

No Uniform Linear Threshold for Hook Monotonicity in Self-Conjugate and Doubled Distinct Partitions

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Abstract

Let $H_t^{\text{SC}}(n)$ and $H_t^{\text{DD}}(n)$ count hooks of length t among self-conjugate and doubled distinct partitions of n , respectively. A 2026 Oberwolfach conjecture asks whether $H_t(n) \geq H_{t+1}(n)$ holds for $n \gg t$ with an implied constant independent of t . We disprove this uniform assertion for both partition families. For every fixed integer $K \geq 3$, the self-conjugate ratio $H_{t+1}^{\text{SC}}(Kt)/H_t^{\text{SC}}(Kt)$ tends to infinity along even t . The same holds for doubled distinct partitions along odd t when $K \geq 4$ is even. More strongly, for every $0 < \alpha < \pi^2/24$, the ratios still diverge at sizes $n = \alpha t^2/(\log t)^2 + O(1)$, with the necessary parity. The proof combines known exact hook generating functions, classical partition asymptotics, and a lower bound obtained from diagonal hooks. The known fixed- t eventual inequalities and the opposite parity cases are unaffected.

1 The uniform question and the results

For a partition λ , let $h_t(\lambda)$ be the number of cells of its Young diagram whose hook length is exactly t . For $\mathcal{F} = \text{SC}, \text{DD}$, write

$$a_{\mathcal{F}}(n) = \#\{\lambda \in \mathcal{F} : |\lambda| = n\}, \quad H_t^{\mathcal{F}}(n) = \sum_{\substack{\lambda \in \mathcal{F} \\ |\lambda| = n}} h_t(\lambda).$$

We set $a_{\mathcal{F}}(0) = 1$ and $a_{\mathcal{F}}(n) = 0$ for $n < 0$. Self-conjugate partitions satisfy $\lambda = \lambda'$. Doubled distinct partitions have Frobenius symbols

$$\begin{pmatrix} b_1 & \cdots & b_r \\ b_1 - 1 & \cdots & b_r - 1 \end{pmatrix}, \quad b_1 > \cdots > b_r > 0.$$

Their size is $2 \sum_i b_i$, so $a_{\text{DD}}(n) = 0$ for odd n .

Conjecture 3 in Ae Ja Yee's report [4, p. 142], based on joint work with Hyunsoo Cho, Byungchan Kim and Eunmi Kim, proposes

$$H_t^{\text{SC}}(n) \geq H_{t+1}^{\text{SC}}(n), \quad H_t^{\text{DD}}(n) \geq H_{t+1}^{\text{DD}}(n) \quad (n \gg t), \quad (1)$$

explicitly requiring the implied constant to be independent of t . This is the uniform linear threshold assertion that there exists an absolute C such that the inequalities hold whenever $n \geq Ct$. The report separately notes the known fixed- t eventual inequalities. It also states the proved self-conjugate odd-to-even and doubled distinct even-to-odd cases; see its Theorems 4 and 5. Our counterexamples occur in the other parities.

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Theorem 1. *For every fixed integer $K \geq 3$,*

$$\frac{H_{t+1}^{\text{SC}}(Kt)}{H_t^{\text{SC}}(Kt)} \longrightarrow \infty \quad (t \longrightarrow \infty, t \text{ even}).$$

For every fixed even integer $K \geq 4$,

$$\frac{H_{t+1}^{\text{DD}}(Kt)}{H_t^{\text{DD}}(Kt)} \longrightarrow \infty \quad (t \longrightarrow \infty, t \text{ odd}).$$

The denominators are positive for all sufficiently large t . In particular, neither inequality in (1) admits a uniform linear threshold.

The failure persists even when n/t tends to infinity. All logarithms below are natural.

Theorem 2. *Fix $0 < \alpha < \pi^2/24$. For self-conjugate partitions let t tend to infinity through even integers and set*

$$n_t = \left\lfloor \frac{\alpha t^2}{(\log t)^2} \right\rfloor.$$

For doubled distinct partitions let t tend to infinity through odd integers and set

$$n_t = 2 \left\lfloor \frac{\alpha t^2}{2(\log t)^2} \right\rfloor.$$

For the respective family, in either case,

$$\liminf_{t \rightarrow \infty} \frac{H_{t+1}^{\mathcal{F}}(n_t)}{t^{\pi/(2\sqrt{6\alpha})-1} H_t^{\mathcal{F}}(n_t)} \geq 1. \quad (2)$$

Consequently $H_{t+1}^{\mathcal{F}}(n_t)/H_t^{\mathcal{F}}(n_t) \rightarrow \infty$.

We use established coefficient formulas with their full attribution; no new partition asymptotic theorem is required. The argument concerns a simultaneous limit in n, t , rather than a substitution of growing t into an error term proved only for fixed t .

