Differences of Signed Representatives: Sharp Answers to Two Questions of Bernert and Arala Santos

Alper Ferudun*

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Abstract

Let $A \subset \{-n, \ldots, n\} \setminus \{0\}$ contain exactly one of k and $-k$ for every $1 \le k \le n$. At the 2025 Oberwolfach workshop on analytic number theory, C. Bernert and N. Arala Santos asked two questions about such sets. First, must $A - A$ contain $(1-o(1))n$ elements of $\{1, \ldots, n\}$? Second, for every $B \subseteq \{1, \ldots, n\}$ with $|B| \ge \varepsilon n$, must the number of pairs $(a, b) \in A^2$ with $a - b \in B$ be $\gg_\varepsilon n^2$? We answer both questions affirmatively, with sharp bounds. At most two elements of $\{1, \ldots, n\}$ are missing from $A - A$. For every $B \subseteq \{1, \ldots, n\}$, the number of pairs $(a, b) \in A^2$ with $a - b \in B$ is at least $\lfloor (|B| - 1)^2/4 \rfloor$. Both bounds are attained by sets of the form $\{1, \ldots, a\} \cup \{-(a+1), \ldots, -n\}$, the second for all $|B| \le \lfloor 2n/3 \rfloor + 1$. The proof is a short double-counting argument. This is an unrefereed note.

1 Introduction

Throughout, $n \ge 1$ and $A \subset \{-n, \ldots, n\} \setminus \{0\}$ is a set containing exactly one of k and $-k$ for every $1 \le k \le n$. Such sets are called *signed sets* here. For $d \in [1, n] = \{1, \ldots, n\}$ let

$$r(d) = \#\{(a, b) \in A^2 : \ a - b = d\},$$

so that $d \in A - A$ if and only if $r(d) > 0$, and $\sum_{a,b \in A} \mathbf{1}_B(a - b) = \sum_{d \in B} r(d)$ for $B \subseteq [1, n]$.

The following two questions were posed by Christian Bernert and Nuno Arala Santos at the problem session of the 2025 Oberwolfach workshop on analytic number theory [MMSW25, p. 2756].

(Q1) Is $|(A - A) \cap [1, n]| = (1 - o(1))n$?

(Q2) Is $\sum_{d \in B} r(d) \gg_\varepsilon n^2$ for every $B \subseteq [1, n]$ with $|B| \ge \varepsilon n$?

The report notes that the analogous questions for sets containing exactly one of k and $1 - k$ have a negative answer, since such a set can consist of even numbers only. It adds that with k and $-k$ there do not seem to be any congruence obstructions. We show that both answers are yes, in a sharp form.

Theorem 1.1. *Every signed set A satisfies $|(A - A) \cap [1, n]| \ge n - 2$. For $n \ge 2$ equality holds for $A = \{1, \ldots, a\} \cup \{-(a + 1), \ldots, -n\}$ whenever $n/2 \le a \le n - 1$.*

Theorem 1.2. *Every signed set A and every $B \subseteq [1, n]$ satisfy*

$$\sum_{a,b \in A} \mathbf{1}_B(a - b) \ \ge \ \left\lfloor \frac{(|B| - 1)^2}{4} \right\rfloor.$$

If $n \ge 2$, then for every $1 \le m \le \lfloor 2n/3 \rfloor + 1$ there are a signed set A and a set B with $|B| = m$ for which equality holds.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.

Theorem 1.2 answers (Q2): if $|B| \geq \varepsilon n \geq 4$, then the number of pairs is at least $\varepsilon^2 n^2/8$. More precisely it is at least $(\frac{1}{4} - o(1))\varepsilon^2 n^2$ as $n \to \infty$, and for every fixed $\varepsilon \leq \frac{2}{3}$ the constant $\frac{1}{4}$ is best possible.[1] The lower bounds in both theorems follow from the next proposition; for Theorem 1.1 only the case $|S| = 2$ is needed.

