

Optimizing over Nonzero Littlewood–Richardson Coefficients: The Quantum Max d -Cut of Complete Bipartite Graphs

Alper Ferudun*

October 15, 2026

Abstract

Konvalinka asked (Oberwolfach Report 2/2026, p. 150) for the maximum of $\Delta(\lambda, \mu, \nu) = \eta(\lambda) - \eta(\mu) - \eta(\nu)$, where $\eta(\lambda) = |\lambda|^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{i=1}^d (\lambda_i - i + 1)^2$, over the triples of partitions with $|\lambda| = n$, $|\mu| = k$, $|\nu| = n - k$, at most d parts and nonzero Littlewood–Richardson coefficient $c_{\mu\nu}^\lambda$. By the work of Klep, Štrekelj and Volčič this maximum is the quantum max d -cut of the complete bipartite graph $K_{n-k,k}$, that is, the largest eigenvalue of the sum of $2(I - \text{SWAP}_{ij})$ over its edges acting on $(\mathbb{C}^d)^{\otimes n}$. They determined it for $d \leq 3$ and for $k \leq 4$ and conjectured that it is always attained at a triple in which at most one part of λ is the sum of a part of μ and a part of ν (their Conjecture 7.7). We determine the maximum for all n, k, d . An explicit Littlewood–Richardson tableau and a level-set argument reduce the problem to the maximization of an elementary function $\Psi(\mu, x)$ over partitions μ of k with at most d parts and vectors $x \in \mathbb{Z}_{\geq 0}^d$ with sum $n - k$, an optimal λ being the decreasing rearrangement of $\mu + x$. Exchange and convexity arguments then give an optimum with at most one row in which both μ and x are nonzero and with balanced remaining rows. Consequently, for $d \geq 2$ the maximum equals $2k(n - k) + 2 \max\{\Psi_A, \Psi_B\}$ over an explicit list of at most $(d - 1)(1 + k(n - k))$ integers built from balanced partitions; it is 0 for $d = 1$ and $4k(n - k)$ for $d \geq n$. As a corollary, Conjecture 7.7 of Klep, Štrekelj and Volčič holds, with explicit optimal triples $(\lambda, \mu, \nu) = (P \uplus Z, P, Z)$ or $(P \uplus Z \uplus (\pi + y), P \uplus (\pi), Z \uplus (y))$, where P and Z are balanced partitions and $\uplus$ denotes the union of the parts. The answer is a maximum over a candidate list of polynomial size, not a single closed formula. The known cases ($d = 2$, which is the Lieb–Mattis theorem; $d = 3$; $k \leq 4$; the star graph) are recovered, and the results agree with brute-force computations over all Littlewood–Richardson coefficients for $n \leq 22$ and with the reduction for $n \leq 80$. The problem is Konvalinka’s; the reduction of the quantum max d -cut of $K_{n-k,k}$ to this maximum, the cases $d \leq 3$ and $k \leq 4$ and the conjecture are due to Klep, Štrekelj and Volčič. New, to our knowledge, are the general solution and the proof of the conjecture. This is an unrefereed note.

1 Introduction

1.1 The problem

The following problem of Matjaž Konvalinka, communicated by Igor Klep, is Problem 10 of the report on the open problem session of the Oberwolfach workshop *Enumerative Combinatorics* (January 2026) [CEF26, p. 150]. For a partition λ of length d define

$$\eta(\lambda) = n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{i=1}^d (\lambda_i - (i-1))^2, \quad (1)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

and for λ, μ, ν define $\Delta(\lambda, \mu, \nu) = \eta(\lambda) - \eta(\mu) - \eta(\nu)$. Given n, k, d with $k \leq n/2$, the problem is to maximize $\Delta(\lambda, \mu, \nu)$ over the triples (λ, μ, ν) with $|\lambda| = n$, $|\mu| = k$, $|\nu| = n - k$ for which the Littlewood–Richardson coefficient $c_{\mu\nu}^\lambda$ is nonzero. This is the problem as stated in the source, with (1) as printed there; the source states no partial results. The problem is record OWR-14299904-015 of the dataset [ula26], where it is listed as open.

Reading. The source does not say what n means in $\eta(\mu)$ and $\eta(\nu)$, and “length d ” cannot mean “exactly d nonzero parts” for μ when $k < d$. We use the reading of the paper in which the problem arose, by Klep, Štrekelj and Volčič [KŠV26]:

(R1) partitions are written as d -tuples, padded with zeros; thus λ has at most d nonzero parts, and η is applied in the same way to μ and ν , which have at most d parts when $c_{\mu\nu}^\lambda \neq 0$ because $\mu, \nu \subseteq \lambda$. The authors of [KŠV26] note that (1) is valid for partitions with at most d rows (last sentence of the proof of their Proposition 6.7);

(R2) in $\eta(\pi)$ the letter n denotes $|\pi|$, as in [KŠV26, (6.4)].

With this reading $\eta(\lambda)$ is the eigenvalue of the quantum max d -cut Hamiltonian of the complete graph on $|\lambda|$ vertices on its λ -isotypic component [KŠV26, Lemma 6.4, Proposition 6.7], and the numbers $\Delta(\lambda, \mu, \nu)$ with $\ell(\lambda) \leq d$ and $c_{\mu\nu}^\lambda \neq 0$ are the eigenvalues of

$$H = \sum_{i \in A, j \in B} 2(I - \text{SWAP}_{ij}) \quad \text{on } (\mathbb{C}^d)^{\otimes n}, \quad |A| = n - k, \quad |B| = k, \quad (2)$$

where SWAP_{ij} exchanges the i -th and the j -th tensor factor [KŠV26, (7.3)–(7.5)]. So the maximum is the largest eigenvalue of the quantum max d -cut Hamiltonian [KŠV26, Definition 1.5] of the complete bipartite graph $K_{n-k, k}$, called the solution of the d -QMC problem for $K_{n-k, k}$ in [KŠV26]. We denote it by

$$\Delta_{\max}(n, k, d) = \max\{\Delta(\lambda, \mu, \nu) : |\lambda| = n, |\mu| = k, |\nu| = n - k, \ell(\lambda) \leq d, c_{\mu\nu}^\lambda \neq 0\}. \quad (3)$$

Under the other readings the answer changes only trivially. If the given n is used in all three values of η , then every Δ decreases by the constant $n^2 + 2k(n - k)$ (Lemma 2.1), and the maximizers do not change. If λ is required to have exactly d nonzero parts, the maximum is the same for $d \leq n$ (Remark 6.2), and there is no admissible triple for $d > n$. Since Δ and $c_{\mu\nu}^\lambda$ are symmetric in μ and ν , the hypothesis $k \leq n/2$ is a normalization, and our results hold for all $1 \leq k \leq n - 1$. (For $k = 0$ the only triples are $(\lambda, \emptyset, \lambda)$, with $\Delta = 0$.)

1.2 Results

Partitions are written with as many zeros as needed; $\ell(\mu)$ is the number of nonzero parts, $\mu'_s = \#\{r : \mu_r \geq s\}$ is the conjugate partition, and $\mu \uplus \nu$ is the partition whose parts are the parts of μ together with those of ν . For integers $N \geq 0$ and $g \geq 1$ let $\text{bal}(N, g)$ be the *balanced* g -tuple $(\lceil N/g \rceil, \dots, \lfloor N/g \rfloor)$: the decreasing g -tuple of nonnegative integers with sum N whose entries differ by at most one (it has zero entries if $N < g$). Throughout, $m = n - k$.

Theorem A (reduction). *Let $n \geq 1$, $0 \leq k \leq n$ and $d \geq 1$. For a partition μ of k with at most d parts and $x = (x_1, \dots, x_d) \in \mathbb{Z}_{\geq 0}^d$ put*

$$\Psi(\mu, x) = \sum_{r=1}^d \sum_{s=\mu_r+1}^{\mu_r+x_r} (\mu'_s - \mu_r). \quad (4)$$

Then

$$\Delta_{\max}(n, k, d) = 2km + 2 \max \left\{ \Psi(\mu, x) : \mu \vdash k, \ell(\mu) \leq d, x \in \mathbb{Z}_{\geq 0}^d, x_1 + \dots + x_d = m \right\}. \quad (5)$$

If (μ, x) is a maximizer, λ is the decreasing rearrangement of $(\mu_1 + x_1, \dots, \mu_d + x_d)$, and ν is the partition whose conjugate is the decreasing rearrangement of the column lengths $\lambda'_s - \mu'_s$ of λ/μ , then $c_{\mu\nu}^\lambda \geq 1$ and (λ, μ, ν) is an optimal triple.

In words: x_r cells are appended to row r of the diagram of μ inside a strip of d rows, and each appended cell earns the number of rows of μ that reach its column, minus the length of μ in its own row. For fixed μ the best x is obtained greedily (Remark 4.3). We call the pair (μ_r, x_r) the r -th row of (μ, x) , and the row *mixed* if $\mu_r > 0$ and $x_r > 0$.

Theorem B (structure of an optimum). *Let $d \geq 2$ and $1 \leq k \leq n - 1$. Up to the order of the rows, some maximizer (μ, x) in (5) has one of the following two shapes, where $1 \leq f \leq d - 1$:*

- (A) f rows $(P_i, 0)$ with $(P_1, \dots, P_f) = \text{bal}(k, f)$, where $f \leq k$, and $d - f$ rows $(0, Z_j)$ with $(Z_1, \dots, Z_{d-f}) = \text{bal}(m, d - f)$;
- (B) f rows $(P_i, 0)$ with $(P_i)_{i \leq f} = \text{bal}(k - \pi, f)$, one mixed row (π, y) , and $g = d - 1 - f$ rows $(0, Z_j)$ with $(Z_j)_{j \leq g} = \text{bal}(m - y, g)$, where $\pi \geq 1$, $y \geq 1$, $\pi \leq \min_i P_i$, and $\pi + y \leq \min_j Z_j$ if $g \geq 1$ (if $g = 0$, there are no rows $(0, Z_j)$, and $y = m$).