2 Coefficient identities and diagonal hooks

The diagonal hooks of a self-conjugate partition are distinct odd positive integers and determine the partition uniquely. The diagonal hooks of a doubled distinct partition with the symbol above are $2b_1, \dots, 2b_r$. Thus

$$A_{\text{SC}}(q) := \sum_{n \geq 0} a_{\text{SC}}(n) q^n = \prod_{j \geq 1} (1 + q^{2j-1}), \quad A_{\text{DD}}(q) := \sum_{n \geq 0} a_{\text{DD}}(n) q^n = \prod_{j \geq 1} (1 + q^{2j}). \quad (3)$$

Lemma 3. *For even t in SC, and for odd t in DD, respectively,*

$$H_t^{\mathcal{F}}(n) = t \sum_{j \geq 1} a_{\mathcal{F}}(n - 2tj). \quad (4)$$

If h is odd for SC, or even for DD, then

$$H_h^{\mathcal{F}}(n) \geq a_{\mathcal{F}}(n - h) - a_{\mathcal{F}}(n - 2h). \quad (5)$$

Proof. For SC, formula (4) is Amdeberhan–Andrews–Ono–Singh [1, Theorem 1.2(1)]. For DD, Cho–Kim–Kim–Yee [2, Section 6.1] give the first-moment generating function for odd t as

$$\sum_{m \geq 1} H_t^{\text{DD}}(2m)q^m = \prod_{j \geq 1} (1 + q^j) \frac{tq^t}{1 - q^t}.$$

Coefficient extraction gives (4), including the zero coefficients at odd sizes.

For (5), use the distinct diagonal-hook parametrization in (3). Adding a part of size h to a diagonal-hook partition of $n - h$ that does not already contain h produces a partition of n with a diagonal h -hook. The number of excluded partitions is at most $a_{\mathcal{F}}(n - 2h)$: remove their unique part of size h , an injective operation. Each resulting partition contributes at least one h -hook. $\square$

We record the classical asymptotics in the size convention used here. Let

$$c = \frac{\pi}{\sqrt{6}}, \quad C_{\text{SC}} = \frac{1}{27/43^{1/4}}, \quad C_{\text{DD}} = \frac{1}{25/43^{1/4}}.$$

Then, along all integer sizes for SC and even sizes for DD,

$$a_{\mathcal{F}}(m) = C_{\mathcal{F}} m^{-3/4} e^{c\sqrt{m}} (1 + o(1)) \quad (m \rightarrow \infty). \quad (6)$$

For SC, this follows by setting the marking variable to 1 in Craig–Ono–Singh [3, Proposition 3.1]. One may fix the hook parameter in that proposition, since at marking variable 1 the coefficient is simply $a_{\text{SC}}(m)$. For DD, $a_{\text{DD}}(2m)$ is the number of partitions of m into distinct parts. The asymptotic in the remark following [2, Theorem 5.1], after replacing m by $m/2$, gives (6). In particular, no uniformity in a hook parameter is being assumed in (6).

3 Counterexamples on every fixed linear scale

Proof of Theorem 1. Fix the indicated K , and use the family and parity of t in the theorem. Write $a = a_{\mathcal{F}}$. Formula (4) is a sum of at most $\lfloor K/2 \rfloor$ nonzero terms. Its first term is $t a((K-2)t)$. Every later term with positive argument is exponentially smaller than the first by (6); an argument of zero contributes only 1. Thus

$$H_t^{\mathcal{F}}(Kt) \sim t a((K-2)t). \quad (7)$$

Apply (5) with $h = t + 1$:

$$H_{t+1}^{\mathcal{F}}(Kt) \geq a((K-1)t-1) - a((K-2)t-2) \sim a((K-1)t-1).$$

The last equivalence refers to the right-hand difference. All these arguments have the correct parity for DD. Again by (6),

$$\frac{a((K-1)t-1)}{t a((K-2)t)} \sim \frac{1}{t} \left(\frac{K-2}{K-1} \right)^{3/4} \exp\left(c(\sqrt{K-1} - \sqrt{K-2})\sqrt{t}\right),$$

which tends to infinity. This proves the ratio assertions. Given any proposed absolute C , choose an integer $K > C$ with the parity requirement in the doubled distinct case. For arbitrarily large admissible t , the inequality fails at $n = Kt > Ct$. $\square$

4 Counterexamples beyond linear size

The following elementary estimate controls the growing number of summands in (4).

Lemma 4. *Fix $\alpha > 0$ and let $n = \alpha t^2/(\log t)^2 + O(1)$, with the required parity in the doubled distinct case. For either coefficient sequence in (6),*

$$\sum_{j \geq 1} a_{\mathcal{F}}(n - 2tj) = a_{\mathcal{F}}(n - 2t)(1 + o(1)).$$

Proof. Put $N = n - 2t$, which tends to infinity and satisfies $N \sim n$. Terms after the first have argument $m = N - 2t\ell$, $\ell \geq 1$. For $N/2 \leq m < N$, the relative asymptotic in (6) implies, uniformly in these arguments,

$$\frac{a_{\mathcal{F}}(m)}{a_{\mathcal{F}}(N)} \leq B \left(\frac{N}{m} \right)^{3/4} e^{-c(\sqrt{N} - \sqrt{m})} \leq B 2^{3/4} e^{-ct\ell/\sqrt{N}},$$

where B is independent of t, ℓ for sufficiently large t . This uses $\sqrt{N} - \sqrt{m} \geq (N - m)/(2\sqrt{N})$. Since $t/\sqrt{N} \sim (\log t)/\sqrt{\alpha}$, the geometric sum over $\ell \geq 1$ tends to zero.