Proposition 1.3. *Let $S \subseteq [1, n]$ be a set of integers of the same parity. Then $\sum_{d\in S} r(d) \geq \binom{|S|}{2}$.*

Questions about sets of integers that avoid prescribed differences go back to Motzkin; see for example [CG73]. Here the set is constrained by the sign condition, and the conclusion is that it can avoid at most two differences.

2 Proof of the main proposition

Let $I = [-n, n] \setminus \{0\}$ and define $g\colon I \to \{1, -1\}$ by $g(x) = 1$ if $x \in A$ and $g(x) = -1$ otherwise. The sign condition says exactly that g is odd: $g(-x) = -g(x)$ for all $x \in I$. For $d \in [1, n]$ let $I_d = \{x \in I : \ x + d \in I\}$, and define the set of *defects*

$$D(d) = \{x \in I_d : \ g(x) = g(x + d)\}.$$

All intervals denote sets of integers; for real $\alpha < \beta$, (α, β) is the set of integers strictly between α and β.

Lemma 2.1. *The map $\mu_d(x) = -x - d$ is an involution of $D(d)$ without fixed points, and $|D(d)| = 2r(d)$. It maps $D(d) \cap [1, n - d]$ onto $D(d) \cap [-n, -d - 1]$, and maps $D(d) \cap [1 - d, -1]$ onto itself. Hence*

$$|D(d) \cap [1, n - d]| + |D(d) \cap (-d/2, 0)| = r(d). \tag{1}$$

Proof. Since $I = -I$, for $x \in I_d$ both $\mu_d(x) = -(x + d)$ and $\mu_d(x) + d = -x$ lie in I, so μ_d maps I_d to itself. Since g is odd, $g(\mu_d(x)) = -g(x + d)$ and $g(\mu_d(x) + d) = g(-x) = -g(x)$. So μ_d maps $D(d)$ to itself, and it exchanges the defects with $g(x) = g(x+d) = 1$ and those with $g(x) = g(x+d) = -1$. The defects with $g(x) = g(x+d) = 1$ correspond to the pairs $(x+d, x) \in A^2$, so there are $r(d)$ of each kind, and $|D(d)| = 2r(d)$. A fixed point would be $x = -d/2$. It is not a defect, since $g(-d/2) = -g(d/2)$.

The set I_d is the union of the three intervals $[1, n - d]$, $[-n, -d - 1]$ and $[1 - d, -1]$. The map μ_d exchanges the first two and maps the third to itself. On the third it exchanges $(-d/2, 0)$ with $(-d, -d/2)$. So exactly half of $D(d)$ lies in $[1, n - d] \cup (-d/2, 0)$, which is (1). ☐

Lemma 2.2. *Let $1 \leq d < d' \leq n$ with $d' - d$ even, and let $j = (d' - d)/2$. Then exactly one of $j \in D(d)$ and $-j \in D(d')$ holds. Moreover $j \in [1, n - d]$ and $-j \in (-d'/2, 0)$.*

Proof. We have $1 \leq j < d'/2$ and $j + d = d' - j = (d + d')/2 \in [1, n]$. So $j \in I_d$, $j \in [1, n - d]$, $-j \in I_{d'}$ and $-j \in (-d'/2, 0)$. Now $j \in D(d)$ means $g(j) = g(j + d)$. Also $-j \in D(d')$ means $g(-j) = g(d' - j)$, that is, $-g(j) = g(j + d)$. Exactly one of these two equations holds. ☐

Proof of Proposition 1.3. Let $s = |S|$. Take a pair $d < d'$ of elements of S and let $j = (d' - d)/2$. By Lemma 2.2, either $j \in D(d) \cap [1, n - d]$ or $-j \in D(d') \cap (-d'/2, 0)$. Assign the pair to (d, j) in the first case and to $(d', -j)$ in the second. This assignment is injective. The couple (d, j) with $j > 0$ can only come from the pair $\{d, d + 2j\}$, and the couple $(d', -j)$ with $-j < 0$ only from the pair $\{d' - 2j, d'\}$. Every couple with first entry δ has its second entry in $(D(\delta) \cap [1, n - \delta]) \cup (D(\delta) \cap (-\delta/2, 0))$, and by (1) there are at most $r(\delta)$ such couples. Hence $\binom{s}{2} \leq \sum_{\delta\in S} r(\delta)$. ☐

