

Conjugacy Expansion in Alternating Groups: All Degrees up to 24 and Two Infinite Families of Classes

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Abstract

A finite group G is *conjugacy-expansive* if for every normal subset S and every conjugacy class C of G the normal set SC contains at least as many conjugacy classes of G as S does. The notion is due to Halasi, Maróti, Sidki and Bezerra, who asked whether every finite simple group is conjugacy-expansive (Kourovka Notebook, Problem 19.12); in Oberwolfach Report 27/2025, Dona asked for the case of the alternating groups. We give a partial answer: A_n is conjugacy-expansive for every $5 \leq n \leq 24$ (exact computation with character-free certificates), and for every $n \geq 5$ the defining condition holds for every class of support at most 6 and, for odd n , for both classes of n -cycles (the latter using the realizability theorem of Thom and Edmonds–Kulkarni–Stong); the general case $n \geq 25$ stays open. The realizability theorem is quoted from Pervova and Petronio; the original papers of Thom and of Edmonds, Kulkarni and Stong were not read. By Hall’s theorem, as observed by Halasi et al., the condition for a class C asks for a permutation π of the classes with $\pi(K) \subseteq KC$ for every class K . For $n \leq 24$ and every C we give explicit pairs $x \in K$, $c \in C$ with $xc \in \pi(K)$, which are checked by multiplying permutations; those for $19 \leq n \leq 24$ were produced by an independent verification run and, because of their size, are regenerated by a program of the package. An exact character computation gives a second proof. For classes of small support the permutation is built from local configurations and a few explicit partners; for n -cycles it rests on a lemma saying that the two halves of a split class behave alike next to an n -cycle. We also give the short proofs of the known facts that S_n ($n \geq 3$) and A_4 are not conjugacy-expansive. No printed answer to the question was found in the sources we searched; a search that finds nothing is not a proof of novelty, and no priority is claimed. This is an unrefereed note.

1 Introduction

1.1 The question

Let G be a finite group. A *normal subset* of G is a union of conjugacy classes; for a normal subset X we write $k(X)$ for the number of classes contained in X , and $\text{Cl}(G)$ for the set of classes. If S is a normal subset and C a class, then $SC = \{sc : s \in S, c \in C\}$ is again a normal subset, and $SC = CS$.

Definition 1.1 ([HMSB12]). G is *conjugacy-expansive* if $k(SC) \geq k(S)$ for every normal subset S and every class C of G . For a fixed class C , G is *expansive for C* if $k(SC) \geq k(S)$ for every normal subset S .

Halasi, Maróti, Sidki and Bezerra introduced this notion [HMSB12] and asked whether every finite simple group is conjugacy-expansive; this is Problem 19.12 of the Kourovka Notebook [KM26] (which

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lists the four authors alphabetically), still open in its latest version. In his contribution [Don25] to the Oberwolfach workshop on computational group theory, Dona discusses products of three conjugacy classes of A_n , starting from his theorem that $C_1 C_2 C_3 = A_n$ for large n and classes of size at least $|A_n|^{1-\delta}$ [Don27]. His last paragraph asks, among other things, whether one can “settle the case $G = \text{Alt}(n)$ of Problem 19.12 of the Kourovka Notebook” [Don25, p. 1433]. This is the precise part of the problem record OWR-14299292-010 of [ula26]:

Problem 1.2. *Is A_n conjugacy-expansive for every $n \geq 5$?*

The range $n \geq 5$ is that of Problem 19.12, which concerns simple groups: A_3 is abelian, hence conjugacy-expansive, and A_4 is not (Proposition 1.3). The other questions of the same paragraph (to decide constructively whether $C_1 C_2 \supseteq C_3$ for all triples of classes, and effective versions of the large-class theorems) are programmes rather than questions with a yes-or-no answer, and are not treated here. Classes are always classes of A_n : an S_n -class of even permutations whose cycle lengths are odd and pairwise distinct splits into two classes of A_n , and both count.

1.2 Results

Theorem A. *For every n with $5 \leq n \leq 24$, the group A_n is conjugacy-expansive.*

Theorem B. *Let $n \geq 5$, and let C be a class of A_n whose elements move at most six points, that is, of cycle type $(3, 1^{n-3})$, $(2, 2, 1^{n-4})$, $(5, 1^{n-5})$, $(3, 3, 1^{n-6})$ or $(4, 2, 1^{n-6})$ (for $n \in \{5, 6\}$ each half of the class of 5-cycles). Then A_n is expansive for C .*

Theorem C. *For every odd $n \geq 5$, the group A_n is expansive for each of its two classes of n -cycles.*

Theorem A has two computer-assisted proofs (Section 3): explicit certificates which use no character theory, and an exact computation of the class multiplication coefficients. For $n \geq 24$, Theorem B is proved without computer from finite data printed in Section 4; for $n \leq 23$ it is part of Theorem A. The proof of Theorem C (Section 5) is elementary except for one input from the theory of branched coverings, the realizability theorem of Thom [Tho65] and of Edmonds, Kulkarni and Stong [EKS84], which we quote in the form stated by Pervova and Petronio [PP06, Theorems 2.7 and 3.1]; we did not read the two original papers. In the terms of the problem record the status is: partial answer: A_n is conjugacy-expansive for every $5 \leq n \leq 24$ (exact computation with character-free certificates), and for every $n \geq 5$ the defining condition holds for every class of support at most 6 and, for odd n , for both classes of n -cycles (the latter using the realizability theorem of Thom and Edmonds–Kulkarni–Stong); the general case $n \geq 25$ stays open.

Proposition 1.3. (a) [HMSB12, Theorem 1.1] *For $n \geq 3$, S_n is not conjugacy-expansive.* (b) A_4 *is not conjugacy-expansive.*

1.3 What is known, what is new, and what stays open

Halasi et al. [HMSB12] proved that a group is *normal expansive* (the condition of Definition 1.1 for normal subgroups S) if and only if it is a direct product of simple or abelian groups, the proof of one direction using the classification of finite simple groups [HMSB12, Theorem 1.1]; that a group is conjugacy-expansive if and only if it is a direct product of conjugacy-expansive simple or abelian groups [HMSB12, Theorem 1.2]; and that $\text{PSL}(2, q)$ for $q \geq 4$, the Suzuki groups $\text{Sz}(q)$ for $q = 2^{2m+1} \geq 8$, and the 26 sporadic groups are conjugacy-expansive [HMSB12, Theorem 1.3]. With the character tables in GAP they checked that “all the 138 simple groups” of the GAP character table library are conjugacy-expansive [HMSB12, Section 1], and they stated the conjecture that every finite simple group is [HMSB12, Conjecture 1.4]. The list of these groups is not printed, and the web

page with the computations cited there was not available to us. Halasi, Maróti and Petényi [HMP13] study character expansiveness, a different notion. Since $A_5 \cong \text{PSL}(2, 4)$ and $A_6 \cong \text{PSL}(2, 9)$, the cases $n = 5, 6$ of Theorem A are in [HMSB12, Theorem 1.3]; further small alternating groups are presumably among the 138 groups, but we could not check which. The tools are standard: Hall's theorem in the form of [HMSB12, Lemma 3.1], the classes and characters of S_n and A_n , and the Murnaghan–Nakayama rule [JK84].