Theorem C (the value). *Let $d \geq 2$ and $1 \leq k \leq n - 1$. For $1 \leq f \leq d - 1$ put*

$$\Psi_A(f) = \sum_{i=1}^f \sum_{j=1}^{d-f} \min(P_i, Z_j), \quad (P, Z) = (\text{bal}(k, f), \text{bal}(m, d - f)),$$

and for $1 \leq f \leq d - 1$, $g = d - 1 - f$, integers $\pi \geq 1$ with $\pi \leq \lfloor (k - \pi)/f \rfloor$ and $1 \leq y \leq m$, where $y = m$ if $g = 0$, put $(P, Z) = (\text{bal}(k - \pi, f), \text{bal}(m - y, g))$, with Z empty if $g = 0$, and

$$\Psi_B(f, \pi, y) = \sum_{i=1}^f \sum_{j=1}^g \min(P_i, Z_j) + \sum_{j=1}^g \min(Z_j, \pi) + \sum_{i=1}^f \min(y, P_i - \pi) - \pi y.$$

Then

$$\Delta_{\max}(n, k, d) = 2k(n - k) + 2 \max \left(\{ \Psi_A(f) \} \cup \{ \Psi_B(f, \pi, y) \} \right), \quad (6)$$

the maximum being taken over the parameters just described; there are at most $(d - 1)(1 + k(n - k))$ of them. The list contains the values of the two shapes of Theorem B; it is larger, since the condition $\pi + y \leq \min_j Z_j$ is not imposed and Z , and in Ψ_A also P , may have zero entries, but every entry is a lower bound: for every entry ψ of the list there is an explicit triple with $c_{\mu\nu}^\lambda \neq 0$, $\ell(\lambda) \leq d$ and $\Delta(\lambda, \mu, \nu) \geq 2k(n - k) + 2\psi$ (Remark 6.1). Moreover $\Delta_{\max}(n, k, 1) = 0$, and $\Delta_{\max}(n, k, d) = 4k(n - k)$ for $d \geq n$.

Corollary D (optimal triples; Conjecture 7.7 of [KŠV26]). *Let $d \geq 2$ and $1 \leq k \leq n - 1$. The maximum $\Delta_{\max}(n, k, d)$ is attained at a triple of one of the forms*

$$(A) \quad \lambda = P \uplus Z, \quad \mu = P, \quad \nu = Z; \quad (B) \quad \lambda = P \uplus Z \uplus (\pi + y), \quad \mu = P \uplus (\pi), \quad \nu = Z \uplus (y),$$

with P, Z, f, π, y as in the corresponding shape of Theorem B (zero parts omitted), and $c_{\mu\nu}^\lambda \geq 1$ for these triples. In form (B), π is a smallest part of μ and y is the smallest part of ν . In particular, Conjecture 7.7 of Klep, Štrelj and Volčič [KŠV26] holds: the maximum is attained at a triple in which at most one part of λ is the sum of a part of μ and a part of ν , while the other parts of λ are distributed in μ and ν .

KSV state the conjecture with $|\mu| = n - k$ and $|\nu| = k$; by the symmetry of Δ and $c_{\mu\nu}^\lambda$ in μ and ν this makes no difference. A triple with $\lambda = \mu \uplus \nu$, such as those of form (A), is called *merged*. For example, $\Delta_{\max}(10, 5, 5) = 72$ is attained at $\lambda = (2, 2, 2, 2, 2)$, $\mu = \nu = (2, 2, 1)$, which has form (B) with $P = Z = (2, 2)$ and $\pi = y = 1$, while the merged triples give at most 70 [KŠV26, Remark 7.9]. The answer (6) is a maximum over a list of polynomial size, each entry of which is an explicit integer; it is not a single closed formula, and KSV remark that a straightforward formula is not to be expected [KŠV26, p. 36]. In the language of Werner states (Section 1.3), the smallest eigenvalue of $\frac{1}{k(n-k)} \sum_{i \in A, j \in B} \text{SWAP}_{ij}$ is $1 - \Delta_{\max}/(2k(n-k)) = -\max \Psi/(k(n-k))$.

1.3 Known cases and related work

The quantum max d -cut problem was introduced by Carlson, Jorquera, Kolla, Kordonowy and Wayland [CJK⁺23], who define the Hamiltonian with the Gell-Mann matrices; Klep, Štrelj and Volčič [KŠV26] (KSV below) show that this definition is equivalent to their Definition 1.5 in terms of swap operators [KŠV26, Section 1.5] and study the problem with the algebra of swap operators and the representation theory of the symmetric groups. They solved it for complete graphs [KŠV26, Theorem 1.6] and for star graphs $K_{n-1,1}$, where the value is $2(n+d-2)$ for $2 \leq d \leq n$ [KŠV26, Theorem 7.2], and reduced the case of $K_{n-k,k}$ to the problem above by a clique decomposition and the Littlewood–Richardson rule [KŠV26, Section 7.2]. For $2k \leq n$ they proved that $\Delta_{\max}(n, k, 2) = 2k(n-k+1)$, which is the theorem of Lieb and Mattis [LM62] [KŠV26, Remark 1.8, Corollary 7.13], and that $\Delta_{\max}(n, k, 3)$ equals $2(k+1)(n-k)$ if $n < 3k$ and $2k(n-k+2)$ if $n \geq 3k$ [KŠV26, Corollary 7.29]. They proved that a merged triple is optimal when $k \leq 4$ or $d \leq 3$ [KŠV26, Proposition 7.8], by a case analysis in their Appendix B; determined the maximum over merged triples, which is attained at balanced μ and ν of one of three explicit heights [KŠV26, Theorem 7.27]; and thus solved the cases $k \leq 4$ [KŠV26, Theorem 1.7, Corollary 7.28]. They observed that $(n, k, d) = (10, 5, 5)$ is the first case in which no merged triple is optimal [KŠV26, Remark 7.9], stated Conjecture 7.7 (supported by computations for $2k \leq n$, $d < n$, $n \leq 26$), and remarked that a straightforward formula for the value is not to be expected. Their identity $\eta(\lambda) = |\lambda|(|\lambda| - 1) - 2c(\lambda)$, with $c(\lambda)$ the content sum [KŠV26, (B.2)], is the starting point of our proof (Lemma 2.1). Section 7 recovers from Theorem C their results for $d = 2$, $d = 3$ and star graphs and their Proposition 7.8; the cases $d \leq 3$ and $k \leq 4$ of the latter follow from a simple criterion for merged optima (Proposition 6.4).

Jakab, Solymos and Zimborás [JSZ22] study the two-sided extendibility of Werner states, which leads to the same eigenvalue problem: with $n_A = n - k$ and $n_B = k$, their parameter α_{n_A, n_B} is the smallest eigenvalue of $\frac{1}{n_A n_B} \sum_{i \in A, j \in B} \text{SWAP}_{ij}$ [KŠV26, Example 1.9], that is, $-\max \Psi/(k(n-k))$. They give a formula for α_{n_A, n_B} for all n_A, n_B, d . For fixed μ, ν they take the dominance-minimal constituent sort $(\mu_i + \nu_{d+1-i})$ (their eq. (11)); they state that the minimum is attained at a pair of balanced partitions with $\lambda = \mu \uplus \nu$ at one of $d-1$ heights (eqs. (13), (14)); they give a closed form for the value of such a pair (eq. (15)) and restrict the heights to three (eq. (16)). KSV report that both variants of (16) give $\alpha = -\frac{1}{3}$ at $(n, k, d) = (6, 3, 4)$, where the exact value is $-\frac{5}{9}$ [KŠV26, Example 1.9], and that no merged triple is optimal at $(10, 5, 5)$ [KŠV26, Remark 7.9]. In our computations (Section 8), with the exact values of Theorem C, for the 1,185 triples with $3 \leq n \leq 20$, $1 \leq k \leq n/2$, $2 \leq d \leq n-1$: the reduction through eq. (11) agrees with the exact value in all cases (as in all cases with $n \leq 30$, $2 \leq d \leq n$ computed in a verification run); the best merged balanced pair (eq. (13), evaluated with the exact Δ at all heights) differs from the exact value in 24 cases, exactly those without a merged optimum (Table 1); eq. (16) as printed, with the three heights around the integer nearest to $n_A/(n_A + n_B)$, differs in 956 cases (943 if ties are rounded up), and with the centre $dn_A/(n_A + n_B)$ suggested in [KŠV26, Example 1.9] in the same 24 cases; the closed form (15) used with (16) differs in 956 cases with the printed centre and in 249 cases with the other one (947 and 251 if ties are rounded up). The second case of (15) as printed differs from the exact value of the merged balanced pair in

11,753 of the 149,530 quadruples $(n, k, d, \hat{d})$ with $n \leq 40$, $k \leq n/2$, $d \leq n-1$, $1 \leq \hat{d} \leq d-1$, and agrees with it in all of them after the sign before the product $(n_A \bmod \hat{d})(n_B \bmod (d-\hat{d}))$ is changed from $-$ to $+$.

For complete graphs the maximum [KŠV26, Theorem 1.6] was also obtained, for Werner states, by Allerstorfer, Christandl, Grinko, Nechita, Ozols, Rochette and Verduyn Lunel [ACG⁺25, Theorem 4.1], as noted in [KŠV26, Section 1.4.3]. Björnberg, Rosengren and Ryan [BRR23] analyse interchange and related spin systems on complete bipartite graphs by Schur–Weyl duality; according to their abstract they compute the free energy and other thermodynamic limits, not the exact maximum for finite n . Štrelkelj [Š25] solves the d -QMC problem for complete tripartite graphs for $d \leq 3$. Taşkın, Gümüş, Işık and Ulvi [TGIU23], the literature reference of the corpus record, give lower and upper bounds, in a generalized dominance order, for the partitions ν with $c_{\mu\nu}^\lambda > 0$ for a given skew shape λ/μ ; we need only the classical upper bound (Lemma 3.1).

2 Notation and the function η

A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ has finitely many nonzero parts; $|\lambda| = \sum_i \lambda_i$, $\ell(\lambda)$ is the number of nonzero parts, and $\lambda \vdash N$ means $|\lambda| = N$. Its diagram is the set of cells $(r, s) \in \mathbb{Z}_{\geq 0}^2$ with $s \leq \lambda_r$ (row r , column s), and $\lambda'_s = \#\{r : \lambda_r \geq s\}$ is the length of column s . We write $\mu \subseteq \lambda$ if $\mu_r \leq \lambda_r$ for all r , equivalently $\mu'_s \leq \lambda'_s$ for all s ; then λ/μ is the set of cells of λ that are not cells of μ . The *content* of the cell (r, s) is $s - r$, and

$$c(\lambda) = \sum_{(r,s) \in \lambda} (s - r)$$

is the content sum. For $|\nu| = |\kappa|$ we write $\nu \leq \kappa$ (ν is dominated by κ) if $\nu_1 + \dots + \nu_i \leq \kappa_1 + \dots + \kappa_i$ for all i .

