broader question of which local removal statements are available. In particular, they do not address the validity of an unpublished intended formulation that differs from (1).

2 A core obstruction

A graph is a *core* if every endomorphism is an automorphism. A graph is *well-covered* if all its inclusion-maximal independent sets have size equal to its independence number $\alpha(H)$. Maximality here means that no vertex can be added; it does not mean maximum cardinality.

Lemma 1. *Let H be a core on k vertices, and replace each vertex v by a nonempty independent part P_v , with complete joins exactly between parts indexed by edges of H . Every H -copy in this blow-up contains exactly one vertex from each part. Consequently there are exactly $\prod_v |P_v|$ copies.*

Proof. Compose an injective embedding of H into the blow-up with the projection onto H . The result is an endomorphism and hence an automorphism, so each part occurs once. Conversely, any choice of one vertex per part induces H . Since every embedding projects to an automorphism, there are no additional types of non-induced copy on these vertices. $\square$

Theorem 2. *Let H be a core on k vertices with at least one edge. Suppose B is a maximal independent set of size $b < a = \alpha(H)$. Let G_m be the blow-up with singleton parts on B and parts of size m elsewhere, and write $n_m = b + (k - b)m$. Then $a(H) = a$ and*

$$d_\infty(G_m, H\text{-free}) \geq \frac{2m}{(k + b)(b + (k - b)m)}, \quad (2)$$

$$R_a^{\max}(H, G_m) \leq m^{k-a-1}. \quad (3)$$

Thus (1) fails for H .

Proof. Every independent set of a core is admissible, because every endomorphism has exactly one preimage at every vertex. Hence $a(H) = \alpha(H)$.

Set $r = k - b$. Lemma 1 gives m^r copies, all containing the b singleton vertices. Suppose a deleted edge set D meets every copy, and write $\Delta = \Delta(D)$. No edge joins two singleton vertices. At most $b\Delta$ deleted edges have a singleton endpoint, and each is in exactly m^{r-1} copies. At most $rm\Delta/2$ deleted edges have two bulk endpoints, and each is in exactly m^{r-2} copies. A union bound therefore gives

$$m^r \leq b\Delta m^{r-1} + \frac{rm\Delta}{2} m^{r-2} = \left(b + \frac{r}{2}\right) \Delta m^{r-1}.$$

If $r = 1$, the second edge type is absent and the same upper bound remains valid. We obtain $\Delta \geq 2m/(k + b)$ and then (2).

Consider an a -set S with positive rooted count. Its part indices are an independent a -set T of H , and no part index is repeated. Maximality of B , together with $b < a$, implies $B \not\subseteq T$. Consequently $|T \cap B| \leq b - 1$, so S fixes at least $a - b + 1$ bulk coordinates. By Lemma 1,

$$R_a(H, G_m; S) = m^{r-|T \cap B|} \leq m^{r-(a-b+1)} = m^{k-a-1}.$$

All other a -sets have rooted count zero. Since n_m is linear in m , (3) is $o(n_m^{k-a})$, while the lower bound in (2) tends to the positive constant $2/((k + b)(k - b))$. $\square$

The obstruction cannot be repaired simply by choosing another single root cardinality.

Corollary 3. *If a core H with at least one edge is not well-covered, then no fixed nonnegative integer q characterizes local H -removability by*

$$R_q^{\max}(H, G_n) = o(n^{k-q}).$$

Proof. Put $a = \alpha(H)$. For $q = a$, use Theorem 2. For $q < a$, keep a *maximum* independent a -set singleton and blow up the other vertices by m . The deletion argument in that theorem applies without the strict inequality $b < a$, so these graphs remain locally far. Their total number of H -copies is m^{k-a} , which is already $o(n^{k-q})$ and bounds every q -root count. For $q > a$, no H -copy has an independent q -set, so every rooted count is zero; the locally far family from Theorem 2 still applies. $\square$

3 A six-vertex wheel

Let $W = K_1 \vee C_5$ be the wheel with five rim vertices and one hub. Here $\alpha(W) = 2$, while the singleton hub is a maximal independent set.

Proposition 4. *There are graphs J_m on $5m + 1$ vertices such that*

$$a(W) = 2, \quad R_2^{\max}(W, J_m) = m^3, \quad d_{\infty}(J_m, W\text{-free}) \geq \frac{2m}{7(5m+1)}.$$

Proof. We first check that W is a core. An endomorphism cannot map the hub to a rim vertex: it would map the rim C_5 into that vertex's neighborhood, which induces a three-vertex path and is bipartite. An odd cycle has no homomorphism to a bipartite graph. Thus the hub is fixed and the rim maps to itself.

Every endomorphism of an odd cycle C_{ℓ} is an automorphism. To see this, write the steps of the associated closed walk as $+1$ or -1 modulo ℓ . Their sum is an odd multiple of ℓ and has absolute value at most ℓ . It is therefore ℓ or $-\ell$, forcing all steps to have the same sign. This proves that W is a core.