Apart from the computer check of [HMSB12], whose list of groups is not printed, we know of no printed proof of Theorem A for $7 \leq n \leq 24$, and of none of Theorems B and C. The note does not settle Problem 1.2: for $n \geq 25$ the question stays open for all classes except those of Theorems B and C. Dona stresses elementary and constructive arguments; the certificates of Theorem A and the proof of Theorem B exhibit the permutation π of Lemma 2.2 and witnesses for it, while the proof of Theorem C uses an existence theorem. Section 6 collects computations beyond the theorems, labelled as evidence, and explains why the tight sets of Section 2 make a general proof harder. The searches behind the statement on novelty are described in the paragraph “Scope and priority” in Section 7.

2 Preliminaries

Conventions. Permutations act on $\{1, \dots, n\}$ and are composed from the right: $(xy)(i) = x(y(i))$. The *type* of x is the partition of n formed by its cycle lengths, fixed points included as parts 1; we write types with exponents for repeated parts, as in $(2^{n/2})$ or $(3, 1^{n-3})$. We write $\ell(x) = \ell(\lambda)$ for the number of cycles (parts) and $\text{supp}(x)$ for the number of points moved by x . A type λ is *even* if $n - \ell(\lambda)$ is even (the types of even permutations), and *split* if its parts are odd and pairwise distinct; split types are even. For classes K, C, L of a group G and a fixed $l \in L$ let $a(K, C; L) = \#\{(x, c) \in K \times C : xc = l\}$. Since KC is a normal subset, $L \subseteq KC$ as soon as L meets KC , that is, as soon as $a(K, C; L) > 0$.

Lemma 2.1 (standard, see [JK84]). (F1) *The permutations of an even type λ form one class of A_n if λ is not split, and two classes of equal size, the two halves of λ , if λ is split.* (F2) *If λ is split and x, y have type λ , then x and y are conjugate in A_n if and only if some, equivalently every, $g \in S_n$ with $gxg^{-1} = y$ is even.* (F3) *Every non-split class of A_n is real, that is, equal to the class of inverses. A split class of type λ is real if and only if $(n - \ell(\lambda))/2$ is even.*

Proof. If λ has an even part, the corresponding cycle of x is an odd element of the centraliser $C_{S_n}(x)$; if two parts are equal to an odd m , the product of m transpositions exchanging the two cycles is. If λ is split, $C_{S_n}(x)$ is generated by the cycles of x , which are even. So $C_{S_n}(x) \subseteq A_n$ if and only if λ is split, which gives (F1) by counting, and (F2) because the conjugators from x to y form a coset of $C_{S_n}(x)$. As x^{-1} has the type of x , (F1) gives the first part of (F3). A permutation r reversing every cycle of x satisfies $rxr^{-1} = x^{-1}$; a cycle of odd length m is reversed by a product of $(m-1)/2$ transpositions, so for split λ the parity of r is that of $\sum_i (\lambda_i - 1)/2 = (n - \ell(\lambda))/2$, and (F2) gives (F3). $\square$

Lemma 2.2 ([HMSB12, Lemma 3.1]). *A finite group G is expansive for a class C if and only if there is a permutation π of $\text{Cl}(G)$ with $\pi(K) \subseteq KC$ for every $K \in \text{Cl}(G)$.*

Proof. Join K to L in a bipartite graph on two copies of $\text{Cl}(G)$ if $L \subseteq KC$. For a set $\mathcal{S}$ of classes with union S , the classes contained in $SC = \bigcup_{K \in \mathcal{S}} KC$ are exactly the neighbours of $\mathcal{S}$, because each KC is a normal subset. So G is expansive for C if and only if every set of classes has at least as many neighbours as elements, which by Hall's theorem [Hal35] means that there is a perfect matching $K \mapsto \pi(K)$. $\square$

We call a class K *tight* for C if $K \not\subseteq KC$ (that is, $C \not\subseteq K^{-1}K$), and write $T(C)$ for the set of tight classes. A permutation π as in Lemma 2.2 moves every tight class.

Remark 2.3 (a sufficient criterion). Let $M_C = (a(K, C; L))_{K, L}$. It is the matrix of multiplication by the class sum of C on the centre of the group algebra, in the basis of class sums, so its eigenvalues are the numbers $|C|\chi(c)/\chi(1)$, $c \in C$, for the irreducible characters χ . If no irreducible character vanishes on C , then $\det M_C \neq 0$, some term $\prod_K a(K, C; \pi(K))$ of the Leibniz expansion is non-zero, and π is as in Lemma 2.2; this is the case $n = k(G)$ of [HMSB12, Proposition 5.1]. For A_n it helps little: for $5 \leq n \leq 20$ only one to four classes of A_n , among them the identity, have no vanishing irreducible character (Section 7).

Proof of Proposition 1.3. (a) Let $S = A_n$, a normal subset of S_n , and let C be the class of transpositions. Every odd permutation σ equals $(\sigma t)t$ with $\sigma t \in A_n$ for a transposition t , so $SC = S_n \setminus A_n$. Hence $k(S)$ and $k(SC)$ are the numbers $e(n)$ and $o(n)$ of partitions of n with an even, respectively odd, number $n - \ell$. Since a cycle of length m has sign $(-1)^{m-1}$, Euler's identity $\prod_{m \text{ odd}} (1 - q^m)^{-1} = \prod_{m \geq 1} (1 + q^m)$ gives

$$\sum_{n \geq 0} (e(n) - o(n))q^n = \prod_{m \text{ odd}} \frac{1}{1 - q^m} \prod_{m \text{ even}} \frac{1}{1 + q^m} = \frac{\prod_{m \geq 1} (1 + q^m)}{\prod_{m \geq 1} (1 + q^{2m})} = \prod_{m \text{ odd}} (1 + q^m).$$