A *Littlewood–Richardson tableau* of shape λ/μ and content ν is a filling of the cells of λ/μ with positive integers, weakly increasing from left to right in each row and strictly increasing from top to bottom in each column, with ν_v entries equal to v for every v , such that the word obtained by reading the rows from top to bottom, each row from right to left, is a lattice word: in every prefix, for every $v \geq 1$, the letter v occurs at least as often as the letter $v + 1$. The Littlewood–Richardson coefficient $c_{\mu\nu}^\lambda$ is the number of these tableaux [Ful97, Chapter 5], [KŠV26, Section 7.2.1]. Hence $c_{\mu\nu}^\lambda \neq 0$ implies $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + |\nu|$; moreover $c_{\mu\nu}^\lambda = c_{\nu\mu}^\lambda$. An *LR triple* is a triple (λ, μ, ν) with $c_{\mu\nu}^\lambda \neq 0$.

Lemma 2.1 ([KŠV26, (B.2), Lemma B.1]). *Let π be a partition with at most d parts. Then (1), with $n = |\pi|$ and π padded with zeros to d entries, gives*

$$\eta(\pi) = |\pi|(|\pi| - 1) - 2c(\pi);$$

in particular $\eta(\pi)$ does not depend on $d \geq \ell(\pi)$. Consequently, if $|\lambda| = n$, $|\mu| = k$, $|\nu| = m = n - k$, then

$$\Delta(\lambda, \mu, \nu) = 2km - 2F(\lambda, \mu, \nu), \quad F(\lambda, \mu, \nu) = c(\lambda) - c(\mu) - c(\nu). \quad (7)$$

If the fixed n is used in all three values of η , then Δ is replaced by $\Delta - n^2 - 2km$.

Proof. We have $\sum_{i=1}^d (\pi_i - (i-1))^2 = \sum_i \pi_i^2 - 2\sum_i (i-1)\pi_i + \sum_{i=1}^d (i-1)^2$, and the last sum is $d(d-1)(2d-1)/6$. Hence $\eta(\pi) = N^2 - \sum_i \pi_i^2 + 2\sum_i (i-1)\pi_i$ with $N = |\pi|$. On the other hand, row i contributes $\sum_{s=1}^{\pi_i} (s-i) = \pi_i(\pi_i+1)/2 - i\pi_i$ to $c(\pi)$, so $2c(\pi) = \sum_i \pi_i^2 - N - 2\sum_i (i-1)\pi_i$, and $N(N-1) - 2c(\pi) = \eta(\pi)$. Then $\Delta = n(n-1) - k(k-1) - m(m-1) - 2F$, and $n(n-1) - k(k-1) - m(m-1) = (k+m)^2 - k^2 - m^2 = 2km$. With the fixed n , $\eta(\mu)$ and $\eta(\nu)$ increase by $n^2 - k^2$ and $n^2 - m^2$, and $2n^2 - k^2 - m^2 = n^2 + 2km$. $\square$

By Lemma 2.1, $\Delta_{\max}(n, k, d) = 2km - 2F_{\min}(n, k, d)$, where $F_{\min}(n, k, d)$ is the minimum of F over the LR triples with $|\lambda| = n$, $|\mu| = k$ and $\ell(\lambda) \leq d$.

3 The best ν for given $\mu \subseteq \lambda$

Let $\mu \subseteq \lambda$. The column lengths of λ/μ are $b_s = \lambda'_s - \mu'_s$ ($s \geq 1$). Let $\nu^* = \nu^*(\lambda, \mu)$ be the partition whose conjugate is the decreasing rearrangement of $(b_s)_{s \geq 1}$; thus $\nu_v^* = \#\{s : b_s \geq v\}$.

Lemma 3.1. *Let $\mu \subseteq \lambda$ and $\nu^* = \nu^*(\lambda, \mu)$.*

- (a) *Writing $r - \mu'_s$ into every cell (r, s) of λ/μ gives a Littlewood–Richardson tableau of shape λ/μ and content ν^* . Hence $c_{\mu\nu^*}^\lambda \geq 1$.*
- (b) *If $c_{\mu\nu}^\lambda \neq 0$, then $\nu \leq \nu^*$.*
- (c) *If $\nu \leq \nu^*$ and $\nu \neq \nu^*$, then $c(\nu) < c(\nu^*)$.*

Consequently $\max\{c(\nu) : c_{\mu\nu}^\lambda \neq 0\} = c(\nu^*)$, attained only at $\nu = \nu^*$.

Proof. (a) Column s of λ/μ consists of the rows $\mu'_s + 1, \dots, \lambda'_s$, which receive the entries $1, \dots, b_s$ from top to bottom; so the columns strictly increase, and the number of entries equal to v is $\#\{s : b_s \geq v\} = \nu_v^*$. If (r, s) and $(r, s+1)$ are cells of λ/μ , then $r - \mu'_s \leq r - \mu'_{s+1}$ because $\mu'_{s+1} \leq \mu'_s$; so the rows weakly increase. If the cell (r, s) has entry $v \geq 2$, then $r - 1 > \mu'_s$, so $(r - 1, s)$ is a cell of λ/μ with entry $v - 1$, and it is read before (r, s) because it lies in an earlier row. The map $(r, s) \mapsto (r - 1, s)$ is injective, so every prefix of the reading word contains at least as many letters $v - 1$ as letters v . By the Littlewood–Richardson rule, $c_{\mu\nu^*}^\lambda \geq 1$.

(b) A Littlewood–Richardson tableau of content ν exists. Column s of it has b_s strictly increasing entries, of which at most $\min(v, b_s)$ are $\leq v$. Hence $\nu_1 + \dots + \nu_v \leq \sum_s \min(v, b_s) = \sum_{t=1}^v \#\{s : b_s \geq t\} = \nu_1^* + \dots + \nu_v^*$ for every v , and $|\nu| = |\nu^*| = |\lambda| - |\mu|$.

(c) Row i of a partition κ contributes $\kappa_i(\kappa_i + 1)/2 - i\kappa_i$ to $c(\kappa)$, so $c(\kappa) = \frac{1}{2} \sum_i \kappa_i^2 - \sum_i (i - \frac{1}{2})\kappa_i$. Let $\kappa = \nu^*$, $\delta_i = \kappa_i - \nu_i$, $D_j = \delta_1 + \dots + \delta_j \geq 0$ and $w_i = \kappa_i + \nu_i$, which is non-increasing in i ; let L exceed the lengths of ν and κ , so that $D_L = 0$. By Abel summation,

$$c(\kappa) - c(\nu) = \frac{1}{2} \sum_{i \leq L} \delta_i w_i - \sum_{i \leq L} (i - \frac{1}{2}) \delta_i = \sum_{j < L} D_j \left(\frac{1}{2} (w_j - w_{j+1}) + 1 \right) \geq \sum_{j < L} D_j,$$

and $\sum_j D_j > 0$ unless $\nu = \kappa$. The last statement follows from (a), (b) and (c). $\square$

Remark 3.2. Parts (a) and (b) are classical. By the symmetry $c_{\mu\nu}^\lambda = c_{\mu'\nu'}^{\lambda'}$ they are equivalent to the statement that the decreasingly sorted row lengths of a skew shape form the dominance-smallest content of its Littlewood–Richardson tableaux, which Jakab, Solymos and Zimborás quote from Azenhas [Aze99] and use for their eq. (11) [JSZ22, Supplemental material, Section B]. We have included the short proofs to keep the note self-contained.

Lemma 3.3 (column formula). *Let $\mu \subseteq \lambda$, $a_s = \mu'_s$ and $b_s = \lambda'_s - \mu'_s$, and put $D(b) = \sum_{u < s} (b_s - b_u)^+$, where $t^+ = \max(t, 0)$. Then*

$$c(\lambda) - c(\mu) - c(\nu^*(\lambda, \mu)) = D(b) - \sum_s a_s b_s.$$

Proof. Column s of a partition κ contains the cells (r, s) , $1 \leq r \leq \kappa'_s$, with contents $(s-1) - (r-1)$; hence $c(\kappa) = \sum_s ((s-1)\kappa'_s - \binom{\kappa'_s}{2})$. Apply this to $\lambda' = a + b$, $\mu' = a$ and $\nu^{*'} = b^\downarrow$, the decreasing rearrangement of b . Since $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy$ and $\sum_s \binom{b_s^\downarrow}{2} = \sum_s \binom{b_s}{2}$,

$$c(\lambda) - c(\mu) - c(\nu^*) = \sum_s (s-1)(b_s - b_s^\downarrow) - \sum_s a_s b_s.$$

For a finite set $S \subset \mathbb{Z}_{>0}$ let $\text{inv}(S) = \#\{(u, s) : u < s, u \notin S, s \in S\}$; if $S = \{s_1 < \dots < s_q\}$, then $\text{inv}(S) = \sum_i (s_i - i) = \sum_{s \in S} (s-1) - \sum_{s=1}^q (s-1)$. The level sets $S_t = \{s : b_s \geq t\}$, $t \geq 1$, satisfy $b_s = \#\{t : s \in S_t\}$ and $b_s^\downarrow = \#\{t : s \leq |S_t|\}$, hence

$$\sum_s (s-1)(b_s - b_s^\downarrow) = \sum_{t \geq 1} \text{inv}(S_t) = \sum_{u < s} \#\{t : b_u < t \leq b_s\} = D(b). \quad \square$$

With $\Phi(a, b) = \sum_s a_s b_s - D(b)$, Lemmas 2.1, 3.1 and 3.3 give

$$F_{\min}(n, k, d) = -\max\{\Phi(\mu', \lambda' - \mu') : \lambda \vdash n, \ell(\lambda) \leq d, \mu \subseteq \lambda, |\mu| = k\}, \quad (8)$$

and for given λ and μ the minimum of $F(\lambda, \mu, \nu)$ over the LR triples is attained at $\nu = \nu^*(\lambda, \mu)$ and only there.

4 The reduction: proof of Theorem A

For a sequence $a = (a_s)_{s \geq 1}$ and a finite set $S \subset \mathbb{Z}_{>0}$ put $\varphi_a(S) = \sum_{s \in S} a_s - \text{inv}(S)$, with inv as in the proof of Lemma 3.3.