[1] The bound $\gg_\varepsilon n^2$ fails for $n \leq 2/\varepsilon$: then B may consist of one or two missing differences, and the sum is 0. For $n > 2/\varepsilon$ it holds with the constant $\varepsilon^2/9$, since $\lfloor (m - 1)^2/4 \rfloor \geq m^2/9$ for $m \geq 3$.

Proof of the lower bounds in Theorems 1.1 and 1.2. If three elements of $[1,n]$ were missing from $A-A$, two of them, say $d<d'$, would have the same parity. Proposition 1.3 with $S=\{d,d'\}$ would then give $0=r(d)+r(d')\geq 1$, a contradiction. So at most two are missing.

For Theorem 1.2, split B into its odd part B_1 and its even part B_2, of sizes $b_1+b_2=|B|$. By Proposition 1.3, $\sum_{d\in B} r(d)\geq \binom{b_1}{2}+\binom{b_2}{2}$. For fixed b_1+b_2 this is smallest when $|b_1-b_2|\leq 1$. The minimum is $\lfloor(|B|-1)^2/4\rfloor$ in both cases: it is $c(c-1)$ if $|B|=2c$, and c^2 if $|B|=2c+1$. □

3 Sharpness

For $n\geq 2$ and $n/2\leq a\leq n-1$ let $A_a=\{1,\dots,a\}\cup\{-(a+1),\dots,-n\}$.

Proposition 3.1. *For $A=A_a$ the values of r are*

$$r(d)=\begin{cases} n-2d, & 1\leq d\leq n-a-1,\\ a-d, & n-a\leq d\leq a,\\ d-a-1, & a+1\leq d\leq n.\end{cases}$$

Proof. There are three kinds of pairs $(x,y)\in A^2$ with $x-y=d\in[1,n]$:

- pairs inside $\{1,\dots,a\}$, which number $(a-d)_+$;
- pairs inside $\{-(a+1),\dots,-n\}$, which number $(n-a-d)_+$;
- pairs with $x\in\{1,\dots,a\}$ and $y=-z$, $z\in\{a+1,\dots,n\}$, for which $d=x+z$. There are $(d-a-1)_+$ of these, because $d\leq n\leq 2a+1$.

Adding the three counts on each range of d gives the formula, using $n-a-1\leq a$, that is, $n\leq 2a+1$. □

Proof of the equality cases. By Proposition 3.1, A_a misses exactly the differences a and $a+1$: on the first range $r(d)=n-2d\geq 2a-n+2>0$, and on the other two ranges $r(d)=0$ only for $d=a$ and for $d=a+1$. This proves the equality case of Theorem 1.1.

Let $q=\min\{2a-n,\ n-a-1\}$. The values $0,1,\dots,q$ each occur at least twice among the $r(d)$: once on $[n-a,a]$ and once on $[a+1,n]$. All other values are larger than q. So for $m\leq 2q+2$, the m smallest values of r are $0,0,1,1,2,2,\dots$. Their sum is $\sum_{i=0}^{m-1}\lfloor i/2\rfloor=\lfloor(m-1)^2/4\rfloor$. Taking for B the corresponding m differences gives equality in Theorem 1.2.