So $e(n) - o(n)$ is the number of partitions of n into distinct odd parts, which is positive for $n \geq 3$ (take (n) or $(n - 1, 1)$), and $k(SC) < k(S)$. (b) Let $S = V_4$, the union of the identity and the class of type $(2, 2)$, and let C be a class of 3-cycles, $c \in C$. Conjugation acts trivially on $A_4/V_4 \cong C_3$, so $C \subseteq V_4 c$, and both sets have four elements. Hence $SC = V_4 V_4 c = C$ and $k(SC) = 1 < 2 = k(S)$. Both statements also follow from [HMSB12, Theorem 1.1], since S_n ($n \geq 3$) and A_4 are not direct products of simple or abelian groups; the witnesses above are normal subgroups. $\square$

3 All degrees up to 24

3.1 Certificates

A *certificate* for a class $C \neq \{1\}$ of A_n is a family of pairs (x_K, c_K) , indexed by the classes K of A_n , with $x_K \in K$ and $c_K \in C$, such that $K \mapsto \pi_C(K) := \text{the class of } x_K c_K$ is a permutation of $\text{Cl}(A_n)$. Since $x_K c_K \in KC \cap \pi_C(K)$, we have $\pi_C(K) \subseteq KC$, and Lemma 2.2 shows that A_n is expansive for C ; for $C = \{1\}$ take $\pi = \text{id}$. Checking a certificate needs only products of permutations and the class test of Lemma 2.1: the type, and for a split type the parity of the permutation which maps the standard element $(12 \cdots \lambda_1)(\lambda_1 + 1 \cdots) \cdots$ of that type to x , cycle by cycle (the cycles of x ordered by length, each written from its smallest point).

Proof of Theorem A. For every $5 \leq n \leq 24$ and every class $C \neq \{1\}$ of A_n there is a certificate:

- (i) for $5 \leq n \leq 18$ and every C , and for $5 \leq n \leq 23$ and the classes of support at most six, the certificates of the package (114,677 pairs, of which 99,720 for the first family). They were found by a random search: for each C a permutation π_C was chosen on the exact supports of Section 3.2, and for each K a pair with $x_K c_K \in \pi_C(K)$ was found among random conjugates;
- (ii) for $19 \leq n \leq 24$ and every C , certificates produced by one of the verification runs of the first round (1,653,478 pairs), by random search alone: random products $x c_0$ ($x \in K$) and $l_0 c^{-1}$ ($c \in C$) for fixed $c_0 \in C$, $l_0 \in L$, and a perfect matching (Hopcroft–Karp) among the edges found. The generator reads the exact supports of Section 3.2 as a guide for a fallback search, which was never used. Because of their size (61.6 MB compressed) these certificates are not in the package; its generator, run with the recorded seed, produces them again (Section 7), as the verification run of the second round did for all six files, with identical content.

The certificates (i) were accepted by five separate checking programs (that of the first version of the results, one of each of the three verification runs, and `check_note.py` of the package), those of (ii) by four (the checker of the run that produced them, that of the first version, `check_note.py`, and that of the run of the second round). Every pair is valid, and for every n every class $C \neq \{1\}$ is certified. $\square$

3.2 An exact computation

The second proof computes, for every $n \leq 24$ and every class C , the set of pairs (K, L) with $a(K, C; L) > 0$ from the character formula

$$a(K, C; L) = \frac{|K||C|}{|A_n|} \sum_{\psi} \frac{\psi(x_K) \psi(x_C) \overline{\psi(x_L)}}{\psi(1)},$$

the sum over the irreducible characters of A_n and $x_K \in K$ etc. These are obtained from the characters χ_μ of S_n (Murnaghan–Nakayama rule) by Frobenius’ rule [JK84]: if $\mu \neq \mu'$, then χ_μ restricts irreducibly (and like $\chi_{\mu'}$); if $\mu = \mu'$ and $h = (h_1, \dots, h_d)$ are the diagonal hook lengths of μ (a split type), then χ_μ restricts to $\psi_\mu^+ + \psi_\mu^-$, where $\psi_\mu^\pm = \chi_\mu/2$ outside the two classes of type h , and $\psi_\mu^\pm$ take the values $(\varepsilon \pm \sqrt{\varepsilon h_1 \cdots h_d})/2$ on them, $\varepsilon = (-1)^{(n-d)/2}$. The sum is evaluated, with the value $\psi(x_L^{-1}) = \overline{\psi(x_L)}$ at the inverse class in place of the complex conjugate, modulo several primes $p \approx 10^8$, $p \equiv 3 \pmod{4}$, for which every number $\varepsilon h_1 \cdots h_d$ is a square modulo p (as $p > n$, the order of A_n and the degrees are invertible modulo p). The two terms of ψ_μ^+ and ψ_μ^- together are a polynomial in $\varepsilon h_1 \cdots h_d$, so the choice of square roots is immaterial; and since $0 \leq a(K, C; L) \leq |C|$ is smaller than the product of the primes, $a(K, C; L) = 0$ if and only if all residues vanish. The program checks the column orthogonality of the table and the identity $\sum_L a(K, C; L)|L| = |K||C|$ modulo every prime, computes one Garner digit more than needed (it is 0), and agrees with a brute force over A_n for $4 \leq n \leq 9$. For every C it found a perfect matching (Hopcroft–Karp), and every saved matching was checked again against the supports (32 minutes in all). The three verification runs repeated this with their own programs. One run of the first round worked without square roots (with $\psi_\mu^\pm = a \pm bs$, $s^2 = \varepsilon h_1 \cdots h_d$, the sum involves only a , b , s^2 and $s\bar{s} = h_1 \cdots h_d$); its supports coincide with those of the note for every $n \leq 24$ and every C , and its structure constants agree with a brute force for $n \leq 10$. The run of the second round also worked without square roots, writing the two terms of $\psi_\mu^\pm$ as terms with $\chi_\mu/2$ plus explicit terms in $h_1 \cdots h_d$; it found a perfect matching for every class of A_n , $n \leq 27$, and its structure constants agree with a brute force for $n \leq 10$. Table 1 lists the class numbers and the largest tight sets.

n	5	6	7	8	9	10	11	12	13	14
$k(A_n)$	5	7	9	14	18	24	31	43	55	72
$\max_C T(C) $	3	3	3	5	6	6	8	10	11	15
n	15	16	17	18	19	20	21	22	23	24
$k(A_n)$	94	123	156	200	254	324	408	513	641	804
$\max_C T(C) $	19	22	24	34	48	56	58	77	101	121
n (evidence)	25	26	27							
$k(A_n)$	997	1236	1526							
$\max_C T(C) $	121	160	227							

Table 1: Number of classes of A_n and the largest number of tight classes $|T(C)|$ over the classes C of A_n . All values for $n \leq 24$ were obtained by the four exact computations of Section 3.2, those for $n = 25, 26, 27$ by two (Section 6).