Lemma 4.1 (families of sets). *Let $b = (b_s)_{s \geq 1}$ be a finitely supported sequence of nonnegative integers, and let $S_1, \dots, S_q$ be finite subsets of $\mathbb{Z}_{>0}$ with $\#\{t : s \in S_t\} = b_s$ for all s . Then $\sum_t \text{inv}(S_t) \geq D(b)$, with equality if the S_t are the level sets $\{s : b_s \geq t\}$. Consequently $\sum_t \varphi_a(S_t) \leq \Phi(a, b)$ for every sequence a , with equality for the level sets.*

Proof. For $u < s$, the number of t with $s \in S_t$ and $u \notin S_t$ is at least $\#\{t : s \in S_t\} - \#\{t : u \in S_t\} = b_s - b_u$, and at least 0. Summing over $u < s$ gives $\sum_t \text{inv}(S_t) \geq D(b)$. Equality for the level sets was shown in the proof of Lemma 3.3. Finally $\sum_t \sum_{s \in S_t} a_s = \sum_s a_s b_s$. $\square$

Lemma 4.2 (intervals). *Let $a_1 \geq a_2 \geq \dots$, let $p \geq 0$, and let $S \subseteq \{p+1, p+2, \dots\}$ with $|S| = q$. Then $\varphi_a(S) \leq \sum_{s=p+1}^{p+q} a_s - pq$, with equality for $S = \{p+1, \dots, p+q\}$.*

Proof. Write $S = \{s_1 < \dots < s_q\}$. Then $s_i \geq p+i$, so $a_{s_i} \leq a_{p+i}$ and $\text{inv}(S) = \sum_i (s_i - i) \geq pq$; for the interval both are equalities. $\square$

Proof of Theorem A. By Lemma 2.1 and (8) it suffices to show that the maximum Φ^* of $\Phi(\mu', \lambda' - \mu')$ over $\lambda \vdash n$ with $\ell(\lambda) \leq d$ and $\mu \subseteq \lambda$, $|\mu| = k$, equals the maximum Ψ^* of $\Psi(\mu, x)$ in (5).

($\Phi^* \leq \Psi^*$.) Let λ and μ be as above, $a = \mu'$, $b = \lambda' - \mu'$, and $S_t = \{s : b_s \geq t\}$. If $s \in S_t$, then $\mu'_s + t \leq \lambda'_s \leq d$; so $S_t = \emptyset$ for $t > d$, and for $t \leq d$ fewer than $d - t + 1$ rows of μ reach column s , that is, $s > p_t := \mu_{d+1-t}$ (μ padded with zeros to d entries). By Lemma 4.1 (equality case) and Lemma 4.2 applied with $p = p_t$,

$$\Phi(a, b) = \sum_{t=1}^d \varphi_a(S_t) \leq \sum_{t=1}^d \sum_{s=p_t+1}^{p_t+|S_t|} (\mu'_s - p_t) = \Psi(\mu, x),$$

where $x_r = |S_{d+1-r}|$, so that $p_{d+1-r} = \mu_r$. Here $\sum_r x_r = \sum_t |S_t| = \sum_s b_s = n - k$; hence $\Phi(a, b) \leq \Psi^*$.
 $(\Phi^* \geq \Psi^*)$ Let $\mu \vdash k$ with $\ell(\mu) \leq d$ and $x \in \mathbb{Z}_{\geq 0}^d$ with $\sum_r x_r = m$, and let λ be the decreasing rearrangement of $(\mu_r + x_r)_{r \leq d}$. Then $|\lambda| = n$, $\ell(\lambda) \leq d$, and $\lambda'_s = \#\{r : \mu_r + x_r \geq s\} \geq \#\{r : \mu_r \geq s\} = \mu'_s$, so $\mu \subseteq \lambda$. Since $\mu_r \geq s$ implies $\mu_r + x_r \geq s$,

$$b_s = \lambda'_s - \mu'_s = \#\{r : \mu_r < s \leq \mu_r + x_r\} = \#\{r : s \in T_r\}, \quad T_r = \{\mu_r + 1, \dots, \mu_r + x_r\}.$$

By Lemma 4.1 and the equality case of Lemma 4.2, $\Phi(\mu', b) \geq \sum_r \varphi_{\mu'}(T_r) = \Psi(\mu, x)$. Hence $\Phi^* \geq \Psi^*$.

So $\Phi^* = \Psi^*$, and (5) follows. If (μ, x) is a maximizer, the triple $(\lambda, \mu, \nu^*(\lambda, \mu))$ just constructed is an LR triple with $\ell(\lambda) \leq d$ by Lemma 3.1(a), and $F = -\Phi(\mu', b) \leq -\Psi^* = F_{\min}$ by Lemma 3.3; so it is optimal. $\square$

Remark 4.3 (greedy maximization; cells). (a) For fixed μ the weights $\mu'_s - \mu_r$, $s = \mu_r + 1, \mu_r + 2, \dots$, of row r are non-increasing in s , and $\Psi(\mu, x)$ is the sum of the first x_r weights of each row r . Hence $\max_x \Psi(\mu, x)$ is the sum of the m largest elements of the multiset of all weights of all d rows; Theorem A thus reduces the problem to a maximization over the partitions $\mu \vdash k$ with $\ell(\mu) \leq d$. (b) If the cells (r, s) with $\mu_r < s \leq \mu_r + x_r$ are called *T-cells*, then $\Psi(\mu, x)$ is the number of pairs (cell of μ , T-cell) in a common column minus the number of such pairs in a common row. (c) The proof shows that every pair (μ, x) , optimal or not, yields an LR triple $(\lambda, \mu, \nu^*(\lambda, \mu))$ with $\ell(\lambda) \leq d$ and $\Delta \geq 2km + 2\Psi(\mu, x)$.

5 Configurations: proof of Theorem B

Definition 5.1. Let $d \geq 1$ and $k, m \geq 0$. A *configuration* is a d -tuple $C = ((p_1, y_1), \dots, (p_d, y_d))$ of pairs of nonnegative integers with $\sum_r p_r = k$ and $\sum_r y_r = m$. The pairs are the *rows* of C , and $\ell_r = p_r + y_r$. Let $\mu'_s = \#\{r : p_r \geq s\}$ (the conjugate of the decreasing rearrangement μ of the p_r), and

$$\Psi(C) = \sum_{r=1}^d \sum_{s=p_r+1}^{\ell_r} (\mu'_s - p_r).$$

The cells (r, s) with $p_r < s \leq \ell_r$ are the *T-cells* of row r ; the T-cell (r, s) has *weight* $\mu'_s - p_r$, and $\Psi(C)$ is the total weight of all T-cells. A row is of type E if $p_r = y_r = 0$, of type T if $p_r = 0 < y_r$, of type P if $p_r > 0 = y_r$, and *mixed* if $p_r > 0$ and $y_r > 0$.

Permuting the rows does not change $\Psi(C)$. If $\mu \vdash k$, $\ell(\mu) \leq d$ and $x \in \mathbb{Z}_{\geq 0}^d$ with $\sum_r x_r = m$, then $((\mu_r, x_r))_{r \leq d}$ is a configuration with value $\Psi(\mu, x)$, and every configuration arises in this way after a permutation of its rows. So the maximum in (5) is the maximum of $\Psi(C)$ over all configurations, and Theorem B is a statement about configurations that maximize Ψ (*maximizing configurations*). Figure 1 shows an example.

Lemma 5.2 (pairwise form). *For every configuration,*

$$\Psi(C) = \sum_{r=1}^d \left(\sum_{i \neq r} (\min(\ell_r, p_i) - p_r)^+ - p_r y_r \right).$$

Proof. $\sum_{s=p_r+1}^{\ell_r} \mu'_s = \sum_i \#\{s : p_r < s \leq \ell_r, s \leq p_i\} = \sum_i (\min(\ell_r, p_i) - p_r)^+$, the term $i = r$ is 0, and $\sum_{s=p_r+1}^{\ell_r} p_r = p_r y_r$. $\square$

We say that row r *receives* $(\min(\ell_r, p_i) - p_r)^+$ from row i . A row with $y_r = 0$ receives nothing, since then $\ell_r = p_r$, and a row with $p_i = 0$ gives nothing.

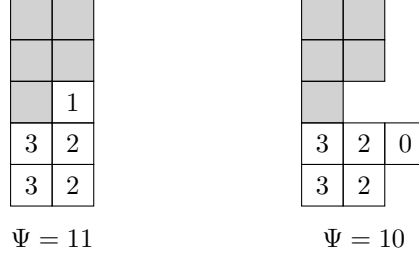


Figure 1: The case $(n, k, d) = (10, 5, 5)$. Shaded: the cells (r, s) with $s \leq p_r$; white: the T-cells, with their weights $\mu'_s - p_r$. Left: the rows $(2, 0), (2, 0), (1, 1), (0, 2), (0, 2)$, of shape (B) with $P = Z = (2, 2)$ and $\pi = y = 1$; they give $\lambda = (2, 2, 2, 2, 2)$, $\mu = \nu = (2, 2, 1)$ and $\Delta = 50 + 2 \cdot 11 = 72$. Right: the best configuration without a mixed row, $(2, 0), (2, 0), (1, 0), (0, 3), (0, 2)$; it gives the merged triple $\lambda = (3, 2, 2, 2, 1)$, $\mu = (2, 2, 1)$, $\nu = (3, 2)$ and $\Delta = 70$.

Lemma 5.3 (moving T-cells). *Let $u \neq v$ be rows of a configuration.*

- (a) *If $\ell_v \leq p_u$, then replacing $(p_u, y_u), (p_v, y_v)$ by $(p_u, 0), (p_v, y_v + y_u)$ does not decrease Ψ .*
- (b) *If $y_u \geq 1$, then replacing them by $(p_u, y_u - 1), (p_v, y_v + 1)$ changes Ψ by $(\mu'_{\ell_v+1} - p_v) - (\mu'_{\ell_u} - p_u)$.*

Proof. The numbers p_r , hence μ' , do not change, and only the moved T-cells change their weights. (a) The j -th T-cell of row u moves from column $p_u + j$ to column $\ell_v + j \leq p_u + j$, and its weight changes from $\mu'_{p_u+j} - p_u$ to $\mu'_{\ell_v+j} - p_v$, which is not smaller, because μ' is non-increasing and $p_v \leq \ell_v \leq p_u$. (b) The last T-cell of row u , in column ℓ_u , moves to column $\ell_v + 1$. $\square$

Lemma 5.4 (one mixed row). *Some maximizing configuration has at most one mixed row.*

Proof. Let C be a maximizing configuration with the smallest number of mixed rows, and suppose that it has at least two. Let a be a mixed row with the largest p , and b another mixed row; so $p_a \geq p_b \geq 1$.

Step 1. If a row $v \neq a$ that is not of type P has $\ell_v \leq p_a$, then Lemma 5.3(a) with $u = a$ turns a into a P-row, while v keeps its type or, if it was an E-row, becomes a T-row; this gives a maximizing configuration with fewer mixed rows, which is impossible. Hence there is no E-row (an E-row has $\ell = 0$), every T-row r has length $z_r := y_r > p_a$, and every mixed row $c \neq a$ satisfies $\ell_c > p_a$.