Now put $t=\lfloor(n-2)/3\rfloor$ and $a=n-1-t$, and write $n-2=3t+\rho$ with $\rho\in\{0,1,2\}$. Then $n/2\leq a\leq n-1$, $n-a-1=t$ and $2a-n=t+\rho$, so $q=t$, and equality holds for every $m\leq 2t+2$. If $\rho\geq 1$, the value $t+1$ also occurs, at $d=a-t-1\in[n-a,a]$, and all remaining values are at least $t+2$. So the $(2t+3)$-th smallest value is $t+1$, and equality holds for $m=2t+3$ as well, since $t(t+1)+(t+1)=\lfloor(2t+2)^2/4\rfloor$. The largest m obtained is $2t+2$ if $\rho=0$ and $2t+3$ if $\rho\geq 1$; in both cases it is $\lfloor 2n/3\rfloor+1$. □

Remark 3.2. (1) Exhaustive computation for $n\leq 26$ shows that the minimum over all signed sets of the sum of the m smallest values of r equals $\lfloor(m-1)^2/4\rfloor$ exactly when $1\leq m\leq\lfloor 2n/3\rfloor+1$, and exceeds it for every larger m. So for these n the range of equality in Theorem 1.2 is best possible, and the sets A_a attain equality in all of it.

(2) Computations for $n\leq 26$, and a separate solver for $n\leq 64$, suggest the following description of the extremal sets in Theorem 1.1. Two differences $d<d'$ can be missing together if and only if $d+d'$ is odd, $d+d'\geq n+1$ and $\gcd(d,d')=1$. Each such pair is then missed by exactly one signed set up to the symmetry $A\mapsto -A$. The parity condition follows from Proposition 1.3; we have not proved the rest.

4 Verification and scope

The following exact computations, with scripts in the source archive, were used.

1. The finder's script checks Theorem 1.1 exhaustively for $n \leq 20$. For every missing difference it also checks the two identities on which the finder's original proof of Theorem 1.1 was based, and it checks the equality example.

2. An independent referee's code, written from scratch, gives the following.

   - It checks Theorem 1.1 exhaustively for $n \leq 26$.
   - It gives the characterisation of Remark 3.2(2) for $n \leq 26$, and with a separate solver for $n \leq 64$.
   - On 69 test sets with $n \in \{30, 61, 120\}$ it checks that $|D(d)| = 2r(d)$, that μ_d maps $D(d)$ onto itself without fixed points, and that at least one of $j \in D(d)$ and $-j \in D(d')$ holds in the setting of Lemma 2.2.
   - It computes the minima of Remark 3.2(1) for $n \leq 26$.

   The referee also found the double-counting argument for (Q2), but applied it only to the sets $\{d : r(d) \leq t\}$; this gave the weaker bound $(|B| - 1)(|B| - 2)/8$.

3. A short program by the author (run as `lead_check.py 20`) checks, for every signed set with $n \leq 20$, the following: Proposition 1.3 for every set of equal parity, the bound of Theorem 1.2 for every m, and the lower bound of Theorem 1.1. It also checks its count of r against a direct count for 50 random signed sets for each n.

4. A second referee's code, written from scratch, checks every clause of Lemma 2.1, including (1), the exclusivity in Lemma 2.2 and the charging in the proof of Proposition 1.3, for all signed sets with $n \leq 14$ and on 100 test sets with $n \leq 121$. It checks Proposition 1.3 and the bound of Theorem 1.2 exhaustively for $n \leq 26$, Proposition 3.1 and the equality cases for $n \leq 40$, and Remark 3.2(2) with its own solver for $n \leq 64$.

Scope and priority. Theorems 1.1 and 1.2 answer both questions of [MMSW25, p. 2756] as stated, under the natural reading of the report. We found no citing works of the report in Crossref or OpenAlex. We found no treatment of the questions on arXiv, in later papers of the proposers, or on erdosproblems.com. The arguments are elementary, and the answers may be known informally. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find and check the arguments, to write and run the verification programs, to referee the proofs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

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[MMSW25] K. Matomäki, J. Maynard, K. Soundararajan, and T. D. Wooley. Analytic Number Theory. *Oberwolfach Reports*, 22(4):2701–2764, 2025. Report No. 51/2025, published

2026; Summary of the problem session (compiled by T. F. Bloom), problem proposed by C. Bernert and N. Arala Santos, p. 2756. doi:10.4171/OWR/2025/51.