4 Classes of support at most six

In this section C is a class of A_n of type $(\mu, 1^{n-s})$, where $\mu \in \{(3), (2, 2), (5), (3, 3), (4, 2)\}$ lists the non-trivial cycle lengths and $s = |\mu| \leq 6$.

Lemma 4.1 (swap lemma). *Let C be a real class of a finite group G , let E be a set of classes containing $T(C)$, and let $P: E \rightarrow \text{Cl}(G) \setminus E$ be injective with $P(K) \subseteq KC$ for every $K \in E$. Then G is expansive for C .*

Proof. Let π exchange K and $P(K)$ for every $K \in E$ and fix all other classes; π is a permutation because P is injective and $P(E) \cap E = \emptyset$. For $K \in E$, $\pi(K) = P(K) \subseteq KC$. For $L = P(K)$ write $xc = l$ with $x \in K$, $c \in C$, $l \in L$; then $x = lc^{-1}$ with $c^{-1} \in C^{-1} = C$, so $\pi(L) = K \subseteq LC$. Every other class is not in E , hence not tight, and $\pi(K) = K \subseteq KC$. Now apply Lemma 2.2. $\square$

Definition 4.2 (gadgets). For a partition τ of r let $x_\tau = (1\ 2 \cdots \tau_1)(\tau_1+1 \cdots \tau_1+\tau_2) \cdots \in \text{Sym}(r)$ be the standard permutation of type τ . A *gadget* for μ is a pair (τ, c) , where τ is a partition of some $r \geq s$ and $c \in \text{Sym}(r)$ has type $(\mu, 1^{r-s})$, such that $x_\tau c$ again has type τ . If τ is split, the gadget is *even* if $x_\tau c = gx_\tau g^{-1}$ for an even $g \in \text{Sym}(r)$; by (F2) in $\text{Sym}(r)$ this does not depend on g . A *long gadget* is one with $\tau = (L)$; then $x_\tau = z_L := (1\ 2 \cdots L)$.

Lemma 4.3 (gadget lemma). *Let $n \geq s + 2$, let (τ, c) be a gadget for μ , and let K be a class of A_n of a type λ which contains τ as a multiset of parts. If λ is split, assume that the gadget is even. Then K is not tight for C .*

Proof. Write $\lambda = \tau \cup \rho$, let $x_0 = x_\tau x'$ with x' a permutation of type ρ of $\{r+1, \dots, n\}$, and let c' be c extended by the identity. The type $(\mu, 1^{n-s})$ of c' has two parts 1, so $c' \in C$ by (F1), and $y_0 = x_0 c' = (x_\tau c)x'$ has type λ . If λ is not split, $x_0, y_0 \in K$ by (F1). If λ is split, then $y_0 = gx_0 g^{-1}$ with g the even permutation of the gadget, extended by the identity; if $x_0 \notin K$, conjugate x_0, c' and y_0 by a transposition t , which keeps c' in C and g even. In all cases we obtain $x \in K$ and $c \in C$ such that xc is conjugate to x by an even permutation, so $xc \in K$ by (F2), and $K \subseteq KC$. $\square$

Lemma 4.4 (locality). *Let c be a permutation of $\{1, \dots, r\}$, regarded as a permutation of $\{1, \dots, L\}$ for $L > r$. Then $(z_L c)(i) = c(i) + 1$ for $i \leq r$ and $(z_L c)(i) = z_L(i)$ for $i > r$. Hence $z_L c$ is an L -cycle if and only if the orbit of 1 under $f: i \mapsto c(i) + 1$ passes through all of $\{1, \dots, r\}$ before it reaches $r+1$, a condition independent of L . In that case the permutation g with $g(z_L^j(1)) = (z_L c)^j(1)$ for all j conjugates z_L to $z_L c$, fixes $r+1, \dots, L$, and does not depend on L .*

Proof. The formula holds because $c(i) \leq r < L$ for $i \leq r$. Since f maps $\{1, \dots, r\}$ bijectively onto $\{2, \dots, r+1\}$, the f -orbit of 1 leaves $\{1, \dots, r\}$, necessarily at $r+1$; from there the cycle of $z_L c$ through 1 continues with $r+2, \dots, L, 1$. This gives the criterion, and in that case $(z_L c)^j(1) = f^j(1)$ for $j \leq r$, with $f^r(1) = r+1$, and $(z_L c)^j(1) = j+1 = z_L^j(1)$ for $r \leq j < L$. $\square$

So a long gadget given by $c \in \text{Sym}(r)$ holds for every $L \geq r$ once the cases $L = r$ and $L = r+1$ are checked, and its conjugator, hence its parity, is the same for all $L > r$. Tables 2 and 3 list the gadgets used below. The odd conjugators of the first long gadgets of $\mu = (3), (2, 2), (5)$ are recorded only to explain why a second, even long gadget is needed.

Proof of Theorem B. For $5 \leq n \leq 23$ this is part of Theorem A (and the certificates (i) of Section 3 cover these classes). Let $n \geq 24$. Then (i) C has at least 18 fixed points, so it is not split and it is real (F1, F3); (ii) every split type of n has a part ≥ 9 , since distinct odd parts at most 7 add up to at most 16; (iii) every type with all parts at most 5 is not split ($1+3+5 < 24$).

μ	c	$z_L c$	L	g	parity
(3)	(1 2 3)	(1 3 2 4 5 $\cdots$ L)	≥ 3	(2 3)	odd
(3)	(1 3 5)	(1 4 5 2 3 6 $\cdots$ L)	≥ 5	(2 4)(3 5)	even
(2, 2)	(1 3)(2 4)	(1 4 3 2 5 $\cdots$ L)	≥ 4	(2 4)	odd
(2, 2)	(1 4)(2 6)	(1 5 6 3 4 2 7 $\cdots$ L)	≥ 6	(2 5 4 3 6)	even
(5)	(1 2 3 4 5)	(1 3 5 2 4 6 $\cdots$ L)	≥ 5	(2 3 5 4)	odd
(5)	(1 2 3 5 6)	(1 3 6 2 4 5 7 $\cdots$ L)	≥ 6	(2 3 6 5 4)	even
(3, 3)	(1 2 3)(4 5 6)	(1 3 2 4 6 5 7 $\cdots$ L)	≥ 6	(2 3)(5 6)	even
(4, 2)	(1 2 3 5)(4 6)	(1 3 6 5 2 4 7 $\cdots$ L)	≥ 6	(2 3 6 4 5)	even

Table 2: Long gadgets: $z_L = (1\ 2\ \cdots\ L)$, and g is the conjugator of Lemma 4.4 from z_L to $z_L c$ (the same for all $L > r$). The even gadgets are used only for split types, that is, for odd L .