Step 2. For an integer t let C_t be obtained from C by replacing the rows a and b by $a_t = (p_a + t, y_a - t)$ and $b_t = (p_b - t, y_b + t)$; the lengths ℓ_a and ℓ_b do not change. Let

$$\begin{aligned} t_+ &= \min(\{y_a, p_b, \ell_b - p_a\} \cup \{\ell_c - p_a\} \cup \{z_r - p_a\}), \\ t_- &= -\min(\{p_a, y_b, \ell_a - p_b\} \cup \{\ell_c - p_b\} \cup \{z_r - p_b\}), \end{aligned}$$

where c runs over the mixed rows other than a and b , and r over the T-rows. By Step 1, $t_- < 0 < t_+$. For $t_- \leq t \leq t_+$ all entries of C_t are nonnegative. We show that $t \mapsto \Psi(C_t)$ is convex on this interval of integers. By Lemma 5.2, the terms of $\Psi(C_t)$ that depend on t are the following.

- (i) What a_t and b_t receive from a row $i \notin \{a, b\}$: $(\min(\ell_a, p_i) - p_a - t)^+$ and $(\min(\ell_b, p_i) - p_b + t)^+$, convex functions of t .
- (ii) What a_t receives from b_t : since $p_b - t \leq \ell_a$ for $t \geq t_-$, it is $(p_b - p_a - 2t)^+$; and what b_t receives from a_t : since $p_a + t \leq \ell_b$ for $t \leq t_+$, it is $(p_a - p_b + 2t)^+$. Both are convex.

(iii) What the other rows receive from a_t and b_t . P-rows receive nothing. A T-row r receives $\min(z_r, p_a + t) + \min(z_r, p_b - t) = p_a + p_b$, since $p_a + t \leq z_r$ and $p_b - t \leq z_r$ on the interval. A mixed row $c \neq a, b$ receives $(p_a + t - p_c)^+ + (p_b - t - p_c)^+$, since $p_a + t \leq \ell_c$ and $p_b - t \leq \ell_c$ on the interval; this is convex.

(iv) The terms $-(p_a + t)(y_a - t) - (p_b - t)(y_b + t)$, which equal a constant plus a linear function of t plus $2t^2$.

Hence $\max(\Psi(C_{t_-}), \Psi(C_{t_+})) \geq \Psi(C_0) = \Psi(C)$, and C_{t_+} or C_{t_-} is maximizing. Consider C_{t_+} (the case of C_{t_-} is the same with the roles of a and b exchanged). If $t_+ = y_a$, then a_t is a P-row; if $t_+ = p_b$, then b_t is a T-row. Otherwise a row v among b_t , the mixed rows c and the T-rows, of type T or mixed, has $\ell_v = p_a + t_+$, the first entry of a_t ; then Lemma 5.3(a) with $u = a_t$ turns a_t into a P-row, if it is not one already, and v keeps its type. In every case we obtain a maximizing configuration with fewer mixed rows, a contradiction. $\square$

Lemma 5.5 (balancing). *Let $u \neq v$ be rows of a configuration.*

- (a) *If $p_u = p_v = 0$, replacing (y_u, y_v) by $\text{bal}(y_u + y_v, 2)$ does not decrease Ψ .*
- (b) *If $y_u = y_v = 0$ and $p_u, p_v \geq p_w$ for every row w with $y_w > 0$, replacing (p_u, p_v) by $\text{bal}(p_u + p_v, 2)$ does not decrease Ψ .*

Proof. If h is concave on an interval of integers containing integers $\alpha \geq \beta + 2$, then $h(\alpha - 1) + h(\beta + 1) \geq h(\alpha) + h(\beta)$. Repeating such steps turns (α, β) into the balanced pair; each step keeps both values in $[\beta, \alpha]$ and decreases $\alpha^2 + \beta^2$. (a) The rows u and v give nothing, pay nothing, and receive $h(y_u) + h(y_v)$, where $h(y) = \sum_{i \notin \{u, v\}} \min(y, p_i)$ is concave; no other term depends on (y_u, y_v) . (b) The rows u and v receive and pay nothing. To a row w with $y_w > 0$ they give $g_w(p_u) + g_w(p_v)$, where $g_w(p) = (\min(\ell_w, p) - p_w)^+$ equals $\min(\ell_w, p) - p_w$ for $p \geq p_w$, a concave function on $[p_w, \infty)$, which contains all values occurring in the balancing steps. Rows with $y_w = 0$ receive nothing. $\square$

Proof of Theorem B. By Lemma 5.4 there is a maximizing configuration C with at most one mixed row.

Suppose that C has a mixed row $w = (\pi, y)$. As long as some row $v \neq w$ has $\ell_v \leq \pi$ and $p_v < \pi$, apply Lemma 5.3(a) with $u = w$. If $p_v = 0$, the new configuration has no mixed row; otherwise v is a P-row with $0 < p_v < \pi$, and it becomes the only mixed row, with a smaller p . So the process stops, with a maximizing configuration. If this still has a mixed row $w = (\pi, y)$, then every other row v has $\ell_v > \pi$ or $p_v \geq \pi$. Hence every P-row has $p \geq \pi$ (for a P-row $\ell_v = p_v$), there is no E-row, and every T-row has length larger than π . Now balance the P-rows pairwise by Lemma 5.5(b), which applies because the rows with $y > 0$ are w (with $p = \pi$) and the T-rows (with $p = 0$), and the T-rows pairwise by Lemma 5.5(a). Every step keeps the configuration maximizing and keeps the new values between the two old ones, so the properties just listed persist; the P-rows end as $\text{bal}(k - \pi, f)$ and the T-rows as $\text{bal}(m - y, g)$, where f is the number of P-rows and $g = d - 1 - f$. So $\pi \leq \min_i P_i$, and $y = m$ if $g = 0$. Moreover:

- $f \geq 1$. Otherwise $\mu = (\pi)$, so $\mu'_s = 0$ for $s > \pi$, and there is a T-row $v = (0, z)$ since $d \geq 2$. By Lemma 5.3(b), moving the last T-cell of w (in column $\pi + y$, of weight $-\pi$) to the end of v (weight $\mu'_{z+1} \geq 0$) increases Ψ by at least $\pi > 0$, contradicting maximality.
- $\pi + y \leq \min_j Z_j$. If a T-row $v = (0, z)$ had $z < \pi + y$, then by Lemma 5.3(b) moving the last T-cell of w from column $\pi + y$ to column $z + 1 \leq \pi + y$ would change Ψ by $\mu'_{z+1} - (\mu'_{\pi+y} - \pi) \geq \pi > 0$.

So the configuration has shape (B).

If C has no mixed row, initially or after the process, balance the P-rows by Lemma 5.5(b) (all rows with $y > 0$ now have $p = 0$) and the rows with $p = 0$ by Lemma 5.5(a). Let f be the number of P-rows. They form $\text{bal}(k, f)$ with $1 \leq f \leq k$, since $k \geq 1$. The $m \geq 1$ T-cells lie in rows with $p = 0$, so $f \leq d - 1$, and these $d - f$ rows form $\text{bal}(m, d - f)$. This is shape (A). $\square$

6 The value and the optimal triples: proofs of Theorem C and Corollary D

Proof of Theorem C. Let $d \geq 2$ and $1 \leq k \leq n - 1$. For every admissible f let C_A be the configuration with the rows $(P_i, 0)$, $i \leq f$, and $(0, Z_j)$, $j \leq d - f$, where $(P, Z) = (\text{bal}(k, f), \text{bal}(m, d - f))$. By Lemma 5.2, the row $(0, Z_j)$ receives $\sum_i \min(Z_j, P_i)$ and the other rows receive nothing, so $\Psi(C_A) = \Psi_A(f)$. For every admissible (f, π, y) let C_B be the configuration with the rows $(P_i, 0)$, (π, y) and $(0, Z_j)$, where $(P, Z) = (\text{bal}(k - \pi, f), \text{bal}(m - y, g))$. The row $(0, Z_j)$ receives $\sum_i \min(Z_j, P_i) + \min(Z_j, \pi)$; the mixed row receives $\sum_i (\min(\pi + y, P_i) - \pi)^+ = \sum_i \min(y, P_i - \pi)$, because $P_i \geq \pi$, and pays πy ; the rows $(P_i, 0)$ receive nothing. So $\Psi(C_B) = \Psi_B(f, \pi, y)$. Since C_A and C_B are configurations (they have d rows, and the sums are k and m), every entry of the list is at most the maximum Ψ^* of Ψ . Conversely, Theorem B gives a maximizing configuration of shape (A) or (B); its value is $\Psi_A(f)$ or $\Psi_B(f, \pi, y)$ with parameters in the stated ranges, since $\pi \leq \min_i P_i = \lfloor (k - \pi)/f \rfloor$, $y \leq m$, and $y = m$ if $g = 0$. So the maximum of the list is Ψ^* , and (6) follows from Theorem A. There are at most $d - 1$ values of f , and for each f at most k values of π and m values of y .

For $d = 1$ the only pair in Theorem A is $\mu = (k)$, $x = (m)$, and $\Psi = \sum_{s=k+1}^n (0 - k) = -km$; so $\Delta_{\max}(n, k, 1) = 0$. For every configuration, Lemma 5.2 and $(\min(\ell_r, p_i) - p_r)^+ \leq \min(y_r, p_i) \leq y_r p_i$ give $\Psi(C) \leq \sum_r \sum_i y_r p_i = km$. If $d \geq n$, then $k \leq d - 1$ and $d - k \geq m$, so $\text{bal}(k, k) = (1, \dots, 1)$ and $\text{bal}(m, d - k)$ consists of m ones and $d - k - m$ zeros; hence $\Psi_A(k) = km$, and $\Delta_{\max}(n, k, d) = 4km$. $\square$

Remark 6.1 (every entry is realized). By Remark 4.3(c), the configuration C_A or C_B of an entry ψ of the list, read as a pair (μ, x) , yields the LR triple $(\lambda, \mu, \nu^*(\lambda, \mu))$ with λ the decreasing rearrangement of $\mu + x$, $\ell(\lambda) \leq d$ and $\Delta \geq 2k(n - k) + 2\psi$. For the entries that satisfy the conditions of Theorem B, this is the triple of Corollary D. The bound $\Delta_{\max}(n, k, d) \leq 4k(n - k)$ for all d , shown in the proof, is also clear from (2), since every SWAP_{ij} has eigenvalues ± 1 .