Keep the hub singleton and blow up each rim vertex to m vertices. Theorem 2, with $k = 6$ and $b = 1$, gives the distance bound. By Lemma 1, an independent pair with positive rooted count consists of vertices from distinct nonadjacent rim parts. Each such pair lies in m^3 copies, and all other pairs have count zero. $\square$

The distance lower bound tends to $2/35$, while $m^3 = o((5m+1)^4)$. Thus a connected graph on six vertices already contradicts the printed criterion.

4 Nine-cycles and an exact distance

Take cyclically indexed parts $P_0, \dots, P_8$, with P_0, P_3, P_6 singletons and all other parts of size m . Join consecutive parts completely and include no other edges. Denote this graph by G_m .

Theorem 5. *For every positive integer m ,*

$$|V(G_m)| = 6m + 3, \quad a(C_9) = 4, \quad R_4^{\max}(C_9, G_m) = m^4,$$

and

$$d_{\infty}(G_m, C_9\text{-free}) = \frac{\lceil m/2 \rceil}{6m+3}. \quad (4)$$

Proof. The odd-cycle argument in Proposition 4 shows that C_9 is a core, and therefore $a(C_9) = \alpha(C_9) = 4$. Every nine-cycle in G_m uses one vertex from each part; there are exactly m^6 such cycles.

The index set $\{0, 3, 6\}$ is maximal independent, of size three. No independent four-set in a nine-cycle contains all three singleton vertices, because every other cycle position is adjacent to one of them. Such a four-set therefore fixes at least two bulk coordinates and has at most m^4 extensions. Equality holds for the singleton vertices in parts 0 and 6 together with one vertex from each of parts 2 and 4.

For the distance, view G_m as three successive gadgets between singleton vertices. Each gadget has the form

$$s \text{ --- } U \text{ --- } V \text{ --- } t, \quad |U| = |V| = m,$$

with all edges indicated by the three joins. If each gadget retains one length-three path between its terminals, the three paths form a nine-cycle: their internal vertices are in disjoint parts. Hence any deletion destroying all nine-cycles must destroy every such path in at least one gadget.

In that gadget, suppose a edges from s into U and c edges from t into V are deleted. If $a = m$ or $c = m$, the maximum deletion degree is at least m . Otherwise, there remain terminal-neighbor sets U' and V' of sizes $m - a$ and $m - c$. Every edge between U' and V' must be deleted. Thus the maximum deletion degree is at least

$$\max\{a, c, m - a, m - c\} \geq \lceil m/2 \rceil.$$

For the matching upper bound, delete $\lfloor m/2 \rfloor$ terminal edges at each end of one gadget, and then delete all edges between the remaining terminal-neighbor sets U' and V' . The terminal deletion degrees are $\lfloor m/2 \rfloor$; the retained bulk vertices have deletion degree $\lceil m/2 \rceil$; and other affected bulk vertices have degree one. This construction also works when $m = 1$. It eliminates all length-three paths in the chosen gadget and hence every nine-cycle. The minimum maximum deletion degree is exactly $\lceil m/2 \rceil$, which proves (4). $\square$

It follows that the independent-four-root count is $o((6m+3)^5)$, whereas the removal distance tends to $1/12$. The three singleton vertices belong to all m^6 cycles. Thus a test that also checks independent three-root counts detects this example. This observation is not a claim that checking all smaller root sizes is sufficient in general.

5 Interpretation and verification

The sentence following Conjecture 8 in [1] states that odd cycles are among the covered cases. Theorem 5 shows that this cannot hold with the fixed-cardinality independent-root formula printed there. The formula requires a qualification or correction. We make no claim about the intended statement or proof of the unpublished graph removal work; at the time of our bounded literature check, the author's publication list still described that manuscript as forthcoming [2].

The distinction between independent and unrestricted roots is decisive. In G_m , the three singleton vertices together with one vertex of P_1 lie in m^5 cycles, but never form an independent set in a cycle. Our family therefore does not satisfy the corresponding sparsity condition for *unrestricted* four-root sets. The result concerns the literal criterion (1), not the broad existence of local removal lemmas or all questions in the associated UnsolvedMath record.

The accompanying exact programs check the parameters and lift counts, independently enumerate the nine-cycles for $m \leq 3$, and compare the optimal deletion degree with finite Boolean satisfiability calculations for $m \leq 4$. They also retain a bounded atlas investigation and negative controls for the root convention. These checks supplement the written all-parameter proofs; they are not formal verification or independent referee review. This unrefereed note was developed with AI assistance, with the statements, proofs, computations, and scope checked by the author's research workflow. No absolute priority claim is made.

References

- [1] F. Wei, Local removal lemmas (graph results joint with J. Fox), in *Combinatorics*, Oberwolfach Reports **23** (2026), no. 1, 58–61. [doi:10.4171/OWR/2026/1](https://doi.org/10.4171/OWR/2026/1).
- [2] F. Wei, Research publications, personal website, <https://sites.google.com/view/fan-wei/home>, accessed 27 October 2026.