Step 1: the tight classes. Let λ be an even type of n and K a class of type λ . We show that K is not tight unless λ lies in the set $E(\mu, n)$ defined by

$$E((3), n) = E((5), n) = \{(1^n), (2^{n/2})\}, \quad E((2, 2), n) = E((3, 3), n) = \{(1^n)\},$$

$$E((4, 2), n) = \{(1^n), (3, 1^{n-3}), (2^{n/2}), (2^{(n-1)/2}, 1), (3^{n/3})\},$$

where a member is present only if it is an even type of n (that is, $(2^{n/2})$ if $4 \mid n$, $(2^{(n-1)/2}, 1)$ if $n \equiv 1 \pmod{4}$, and $(3^{n/3})$ if $3 \mid n$). Let $\lambda \neq (1^n)$. If λ is split, it has a part ≥ 9 by (ii), and the even long gadget of μ in Table 2 applies (Lemmas 4.3 and 4.4). Let λ be non-split, with largest part m . If m is at least the bound $L \geq 3, 4, 5, 6$ of the first long gadget of $\mu = (3), (2, 2), (5), (3, 3), (4, 2)$, that gadget applies. Otherwise λ contains a small gadget τ of Table 3, as follows, or lies in $E(\mu, n)$; write q for a part of λ other than one largest part, which exists since $n > m$.

- $\mu = (3)$, $m \leq 2$: $\lambda = (2^a, 1^b)$ with $a \geq 1$; if $b \geq 1$, $\tau = (2, 1)$, and otherwise $\lambda = (2^{n/2})$.
- $\mu = (2, 2)$: if $m = 3$, $\tau = (3, q)$; if $m = 2$, $\lambda = (2^a, 1^b)$ with $a \geq 2$ even, and $\tau = (2, 2)$.
- $\mu = (5)$: if $m = 4$, $\tau = (4, q)$. If $m = 3$, write $\lambda = (3^a, 2^b, 1^c)$: $a \geq 2$ gives $(3, 3)$; $a = 1$, $b \geq 1$ gives $(3, 2)$; $a = 1$, $b = 0$ gives $(3, 1, 1)$. If $m = 2$, $\lambda = (2^a, 1^b)$ with $a \geq 2$: $b \geq 1$ gives $(2, 2, 1)$, and $b = 0$ is $(2^{n/2})$.
- $\mu = (3, 3)$: if $m = 5$, $\tau = (5, q)$. If $m = 4$, write $\lambda = (4^a, 3^b, 2^c, 1^d)$: $b \geq 1$, $c \geq 1$, $a \geq 2$ give $(4, 3)$, $(4, 2)$, $(4, 4)$, and otherwise $\lambda \supseteq (4, 1, 1)$. If $m = 3$, write $\lambda = (3^b, 2^c, 1^d)$: $b \geq 2$ gives $(3, 3)$, and for $b = 1$ the cases $c \geq 2$, $c = 1$, $c = 0$ give $(3, 2, 2)$, $(3, 2, 1)$, $(3, 1, 1, 1)$. If $m = 2$, $\lambda = (2^c, 1^d)$ with $c \geq 2$ even: $c \geq 4$ gives $(2, 2, 2)$ and $c = 2$ gives $(2, 2, 1, 1)$.
- $\mu = (4, 2)$: $m = 5$ and $m = 4$ as for $(3, 3)$. If $m = 3$, write $\lambda = (3^b, 2^c, 1^d)$: $c \geq 2$ gives $(3, 2, 2)$; $c = 1$, $d \geq 1$ gives $(3, 2, 1)$; $c = 1$, $d = 0$ forces $b \geq 2$ and gives $(3, 3, 2)$; $c = 0$, $b \geq 2$, $d \geq 1$ gives $(3, 3, 1)$; the remaining cases are $(3^{n/3})$ and $(3, 1^{n-3})$. If $m = 2$, $\lambda = (2^c, 1^d)$ with $c \geq 2$: $d \geq 2$ gives $(2, 2, 1, 1)$, and $d \leq 1$ gives $(2^{n/2})$ or $(2^{(n-1)/2}, 1)$.

In each case Lemma 4.3 shows that K is not tight; the small gadgets are applied only to non-split types, so their parity plays no role (one of them, $((5, 3), (1\ 2\ 3\ 6)(4\ 8))$ for $\mu = (4, 2)$, has in fact an odd conjugator). Hence $T(C) \subseteq E(\mu, n)$, and all members of $E(\mu, n)$ are non-split.

Step 2: partners. Define P on $E(\mu, n)$ by Table 4. In each row c has type $(\mu, 1^{n-s})$, so $c \in C$; x has the type of K ; and xc has the type of $P(K)$, which is not split, so $xc \in P(K)$ by (F1) and $P(K) \subseteq KC$. For fixed μ and n the partners have pairwise different types, none of which lies in $E(\mu, n)$; so P is injective with image disjoint from $E(\mu, n)$.