Proof of Corollary D. Take a maximizing configuration of shape (A) or (B) (Theorem B), read it as a pair (μ, x) , and let λ be the decreasing rearrangement of $\mu + x$; then $\lambda = P \uplus Z$, resp. $\lambda = P \uplus Z \uplus (\pi + y)$, and $\mu = P$, resp. $\mu = P \uplus (\pi)$. By Theorem A the triple $(\lambda, \mu, \nu^*(\lambda, \mu))$ is optimal and $c_{\mu\nu^*}^\lambda \geq 1$. It remains to show that $\nu^* = Z$, resp. $\nu^* = Z \uplus (y)$. As in the proof of Theorem A, the column lengths of λ/μ are $b_s = \#\{r : s \in T_r\}$, where the rows $(P_i, 0)$ contribute $T_r = \emptyset$, a row $(0, Z_j)$ contributes $\{1, \dots, Z_j\}$, and the mixed row contributes $\{\pi + 1, \dots, \pi + y\}$. Let $Z_{(1)} \geq Z_{(2)} \geq \dots$ be the entries of Z , and recall that $\nu_t^* = \#\{s : b_s \geq t\}$. In shape (A), $\{s : b_s \geq t\} = \{1, \dots, Z_{(t)}\}$, so $\nu^* = Z$. In shape (B), every $Z_j \geq \pi + y$, so $b_s = g$ for $s \leq \pi$, $b_s = g + 1$ for $\pi < s \leq \pi + y$, and $b_s = \#\{j : Z_j \geq s\} \leq g$ for $s > \pi + y$. Hence $\{s : b_s \geq t\} = \{1, \dots, Z_{(t)}\}$ for $t \leq g$, since $Z_{(t)} \geq \pi + y$, and $\{s : b_s \geq g + 1\} = \{\pi + 1, \dots, \pi + y\}$. So $\nu^* = (Z_{(1)}, \dots, Z_{(g)}, y) = Z \uplus (y)$, in decreasing order because $y < \pi + y \leq Z_{(g)}$. This also shows that y is the smallest part of ν , and π is a smallest part of μ because $P_i \geq \pi$. In form (A) every part of λ is a part of μ or of ν ; in form (B) the same holds except for the single part $\pi + y$, the sum of the part π of μ and the part y of ν . This is the statement of Conjecture 7.7 of [KŠV26]. $\square$

Remark 6.2 (λ with exactly d parts). If $2 \leq d \leq n$ and $1 \leq k \leq n - 1$, some optimal triple has $\ell(\lambda) = d$, so the maximum does not change if λ is required to have exactly d nonzero parts. Indeed,

take a maximizing configuration as in Theorem B. In shape (B), $P_i \geq \pi \geq 1$ and $Z_j \geq \pi + y \geq 2$, so λ has $f + 1 + g = d$ nonzero parts. In shape (A) every $P_i \geq 1$. If some Z_j were 0, then all $Z_j \leq 1$, so there would be exactly m rows $(0, 1)$ and at least one row $(0, 0)$, and $k = n - m \geq d - m > f$ would give some $P_i \geq 2$. Replacing that row by $(P_i - 1, 0)$ and one row $(0, 0)$ by $(1, 0)$ keeps the m T-cells in column 1 and raises the weight of each of them, the number of rows with $p \geq 1$, from f to $f + 1$, contradicting maximality. (For $d = 1$, $\lambda = (n)$.)

Remark 6.3 (merged triples). For all partitions $\mu \vdash k$ and $\nu \vdash m$ with $\ell(\mu) + \ell(\nu) \leq d$, the merged triple satisfies $c_{\mu\nu}^{\mu \uplus \nu} \geq 1$ and

$$\Delta(\mu \uplus \nu, \mu, \nu) = 2km + 2 \sum_{i,j} \min(\mu_i, \nu_j) :$$

the conjugate of $\mu \uplus \nu$ is $\mu' + \nu'$, so $b = \nu'$, $D(b) = 0$ and $\nu^*(\mu \uplus \nu, \mu) = \nu$ in Lemmas 3.1 and 3.3, and $\sum_s \mu'_s \nu'_s = \sum_{i,j} \min(\mu_i, \nu_j)$. In particular

$$\Psi_A(f) = \frac{1}{2}(\Delta(\mu \uplus \nu, \mu, \nu) - 2k(n - k)) \quad \text{for } \mu = \text{bal}(k, f), \nu = \text{bal}(m, d - f)$$

(zero parts omitted). Hence the maximum of Δ over the merged triples with $\ell(\mu) + \ell(\nu) \leq d$ is $2km + 2 \max_f \Psi_A(f)$: such a triple, read as the configuration with the rows $(\mu_i, 0)$, the rows $(0, \nu_j)$ and $d - \ell(\mu) - \ell(\nu)$ rows $(0, 0)$, has the value $\sum_{i,j} \min(\mu_i, \nu_j)$ by Lemma 5.2, and balancing its rows with $p > 0$ by Lemma 5.5(b) (all rows with $y > 0$ have $p = 0$) and its rows with $p = 0$ by Lemma 5.5(a) does not decrease this value and ends at the configuration C_A of $\Psi_A(f)$ with $f = \ell(\mu)$. Klep, Štrekelj and Volčič [KŠV26, Theorem 7.27] show that, for $n \geq 2k$ and $n > d$, this maximum is attained at one of three explicit heights (their height of the partition of $n - k$ is our $d - f$). For example, at $(n, k, d) = (6, 3, 4)$ the values $\Psi_A(1), \Psi_A(2), \Psi_A(3)$ are 3, 5, 3, and every Ψ_B is at most 4; so $\Delta_{\max} = 18 + 10 = 28$, attained at $\lambda = (2, 2, 1, 1)$, $\mu = \nu = (2, 1)$, as in [KŠV26, Example 1.9].

The following criterion shows when merged triples suffice.

Proposition 6.4 (merged optima). *Let $d \geq 2$ and $1 \leq k \leq n - 1$.*

- (a) *If $f = d - 1$ (so $g = 0$ and $y = m$), then $\Psi_B(f, \pi, m) < \Psi_A(d - 1)$ for every admissible π .*
- (b) *If $f \leq d - 2$ and $\sum_{i=1}^f \min(y, P_i - \pi) \leq \pi y$, where $P = \text{bal}(k - \pi, f)$, then $\Psi_B(f, \pi, y) \leq \Psi_A(f + 1)$. This holds in particular if $f = 1$.*
- (c) *If $d \leq 3$ or $k \leq 4$, then $\Delta_{\max}(n, k, d) = 2km + 2 \max_f \Psi_A(f)$, and a merged triple with balanced μ and ν is optimal.*

Proof. (a) The i -th entry of $\text{bal}(N, f)$ is $\lfloor (N + f - i)/f \rfloor$, which is non-decreasing in N ; so $P = \text{bal}(k - \pi, f)$ and $P' = \text{bal}(k, f)$ satisfy $P_i \leq P'_i$. With $g = 0$, $\Psi_B(f, \pi, m) = \sum_i \min(m, P_i - \pi) - \pi m \leq \sum_i \min(m, P'_i) - \pi m = \Psi_A(d - 1) - \pi m$, because $\text{bal}(m, 1) = (m)$.

(b) Let C' consist of the rows $(P_i, 0)$, $i \leq f$, $(\pi, 0)$, $(0, Z_1 + y)$ and $(0, Z_j)$, $2 \leq j \leq g$, where $g = d - 1 - f \geq 1$. It is a configuration without mixed rows, with $f + 1$ rows with $p > 0$, and by Lemma 5.2

$$\begin{aligned} \Psi(C') &= \sum_j \left(\sum_i \min(P_i, Z'_j) + \min(\pi, Z'_j) \right) \geq \sum_j \left(\sum_i \min(P_i, Z_j) + \min(\pi, Z_j) \right) \\ &= \Psi_B(f, \pi, y) + \pi y - \sum_i \min(y, P_i - \pi), \end{aligned}$$

where $Z' = (Z_1 + y, Z_2, \dots, Z_g)$. Balancing C' (Lemma 5.5; the rows with $y > 0$ have $p = 0$) gives the configuration C_A of $\Psi_A(f + 1)$ without decreasing Ψ . If $f = 1$, then $\min(y, P_1 - \pi) \leq y \leq \pi y$.

(c) By Theorem B there is a maximizing configuration of shape (A) or (B). Suppose that it has shape (B). By (a), $g \geq 1$. If $d \leq 3$, then $1 \leq f = d-1-g \leq d-2 \leq 1$, so $f = 1$ and (b) applies. If $k \leq 4$ and $f \geq 2$, then $(f+1)\pi \leq k \leq 4$ gives $\pi = 1$ and $\sum_i \min(y, P_i - 1) \leq \sum_i (P_i - 1) = k - 1 - f \leq 1 \leq y$, so (b) applies; for $f = 1$ (b) applies as well. So the maximum of Ψ is $\max_f \Psi_A(f)$, and by Remark 6.3 the merged triple with $\mu = \text{bal}(k, f)$ and $\nu = \text{bal}(m, d - f)$ for a maximizing f is optimal. $\square$

7 Known cases and examples

Proposition 7.1. *Let $1 \leq k \leq n/2$ and $m = n - k$.*

- (a) $\Delta_{\max}(n, k, 2) = 2k(n - k + 1)$.
- (b) $\Delta_{\max}(n, k, 3) = 2(k + 1)(n - k)$ if $n < 3k$, and $2k(n - k + 2)$ if $n \geq 3k$.
- (c) $\Delta_{\max}(n, 1, d) = 2(n + d - 2)$ for $2 \leq d \leq n$.
- (d) Let $d \geq 2$. If $k \leq 4$ or $d \leq 3$, a merged triple with balanced μ and ν is optimal. (For $d = 1$ the only triple is $((n), (k), (n - k))$, which is not merged.)

Part (a) is the theorem of Lieb and Mattis [LM62] for $d = 2$, as noted in [KŠV26, Remark 1.8, Corollary 7.13]; (b) is [KŠV26, Corollary 7.29]; (c) is [KŠV26, Theorem 7.2]; (d) is [KŠV26, Proposition 7.8], with balancedness as in [KŠV26, Lemma 7.10]. Together with [KŠV26, Theorem 7.27], (d) gives their solution of the cases $k \leq 4$ [KŠV26, Corollary 7.28].