For $0 \leq m < N/2$, (6), with the finitely many small m absorbed into a constant, gives $a_{\mathcal{F}}(m) \leq B' e^{c\sqrt{N/2}}$. There are at most $N + 1$ such terms, whereas $a_{\mathcal{F}}(N)$ is bounded below by a positive constant times $N^{-3/4} e^{c\sqrt{N}}$. Their total relative contribution is at most

$$O\left(N^{7/4} e^{-c(1-1/\sqrt{2})\sqrt{N}}\right) = o(1).$$

Coefficients at negative arguments vanish, proving the lemma. $\square$

Proof of Theorem 2. Use the corresponding sequence $n = n_t$, put $h = t + 1$, and abbreviate $a = a_{\mathcal{F}}$. By Lemmas 3 and 4,

$$H_t^{\mathcal{F}}(n) = t a(n - 2t)(1 + o(1)).$$

Since $t = o(n)$, formula (6) also gives

$$\frac{a(n - 2h)}{a(n - h)} \longrightarrow 0.$$

Indeed, the logarithm of this ratio is $-ct/(2\sqrt{n}) + o(1)$, which tends to minus infinity. Hence the diagonal lower bound is

$$H_{t+1}^{\mathcal{F}}(n) \geq a(n - h)(1 - o(1)).$$

For the remaining ratio, (6) yields

$$\log \frac{a(n - h)}{a(n - 2t)} = \frac{3}{4} \log \frac{n - 2t}{n - h} + c(\sqrt{n - h} - \sqrt{n - 2t}) + o(1) = \frac{ct}{2\sqrt{n}} + o(1).$$

The Taylor remainder here is $O(1/\sqrt{n} + t^2/n^{3/2}) = o(1)$. The rounding in n changes the last expression by $o(1)$, so

$$\frac{a(n - h)}{t a(n - 2t)} = (1 + o(1)) t^{c/(2\sqrt{\alpha}) - 1}.$$

This proves (2); its exponent is positive precisely when $\alpha < c^2/4 = \pi^2/24$. $\square$

Corollary 5. *Suppose $M_{\mathcal{F}}(t)$ is an eventual threshold such that $H_t^{\mathcal{F}}(n) \geq H_{t+1}^{\mathcal{F}}(n)$ for every integer $n \geq M_{\mathcal{F}}(t)$. Then*

$$\liminf_{\substack{t \rightarrow \infty \\ t \text{ even}}} \frac{M_{\text{SC}}(t)(\log t)^2}{t^2} \geq \frac{\pi^2}{24}, \quad \liminf_{\substack{t \rightarrow \infty \\ t \text{ odd}}} \frac{M_{\text{DD}}(t)(\log t)^2}{t^2} \geq \frac{\pi^2}{24}.$$

Proof. For every fixed $\alpha < \pi^2/24$, Theorem 2 produces a counterexample at n_t for every sufficiently large t of the indicated parity. Consequently $M_{\mathcal{F}}(t) > n_t$. Take the lower limit and then let $\alpha \uparrow \pi^2/24$. $\square$

This is a lower bound only. We assert neither an optimal threshold nor a matching upper bound.

5 Exact examples and scope

Table 1 gives small examples with positive denominators. No floating-point arithmetic is used in these counts.

Family	K	t	$n = Kt$	$H_t(n)$	$H_{t+1}(n)$
SC	3	22	66	176	204
SC	4	24	96	2112	2117
SC	10	30	300	50568030	50583984
DD	4	21	84	1617	1644
DD	10	31	310	100255705	100362108

Table 1: Exact reversals of adjacent-hook monotonicity.

The accompanying Python script computes coefficients through size 50000 using Euler’s pentagonal recurrence and product identities. It compares 2402 coefficients against direct multiplication of the two products in (3). It also constructs 4778 Ferrers diagrams of weight at most 60, counts their hooks cell by cell, and verifies 7442 hook identities. Further exact examples have $K = 20, 50, 100$, and two large certificates check the diagonal lower bound itself. These are reproducibility checks; the infinite families are proved by the arguments above, not by finite computation.

The result refutes the uniform linear assertion in the printed Conjecture 3 [4], represented by UnsolvedMath record OWR-14299904-009. It does not refute eventual monotonicity with t fixed, and it does not alter the opposite parity theorems already recorded there. In particular the doubled distinct even-to-odd assertion for $t \geq 8$ is not a target of our counterexamples. The hook identities and coefficient asymptotics are credited to [1, 3, 2]. A bounded literature search did not locate this simultaneous-parameter counterexample; no claim of exhaustive priority verification is made.

This preprint and its reproducibility files were developed with AI assistance in literature analysis, symbolic reasoning, exact computation and manuscript preparation. The written proof was self-audited; it has not undergone independent peer review or formal proof-assistant verification.

References

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