τ	C	$x_{\tau}C$	τ	C	$x_{\tau}C$
$\mu = (3)$			$\mu = (3, 3)$, continued		
(2, 1)	(1 2 3)	(2 3)	(4, 4)	(1 2 3)(5 6 7)	(1 3 2 4)(5 7 6 8)
$\mu = (2, 2)$			(3, 3)	(1 2 3)(4 5 6)	(1 3 2)(4 6 5)
(3, 1)	(1 2)(3 4)	(1 3 4)	(3, 2, 2)	(1 4 6)(2 7 5)	(1 5 3)(2 6)(4 7)
(3, 2)	(1 4)(2 5)	(1 5 3)(2 4)	(3, 2, 1)	(1 2 3)(4 5 6)	(1 3 2)(5 6)
(3, 3)	(1 4)(2 5)	(1 5 3)(2 6 4)	(3, 1, 1, 1)	(1 3 2)(4 5 6)	(4 5 6)
(2, 2)	(1 3)(2 4)	(1 4)(2 3)	(2, 2, 2)	(1 3 5)(2 6 4)	(1 4)(2 5)(3 6)
$\mu = (5)$			(2, 2, 1, 1)	(1 2 5)(3 4 6)	(2 5)(4 6)
(4, 1)	(1 2 4 3 5)	(1 3 5 2)	$\mu = (4, 2)$		
(4, 2)	(1 2 3 4 5)	(1 3)(2 4 6 5)	(5, 1)	(1 2 3 6)(4 5)	(1 3 6 2 4)
(4, 3)	(1 2 4 5 3)	(1 3 2)(4 6 7 5)	(5, 2)	(1 2 3 4)(5 6)	(1 3 5 7 6)(2 4)
(4, 4)	(1 2 5 3 8)	(1 3 5 4)(2 6 7 8)	(5, 3)	(1 2 3 6)(4 8)	(1 3 7 8 5)(2 4 6)
(3, 3)	(1 2 4 3 6)	(1 3 4)(2 5 6)	(5, 4)	(1 2 3 6)(4 7)	(1 3 7 5)(2 4 8 9 6)
(3, 2)	(1 2 4 3 5)	(1 3 4)(2 5)	(5, 5)	(1 2 3 6)(4 8)	(1 3 7 8 5)(2 4 9 10 6)
(3, 1, 1)	(1 3 2 4 5)	(2 4 5)	(4, 1, 1)	(1 4 3 5)(2 6)	(2 6 3 5)
(2, 2, 1)	(1 2 3 5 4)	(2 4)(3 5)	(4, 2)	(1 2 4 5)(3 6)	(1 3 5 2)(4 6)
$\mu = (3, 3)$			(4, 3)	(1 2 3 5)(4 6)	(1 3 6)(2 4 7 5)
(5, 1)	(1 2 3)(4 6 5)	(1 3 2 4 6)	(4, 4)	(1 2 3 5)(4 7)	(1 3 6 7)(2 4 8 5)
(5, 2)	(1 2 4)(3 5 6)	(1 3)(2 5 7 6 4)	(3, 2, 1)	(1 2 4 5)(3 6)	(1 3 6)(2 5)
(5, 3)	(1 2 3)(6 7 8)	(1 3 2 4 5)(6 8 7)	(3, 2, 2)	(1 2 4 6)(5 7)	(1 3)(2 5 6)(4 7)
(5, 4)	(1 2 3)(4 5 6)	(1 3 2 4)(5 7 8 9 6)	(3, 3, 1)	(1 2 4 6)(3 7)	(1 3 7)(2 5 6)
(5, 5)	(1 2 3)(6 7 8)	(1 3 2 4 5)(6 8 7 9 10)	(3, 3, 2)	(1 2 4 7)(5 8)	(1 3)(2 5 7)(4 8 6)
(4, 1, 1)	(1 3 2)(4 5 6)	(1 4 5 6)	(2, 2, 1, 1)	(1 2 3 4)(5 6)	(2 4)(5 6)
(4, 2)	(1 2 5)(3 4 6)	(1 3)(2 6 4 5)			
(4, 3)	(1 2 3)(5 6 7)	(1 3 2 4)(5 7 6)			

Table 3: The 42 small gadgets (τ, c) : x_{τ} is the standard permutation of type τ on $\{1, \dots, |\tau|\}$, for example $x_{(5,3)} = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$, and $x_{\tau}C$ has type τ .

Step 3. C is real by (i), $T(C) \subseteq E(\mu, n)$ by Step 1, and P is as in Lemma 4.1 by Step 2. So A_n is expansive for C . $\square$

Remark 4.5. The proof gives π explicitly: it exchanges the identity with C and every other member of $E(\mu, n)$ with its partner, and fixes all other classes. The products in Tables 2 to 4 and the case analysis of Step 1 were also checked by computer (Section 7). For $12 \leq n \leq 31$ exact computations show $T(C) = E(\mu, n)$ for all five μ , so Step 1 is sharp there (Section 6).

5 The classes of n -cycles

Let n be odd. Then (n) is a split type; we write C_+ and C_- for its two halves and T_3 for the class of 3-cycles. For odd n every even type has an odd number of parts. The only input from outside group theory is the following theorem.

Theorem 5.1 (Thom; Edmonds, Kulkarni and Stong; as stated in [PP06, Theorem 2.7]). *A compatible branch datum is realizable if one of its partitions is (d) .*

Pervova and Petronio attribute this to Thom [Tho65, Theorem 1] for coverings of the sphere by the sphere and to Edmonds, Kulkarni and Stong [EKS84, Proposition 5.2] in general; we did not read these papers. We need the case of a covering of degree n of the sphere, branched over three points with local degrees given by partitions $\lambda, (n), \nu$ of n , all different from (1^n) as in [PP06], by a closed orientable surface of genus $g \geq 0$. Such a datum is compatible, that is, it satisfies conditions (1)–(5) of [PP06, Section 1], if and only if $\ell(\lambda) + 1 + \ell(\nu) - n = 2 - 2g$ (the Riemann–Hurwitz condition

μ	$K \in E(\mu, n)$	$P(K)$	x (local part)	c	xc (local part)
all	(1^n)	C	1	any	c
(3)	$(2^{n/2})$	$(3, 1, 2^{(n-4)/2})$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(2\ 4\ 3)$
(5)	$(2^{n/2})$	$(3, 1, 2^{(n-4)/2})$	$(1\ 2)(3\ 4)(5\ 6)$	$(1\ 2\ 3\ 5\ 4)$	$(2\ 4)(3\ 6\ 5)$
(4, 2)	$(3, 1^{n-3})$	$(4, 3, 2, 1^{n-9})$	$(7\ 8\ 9)$	$(1\ 2\ 3\ 4)(5\ 6)$	$(1\ 2\ 3\ 4)(5\ 6)(7\ 8\ 9)$
(4, 2)	$(2^{n/2})$	$(3, 1, 2^{(n-4)/2})$	$(1\ 2)(3\ 4)(5\ 6)$	$(1\ 2\ 3\ 5)(4\ 6)$	$(2\ 4\ 5)(3\ 6)$
(4, 2)	$(2^{(n-1)/2}, 1)$	$(3, 1, 1, 2^{(n-5)/2})$	$(1\ 2)(3\ 4)(5\ 6)$, 7 fixed	$(1\ 2\ 3\ 4)(5\ 7)$	$(2\ 4)(5\ 7\ 6)$
(4, 2)	$(3^{n/3})$	$(5, 1, 3^{n/3-2})$	$(1\ 2\ 3)(4\ 5\ 6)$	$(1\ 2\ 3\ 4)(5\ 6)$	$(1\ 3\ 5\ 4\ 2)$

Table 4: Partners for the members of $E(\mu, n)$, $n \geq 24$. In each row x is the local part shown times the standard permutation of the remaining type of K on the points not shown, and c fixes those points.