Proof. (a) By Proposition 6.4(c), the maximum of Ψ is $\Psi_A(1) = \min(k, m) = k$. (b) By Proposition 6.4(c), it is $\max(\Psi_A(1), \Psi_A(2))$, where $\Psi_A(1) = \min(k, \lceil m/2 \rceil) + \min(k, \lfloor m/2 \rfloor)$ and $\Psi_A(2) = \sum_i \min(P_i, m) = k$, because $P_i \leq k \leq m$. If $m \geq 2k$, then $\Psi_A(1) = 2k$. If $m < 2k$, then $\lceil m/2 \rceil \leq k$ and $\lfloor m/2 \rfloor < k$, so $\Psi_A(1) = m \geq k$. (c) For $k = 1$ there is no admissible π , because $\pi \leq \lfloor (1 - \pi)/f \rfloor$ fails for $\pi \geq 1$. Further $\text{bal}(1, f) = (1, 0, \dots, 0)$, so $\Psi_A(f) = \#\{j : Z_j \geq 1\} = \min(m, d - f) \leq d - 1$, with equality for $f = 1$ because $m = n - 1 \geq d - 1$. (d) is Proposition 6.4(c). $\square$

Examples. For $(n, k, d) = (6, 3, 4)$ see Remark 6.3; this is [KŠV26, Example 1.9], and the Werner-state parameter of Section 1.3 is $-\max \Psi / (k(n - k)) = -\frac{5}{9}$. For $(10, 5, 5)$, Figure 1 shows the optimal configuration and the best one without a mixed row (72 against 70, as in [KŠV26, Remark 7.9]). For $(14, 7, 5)$ the maximum is $2 \cdot 49 + 2 \cdot 15 = 128$, attained at $\lambda = (3, 3, 3, 3, 2)$, $\mu = \nu = (3, 3, 1)$ (form (B) with $P = Z = (3, 3)$, $\pi = y = 1$), while merged triples give at most 126. Table 1 lists all triples (n, k, d) with $n \leq 20$, $k \leq n/2$ and $2 \leq d \leq n$ for which no merged triple is optimal, with an optimal triple of form (B); in all of them it is the only maximizing configuration of the form of Theorem B. For $n \leq 11$ and all $1 \leq k \leq n - 1$, the triple $(10, 5, 5)$ is the only such case. For $n \leq 30$ ($k \leq n/2$, $2 \leq d \leq n$) there are 141 such cases.

8 Computations

No proof in this note depends on a computation. The computations test the statements as printed against a brute force over all Littlewood–Richardson coefficients, which uses none of the lemmas, and against the reduction of Theorem A. The reproducibility package (`reproducibility/` in the source archive) contains the program of this note, the programs with which the results were first obtained (`original/`), and the programs of the three verification runs (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`), all with their recorded outputs; its `README.md` gives the command lines, the running times, and the ranges of n , k and d covered by each program.

(n, k, d)	$\Delta_{\max}$	merged	λ	μ	ν	(π, y)
(10, 5, 5)	72	70	(2^5)	$(2^2, 1)$	$(2^2, 1)$	(1, 1)
(12, 5, 6)	102	100	(2^6)	$(2^2, 1)$	$(2^3, 1)$	(1, 1)
(14, 5, 7)	132	130	(2^7)	$(2^2, 1)$	$(2^4, 1)$	(1, 1)
(14, 7, 5)	128	126	$(3^4, 2)$	$(3^2, 1)$	$(3^2, 1)$	(1, 1)
(14, 7, 7)	144	140	(2^7)	$(2^3, 1)$	$(2^3, 1)$	(1, 1)
(15, 5, 7)	142	140	$(3, 2^6)$	$(2^2, 1)$	$(3, 2^3, 1)$	(1, 1)
(16, 5, 8)	162	160	(2^8)	$(2^2, 1)$	$(2^5, 1)$	(1, 1)
(16, 7, 8)	186	182	(2^8)	$(2^3, 1)$	$(2^4, 1)$	(1, 1)
(17, 5, 8)	172	170	$(3, 2^7)$	$(2^2, 1)$	$(3, 2^4, 1)$	(1, 1)
(17, 7, 6)	184	182	$(3^5, 2)$	$(3^2, 1)$	$(3^3, 1)$	(1, 1)
(18, 5, 8)	182	180	$(3^2, 2^6)$	$(2^2, 1)$	$(3^2, 2^3, 1)$	(1, 1)
(18, 5, 9)	192	190	(2^9)	$(2^2, 1)$	$(2^6, 1)$	(1, 1)
(18, 7, 6)	200	198	(3^6)	$(3^2, 1)$	$(3^3, 2)$	(1, 2)
(18, 7, 9)	228	224	(2^9)	$(2^3, 1)$	$(2^5, 1)$	(1, 1)
(18, 9, 5)	200	198	$(4^4, 2)$	$(4^2, 1)$	$(4^2, 1)$	(1, 1)
(18, 9, 9)	240	234	(2^9)	$(2^4, 1)$	$(2^4, 1)$	(1, 1)
(19, 5, 9)	202	200	$(3, 2^8)$	$(2^2, 1)$	$(3, 2^5, 1)$	(1, 1)
(19, 7, 9)	242	240	$(3, 2^8)$	$(2^3, 1)$	$(3, 2^4, 1)$	(1, 1)
(20, 5, 9)	212	210	$(3^2, 2^7)$	$(2^2, 1)$	$(3^2, 2^4, 1)$	(1, 1)
(20, 5, 10)	222	220	(2^{10})	$(2^2, 1)$	$(2^7, 1)$	(1, 1)
(20, 7, 7)	240	238	$(3^6, 2)$	$(3^2, 1)$	$(3^4, 1)$	(1, 1)
(20, 7, 10)	270	266	(2^{10})	$(2^3, 1)$	$(2^6, 1)$	(1, 1)
(20, 9, 10)	294	288	(2^{10})	$(2^4, 1)$	$(2^5, 1)$	(1, 1)
(20, 10, 7)	264	260	$(3^6, 2)$	$(3^3, 1)$	$(3^3, 1)$	(1, 1)

Table 1: The 24 triples (n, k, d) with $n \leq 20$, $1 \leq k \leq n/2$ and $2 \leq d \leq n$ for which no merged triple is optimal: the maximum, the maximum over merged triples, and the optimal triple of form (B) of Corollary D (π and y are the smallest parts of μ and ν ; exponents denote repeated parts).

The program of this note. The exact program of this note, `check_note.py` (Python standard library, exact integer arithmetic, about ten seconds), has six parts. (i) It computes all Littlewood–Richardson coefficients with $|\lambda| \leq 12$ without Littlewood–Richardson tableaux: the skew Kostka numbers (numbers of semistandard tableaux of shape λ/μ and given content) are counted as chains of horizontal strips and transformed with the inverse of the unitriangular Kostka matrix. The resulting 20,237 nonzero coefficients, for the 8,312 pairs $\mu \subseteq \lambda$ with $0 < |\mu| < |\lambda| \leq 12$, are symmetric in μ and ν and satisfy the dimension identity $\sum_{\lambda} c_{\mu\nu}^{\lambda} f^{\lambda} = \binom{|\mu|+|\nu|}{|\mu|} f^{\mu} f^{\nu}$, where f^{λ} is the number of standard tableaux. On these data the program checks Lemma 2.1 (for every padding length), Lemmas 3.1 and 3.3 for every pair, and it computes $\Delta_{\max}(n, k, d)$, with η evaluated exactly as printed in (1), by brute force for all 572 triples with $2 \leq n \leq 12$, $1 \leq k \leq n-1$, $1 \leq d \leq n$: the values agree with Theorem C (and with 0 for $d=1$ and $4k(n-k)$ for $d=n$), the other two readings behave as stated in Section 1.1, and the maxima over merged triples agree with Remark 6.3. (ii) For the 506 triples with $2 \leq d \leq n \leq 12$ it builds the triples of Corollary D from every maximizing entry of the list that has the form of Theorem B (575 triples) and checks that they are LR triples with $\ell(\lambda) \leq d$ and maximal Δ ; it also checks the construction of Theorem A. (iii) It compares Theorem C with the maximization of Theorem A, by the greedy rule of Remark 4.3, for all 8,555 triples with $2 \leq n \leq 30$, $1 \leq k \leq n-1$, $2 \leq d \leq n$, and two implementations of the list (one with $O(1)$ arithmetic per entry) with each other for $n \leq 16$ and $2 \leq d \leq n+2$. (iv) It checks Proposition 6.4 on all entries of the lists for $n \leq 24$, Proposition 7.1 for $n \leq 40$, the examples, and Table 1, including the uniqueness of the maximizer of the form of Theorem B there. (v) It recomputes the numbers on the formulas of [JSZ22] given in

Section 1.3. (vi) It enumerates all 10,737 configurations for the 204 triples with $2 \leq d \leq n \leq 9$ and checks Lemma 5.2, Lemma 5.4, Theorem B and Theorem C on them. All 495,166 checks pass.

The first computations (original/) enumerated Littlewood–Richardson tableaux for $|\lambda| \leq 14$ and checked Lemmas 3.1 and 3.3 and the minimum of F over all LR triples for the 483 triples with $n \leq 14$, $k \leq n/2$, $1 \leq d \leq n$; they compared Theorem C with the exhaustive maximization of Theorem A for the 13,959 triples with $n \leq 44$, $k \leq n/2$, $2 \leq d \leq n$; checked the triples of Corollary D by tableau enumeration for the 708 triples with $n \leq 17$, $k \leq n/2$, $2 \leq d \leq n-1$ (for $d = n$ the value $4k(n-k)$ is covered by the comparison with Theorem A, and the triples by the program of this note and by run A below); and tested the lemmas of Section 5 on about 330,000 random instances.

Verification. The results were first obtained, with proofs and with the programs in *original/*. Three independent verification runs, all AI-assisted, followed; each wrote its own programs from the definitions. The first two (A and B) examined the results before this note was written, and neither saw the report of the other; the third examined the final text.

The first run (A) compared the statement with the source and with [KŠV26] and re-derived all lemmas and the proofs of Theorems A, B and C and Corollary D line by line. Its programs computed all Littlewood–Richardson coefficients with $|\lambda| \leq 19$ by skew Kostka numbers and the inverse Kostka matrix (272,896 pairs $\mu \subseteq \lambda$, 1,431,061 nonzero coefficients), found Theorem C equal to the brute force for all 2,280 triples with $n \leq 19$, $1 \leq k \leq n-1$, $1 \leq d \leq n$, and the triples of Corollary D valid; minimized $c(\lambda) - c(\mu) - c(\nu^*(\lambda, \mu))$ over all pairs $\mu \subseteq \lambda$, using Lemma 3.1 only, for all 8,555 triples with $n \leq 30$, $2 \leq d \leq n$ (also for the reading with exactly d parts and for eq. (11) of [JSZ22]); compared Theorem A with Theorem C for the 84,540 triples with $n \leq 80$, $k \leq n/2$, $2 \leq d \leq n$, for 6,000 random triples with $n \leq 500$ and for 1,000 random triples with k up to $n/2$ and $d \leq 8$; checked the proved cases of [KŠV26] ($d = 2$, $d = 3$, the star graph, $k \leq 4$ via their Corollary 7.28, and their Proposition 7.8) for $n \leq 150$, resp. $n \leq 120$; tested the lemmas of Section 5 on about 430,000 random instances; and evaluated the formulas of [JSZ22].