(1); the parity condition (2) is then automatic, (3) holds as the covering surface is orientable, and (4) and (5) concern non-orientable base surfaces). By [PP06, Theorem 3.1] a datum is realizable if and only if there are $s_1, s_2, s_3 \in S_n$ of types $\lambda, (n), \nu$ with $s_1 s_2 s_3 = 1$ generating a transitive group, and transitivity is automatic here. Since $s_1 s_2 = s_3^{-1}$ has type ν , we obtain, after conjugation, the following statement (the cases with (1^n) are trivial).

Corollary 5.2. *Let λ, ν be partitions of n with $\ell(\lambda) + \ell(\nu) \leq n + 1$ and $\ell(\lambda) + \ell(\nu) \equiv n + 1 \pmod{2}$. Then for every l of type ν there are x of type λ and an n -cycle c with $xc = l$.*

Lemma 5.3 (the halves do not matter). *Let $n \geq 5$ be odd, $C \in \{C_+, C_-\}$, and let K, L be classes of A_n whose types λ, ν are different from (n) . Then $a(K, C; L)$ depends only on λ and ν .*

Proof. Use the character formula of Section 3.2; the class sizes do not depend on the halves. The terms of the characters $\chi_\mu|_{A_n}$ with $\mu \neq \mu'$ do not see the halves. Let $\mu = \mu'$ with diagonal hook lengths h . By the Murnaghan–Nakayama rule, $\chi_\mu((n)) = 0$ unless μ is a hook [JK84]. If $h \in \{\lambda, \nu\}$, then $h \neq (n)$ has at least two parts, μ is not a hook, and $\psi_\mu^\pm(x_C) = \chi_\mu((n))/2 = 0$, so both terms vanish. If $h = (n)$, that is $\mu = \eta := ((n+1)/2, 1^{(n-1)/2})$, then $\psi_\eta^\pm$ equal $\chi_\eta/2$ at x_K and x_L , and the two terms add up to

$$\frac{\chi_\eta(\lambda)\chi_\eta(\nu)(\psi_\eta^+(x_C) + \psi_\eta^-(x_C))}{2\chi_\eta(1)} = \frac{\chi_\eta(\lambda)\chi_\eta(\nu)\chi_\eta((n))}{2\chi_\eta(1)},$$

which does not depend on C . If $h \notin \{\lambda, \nu, (n)\}$, all three values are given by $\chi_\mu/2$. $\square$

Corollary 5.4. *Let $n \geq 5$ be odd, and let K, L be classes of A_n whose types λ, ν are not (1^n) or (n) , with $\ell(\lambda) + \ell(\nu) \leq n + 1$. Then $L \subseteq KC_+$ and $L \subseteq KC_-$.*

Proof. Fix $l \in L$. The sum of $a(K', C'; L)$ over the classes K' of type λ and over $C' \in \{C_+, C_-\}$ is the number of pairs (x, c) with x of type λ , c an n -cycle and $xc = l$, which is positive by Corollary 5.2 ($\ell(\lambda)$ and $\ell(\nu)$ are odd). By Lemma 5.3 all summands are equal, so each is positive. $\square$

Lemma 5.5 (an injection). *Let $n \geq 7$ be odd. For an even type $\lambda \neq (1^n)$ with $\ell(\lambda) \geq (n+3)/2$ put $i = n - \ell(\lambda)$, let ρ be the partition of i formed by the numbers $\lambda_j - 1$ with $\lambda_j > 1$, and $r = \ell(\rho)$. Let $P(\lambda) = (n - i, \rho)$ if r is even and $P(\lambda) = (n - i - 1, \rho, 1)$ if r is odd. Then P is injective, $P(\lambda)$ is an even type with $3 \leq \ell(P(\lambda)) \leq (n - 1)/2$, and $\ell(\lambda) + \ell(P(\lambda)) \leq n + 1$.*

Proof. Here i is even, $2 \leq i \leq (n-3)/2$, and $r \geq 1$. Since $n - i - 1 \geq (n+1)/2 > i \geq \rho_1$, both formulas define partitions of n , with largest part $n - i$ (odd) or $n - i - 1$ (even). If r is even, $\ell(P(\lambda)) = r + 1$ is odd, so $P(\lambda)$ is even, and $\ell(\lambda) + \ell(P(\lambda)) = n - i + r + 1 \leq n + 1$ as $r \leq i$. If r is odd, then $r \leq i - 1$, $\ell(P(\lambda)) = r + 2$ is odd, and $\ell(\lambda) + \ell(P(\lambda)) \leq n + 1$. In both cases $3 \leq \ell(P(\lambda)) \leq i + 1 \leq (n - 1)/2$. The parity of the largest part of $P(\lambda)$ tells the case, and then $P(\lambda)$ determines i, ρ and λ . $\square$

Lemma 5.6 (3-cycles change the half). *Let $n \geq 5$ be odd, $z = (a_1 a_2 \cdots a_n)$ and $t = (a_1 a_2 a_3)$. Then $zt = (a_1 a_3 a_2 a_4 \cdots a_n)$, and zt and tz lie in the class of n -cycles that does not contain z .*

Proof. The product is direct. The odd permutation $(a_2 a_3)$ conjugates z to zt , so zt lies in the other half by (F2), and $tz = t(zt)t^{-1}$ with t even. $\square$

Proof of Theorem C. For $n = 5$ use Theorem A; let $n \geq 7$. Fix $C \in \{C_+, C_-\}$ and let C' be the other class of n -cycles; by (F3), $C^{-1} = C$ if $(n-1)/2$ is even and $C^{-1} = C'$ otherwise. Let U be the set of classes $K \neq \{1\}, T_3$ with $\ell(K) \geq (n+3)/2$. They are not split, since a split type with ℓ parts has sum at least $1 + 3 + \cdots + (2\ell - 1) = \ell^2 > n$. For $K \in U$ let $Q(K)$ be a class of type $P(\lambda)$, λ the type of K (Lemma 5.5; either half if that type splits). Let π

- exchange $\{1\}$ with C and T_3 with C' if C is real, and map $\{1\} \mapsto C \mapsto T_3 \mapsto C' \mapsto \{1\}$ if not;
- exchange K and $Q(K)$ for every $K \in U$, and fix all other classes.