The second run (B) enumerated Littlewood–Richardson tableaux for $|\lambda| \leq 22$ (1,022,205 pairs $\mu \subseteq \lambda$; the enumerator was tested by the symmetry and the dimension identity), checked Lemmas 3.1 and 3.3 on all pairs, Theorem A against the brute force for all 3,542 triples with $n \leq 22$, $1 \leq k \leq n-1$, $1 \leq d \leq n$, and the construction of Theorem A for 780 triples; it compared Theorem C with its own implementation of Theorem A for the 13,959 triples with $n \leq 44$, $k \leq n/2$, $2 \leq d \leq n$, and with [KŠV26, Corollary 7.28], implemented from the text of [KŠV26], for 3,590 triples with $k \leq 4$; it tested the lemmas of Section 5 on 40,000 random configurations and Theorem B on all configurations with $n \leq 9$; four deliberately wrong inputs (a wrong η , wrong weights, the list without Ψ_B , unbalanced parts) were all detected, and a fifth (dropping the lattice condition) cannot change the maximum by Lemma 3.1(b); and it re-ran all programs of *original/* from a copy and obtained all 20 recorded outputs byte for byte.

Runs A and B confirmed all theorems, lemmas and the corollary and found no gap in the proofs; run A found that a formula in a remark of the first written version lacked a factor $\frac{1}{2}$. The two runs required corrections of the first written version, which were made: the factor $\frac{1}{2}$ in the relation between Ψ_A and merged triples (Remark 6.3); the description of the formulas of [JSZ22], with the variant behind every count (Section 1.3); the explicit statement that the list of Theorem C is larger than the set of shapes of Theorem B and consists of lower bounds (Remark 6.1); the treatment of rows of type E in the proof of Lemma 5.4; the attributions of Lemma 2.1 and Lemma 3.1 (Remark 3.2); journal references; and the ranges of d covered by each computation. This text was written after the two runs; its proofs are those of the first written version with these corrections, in the present numbering. Remark 6.2, Proposition 6.4, the formula of Remark 6.3 and the proofs of Proposition 7.1 were added when the note was written; runs A and B did not examine them, and the exact program

of this note checks them.

The third run examined the final text and the package line by line, as if nothing had been checked, and these four additions first. It read the source and the quoted parts of [KŠV26], [JSZ22] and [BRR23] again, re-derived every proof, and wrote its own programs before reading `check_note.py`. They compute all Littlewood–Richardson coefficients with $|\lambda| \leq 18$ by a third method, the Brauer–Klimyk formula (without tableaux and without the inverse Kostka matrix), and with them the brute force for all 2,244 triples with $n \leq 18$, $1 \leq k \leq n-1$, $1 \leq d \leq n+2$, the other two readings, Remarks 6.2 and 6.3 and the triples of Corollary D; the largest eigenvalue of (2), computed directly on a weight space of $(\mathbb{C}^d)^{\otimes n}$, for 138 triples with $n \leq 10$ (for $(10, 5, 5)$ on a space of dimension 113,400), and for 81 small triples the whole spectrum, compared with the set of the values $\Delta(\lambda, \mu, \nu)$; Theorem C against Theorem A for 9,425 triples with $n \leq 30$ and all k ; Propositions 6.4 and 7.1 for $n \leq 40$; all 212,612 configurations with $n \leq 13$, including the convexity and the endpoint cases in the proof of Lemma 5.4; and the numbers on [JSZ22] in Section 1.3. Six deliberately wrong variants were all detected. The run re-ran the package from the archive (the program of this note and 47 recorded runs of the earlier programs), which reproduced every recorded output, and repeated the literature search. It found that Proposition 7.1(d) lacked the hypothesis $d \geq 2$ and was false for $d = 1$, no other error, and no gap in the proofs. It required seven corrections, all made: that hypothesis; the deduction of the maximum over merged triples in Remark 6.3, now written out; the description of the source in Section 1.1; the credit for the quantum max d -cut problem [CJK⁺23]; the account of runs A and B in this paragraph; a running time in the package; and the record of this run. Its programs and outputs are in `independent_run_2/`.

Scope and priority. The note answers Konvalinka’s question for all n , k and d (Theorem C), in the reading of Section 1.1; the answer is the maximum of an explicit list of at most $(d-1)(1+k(n-k))$ integers, not a single closed formula. The problem is Konvalinka’s [CEF26]; the quantum max d -cut problem was introduced by Carlson, Jorquera, Kolla, Kordonowy and Wayland [CJK⁺23]. The identification of the maximum with the quantum max d -cut of $K_{n-k,k}$ [KŠV26, (7.5)], Lemma 2.1 ([KŠV26, (B.2), Lemma B.1]), the cases $d \leq 3$ and $k \leq 4$, the maximum over merged triples and Conjecture 7.7 are due to Klep, Štrelelj and Volčič [KŠV26]; the case $d = 2$ is the theorem of Lieb and Mattis [LM62]; the dominance bound of Lemma 3.1(b) is classical (Remark 3.2). New, to our knowledge, are Theorems A, B and C in general and the proof of Conjecture 7.7 of [KŠV26] (Corollary D); Proposition 6.4 also gives a proof of [KŠV26, Proposition 7.8] that does not use their case analysis. No earlier proof of the general case or of Conjecture 7.7 of Klep, Štrelelj and Volčič was found in the sources searched: the arXiv (queries for quantum max d -cut, quantum max cut on complete bipartite graphs and for qudits, Littlewood–Richardson coefficients with quantum max cut, swap operators, Hamiltonians or ground states, extendibility of Werner states, the Lieb–Mattis theorem for $SU(d)$, interchange models on complete bipartite graphs, and the authors Konvalinka, Klep, Štrelelj and Volčič), Crossref, zbMATH, the lists of works citing [KŠV26] and [JSZ22] in OpenAlex and Semantic Scholar, and five general web searches. A search that finds nothing is not a proof of novelty, and no priority is claimed.

What was read: the source [CEF26], its front matter and Problem 10 on p. 150, in the open file of the publisher; of [KŠV26], in arXiv version 5, Definition 1.5, Sections 1.4.3 and 1.4.4, the beginning of Section 1.5, Section 6.2, Theorem 7.2, Section 7.2 and Appendix B, whose case analysis (the proof of their Proposition 7.8) was not checked; of [JSZ22] (version 2), the main text and Section B of the supplemental material; the abstract, the main theorems and the introduction of [BRR23]; and the abstracts of [ACG⁺25], [CJK⁺23], [Š25] and [TGIU23]. The statement of [Aze99] is taken from [JSZ22], and [Ful97] is cited as a standard reference for the Littlewood–Richardson rule. The searches were made on 15 October 2026, when the results were first obtained, by the verification runs A and B,

in part again when this note was written, and by the third verification run; the works citing [KŠV26] that were found are [Š25] only, and none of the works found answers the question in general.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [ACG⁺25] Rene Allerstorfer, Matthias Christandl, Dmitry Grinko, Ion Nechita, Maris Ozols, Denis Rochette, and Philip Verduyn Lunel. Monogamy of highly symmetric states. *Communications in Mathematical Physics*, 407, 2025. Article no. 21; published online 8 December 2025, in no. 1 of the volume (January 2026). [arXiv:2309.16655](#), [doi:10.1007/s00220-025-05483-3](#).
- [Aze99] Olga Azenhas. The admissible interval for the invariant factors of a product of matrices. *Linear and Multilinear Algebra*, 46(1–2):51–99, 1999. [doi:10.1080/03081089908818605](#).
- [BRR23] Jakob E. Björnberg, Hjalmar Rosengren, and Kieran Ryan. Heisenberg models and Schur–Weyl duality. *Advances in Applied Mathematics*, 151:102572, 2023. [arXiv:2201.10209](#), [doi:10.1016/j.aam.2023.102572](#).
- [CEF26] Guillaume Chapuy, Sergi Elizalde, and Ilse Fischer. Enumerative Combinatorics. *Oberwolfach Reports*, 23(1):81–156, 2026. Report No. 2/2026, workshop of 11–16 January 2026. Open problem session (collected by G. Chapuy), Problem 10 (problem of M. Konvalinka, communicated by I. Klep), p. 150. [doi:10.4171/OWR/2026/2](#).
- [CJK⁺23] Charlie Carlson, Zackary Jorquera, Alexandra Kolla, Steven Kordonowy, and Stuart Wayland. Approximation algorithms for Quantum Max- d -Cut, 2023. Preprint, version 2 (21 February 2024). [arXiv:2309.10957](#).
- [Ful97] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. [doi:10.1017/CB09780511626241](#).
- [JSZ22] Dávid Jakab, Adrian Solymos, and Zoltán Zimborás. Extendibility of Werner states, 2022. Preprint, version 2 (22 September 2022). [arXiv:2208.13743](#).
- [KŠV26] Igor Klep, Tea Štrekelj, and Jurij Volčič. Quantum Max d -Cut via qudit swap operators. *Quantum*, 10:2203, 2026. Numbering of statements as in arXiv version 5 (2 September 2026). [arXiv:2503.20942](#), [doi:10.22331/q-2026-09-03-2203](#).
- [LM62] Elliott Lieb and Daniel Mattis. Ordering energy levels of interacting spin systems. *Journal of Mathematical Physics*, 3(4):749–751, 1962. [doi:10.1063/1.1724276](#).
- [Š25] Tea Štrekelj. Quantum Max Cut for complete tripartite graphs, 2025. Preprint. [arXiv:2512.03740](#).
- [TGIU23] Müge Taşkın, R. Bedii Gümüş, Sinan Işık, and M. İkbāl Ulvi. Lower and upper bounds for nonzero Littlewood–Richardson coefficients, 2023. Preprint, version 2 (6 April 2023). [arXiv:2208.06541](#).

[ula26] ulamai. UnsolvedMath. Hugging Face dataset,
<https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record
OWR-14299904-015.