This is a permutation: by Lemma 5.5 the classes $Q(K)$ are pairwise distinct and have $3 \leq \ell \leq (n-1)/2$, so they are not $\{1\}$ ($\ell = n$), T_3 ($\ell = n-2$), C or C' ($\ell = 1$), and not in U . We check $\pi(K) \subseteq KC$ for every class K . (1) $\pi(\{1\}) = C$. If C is real, $\pi(C) = \{1\} \subseteq C^{-1}C = CC$; if not, $\pi(C') = \{1\} \subseteq C^{-1}C = C'C$. (2) C real: take $c \in C$, $z = c^{-1} \in C$, and t a 3-cycle on three consecutive points of z as in Lemma 5.6. Then $x = tz \in C'$ and $xc = t$, so $\pi(C') = T_3 \subseteq C'C$; and $x = tc^{-1}$ with $c^{-1} \in C$, so $\pi(T_3) = C' \subseteq T_3C$. (3) C not real: with $c \in C$, $z = c^{-1} \in C'$ and t as before, $x = tz \in C$ and $xc = t$, so $\pi(C) = T_3 \subseteq CC$; and for $c \in C$ and a 3-cycle t on three consecutive points of c , $tc \in C'$, so $\pi(T_3) = C' \subseteq T_3C$. (4) For $K \in U$, $\ell(K) + \ell(Q(K)) \leq n+1$, and neither type is (1^n) or (n) ; so $Q(K) \subseteq KC$ and $K \subseteq Q(K)C$ by Corollary 5.4. (5) A class K fixed by π is not $\{1\}$, C , C' , T_3 , and not in U ; so its type is not (1^n) or (n) , and $\ell(K) \leq (n+1)/2$. Thus $2\ell(K) \leq n+1$, and $K \subseteq KC$ by Corollary 5.4. By Lemma 2.2, A_n is expansive for C . $\square$

Remark 5.7. For odd $7 \leq n \leq 31$, exact computations show more: for all classes $K, L \neq \{1\}$, including the classes of n -cycles, $L \subseteq KC_+$ if and only if $\ell(K) + \ell(L) \leq n+1$, and likewise for C_- (Section 6). The proof above needs only Corollary 5.4 and the 3-cycles of Lemma 5.6.

Remark 5.8. The classes of n -cycles are the classes with the fewest cycles, not the largest classes of A_n : $|C_+| = n!/(2n)$ is smaller than $n!/(2(n-2))$, the size of the class of type $(n-2, 1, 1)$.

6 Evidence beyond the theorems, and conclusion

The following computations go beyond the theorems. They are reported as evidence, not as results.

- Two exact computations of verification runs, with programs written independently of each other (the method of Section 3.2 without square roots, with five to seven primes and the Garner digits beyond the needed ones checked to vanish), found a perfect matching for every class C of A_{25} , A_{26} and A_{27} , with the numbers of Table 1. So no counterexample exists for $n \leq 27$ according to these computations; for these n no certificates were produced.
- For $12 \leq n \leq 31$ and the five types μ of Theorem B, the exact tight sets are $T(C) = E(\mu, n)$ (four computations for $n \leq 24$, two for $25 \leq n \leq 27$, one for $28 \leq n \leq 31$), and the permutation of Remark 4.5 is valid on the exact supports (two computations for $n \leq 27$, one beyond).
- For odd $7 \leq n \leq 31$ the rule of Remark 5.7 holds, and the permutation of the proof of Theorem C is valid on the exact supports (at least two computations for $n \leq 27$, one beyond); for $3 \leq n \leq 32$, odd and even, Corollary 5.2 and its converse hold for all pairs of partitions of n (computed with the hook characters, the only characters of S_n that do not vanish on n -cycles; two computations for $5 \leq n \leq 31$).

What blocks a general proof. The tight sets grow with n (Table 1), so no argument with a bounded number of exceptional classes, as in Lemma 4.1, works for all C . A general proof needs, uniformly in n , a description of the classes K with $C \not\subseteq K^{-1}K$, an existence problem with genuine exceptions (a product of two fixed-point-free involutions has every cycle length with even multiplicity, so $(2^{n/2}) \in T(C)$ for $\mu = (3)$ and $4 \mid n$); an injective partner map on these classes; and, for non-real split classes C , a substitute for Lemma 4.1, such as the 4-cycle of classes in the proof of Theorem C. An analogue of Theorem C for even n (classes of type $(n-1, 1)$) would need the realizable data with a partition $(d-1, 1)$, classified in [EKS84, Proposition 5.3] according to [PP06], with exceptions such as $(3, 1), (2, 2), (2, 2)$ [PP06, Section 2]; we did not pursue it.

Conclusion. For Problem 1.2 the note gives a partial answer: A_n is conjugacy-expansive for every $5 \leq n \leq 24$ (exact computation with character-free certificates), and for every $n \geq 5$ the defining condition holds for every class of support at most 6 and, for odd n , for both classes of n -cycles (the latter using the realizability theorem of Thom and Edmonds–Kulkarni–Stong); the general case $n \geq 25$ stays open. The case of alternating groups of Problem 19.12 of [KM26] is not settled.

7 Computations and reproducibility

Theorem A rests on computations. The proofs of Theorem B for $n \geq 24$ and of Theorem C do not, but their finite data were also checked by computer. The reproducibility package has five parts.

The programs with which the results were obtained (`original/`; Python 3 with NumPy, SciPy and NetworkX), with their recorded outputs and the certificates (i): the exact computation of Section 3.2 with its brute-force cross-check, the generator and checker of the certificates, the checks of Tables 2 to 4 and of Step 1 for $24 \leq n \leq 47$, and the checks for Section 5 and Remark 2.3.

The programs of the two verification runs of the first round (`independent_runs/`), with their recorded outputs, among them the generator of the certificates (ii) and the SHA-256 checksums of the files it produced.

The programs of the verification run of the second round (`independent_run_2/`), with their recorded outputs: a brute force over A_n for $n \leq 10$, an exact computation of the supports for $n \leq 27$, the checks of Tables 2 to 4 as read from the $\text{\TeX}$ source of the note and of Step 1 for $24 \leq n \leq 52$, a checker of the certificates, Corollary 5.2 for $n \leq 32$, and the record of its re-run of the package.

The program `check_note.py` (standard library only, about 25 seconds). It checks the class test of Lemma 2.1 against conjugation orbits ($n \leq 8$); Hall matchings from brute-force supports for $5 \leq n \leq 8$, and the failure for A_4 ; Proposition 1.3(a); every certificate (i); Tables 2 to 4, Lemma 4.4 for $L \leq 60$ and Step 1 for $24 \leq n \leq 40$; Lemmas 5.5 and 5.6, and by brute force for $n = 7, 9$ Lemma 5.3, Corollary 5.4 and the proof of Theorem C; Corollary 5.2 for $n \leq 9$; and Table 1. With the option `--certs-extra` it checks further certificate files; it accepted all 1,653,478 pairs of the certificates (ii) in about 20 seconds.

Re-runs. When the note was written, the package was run again from a fresh copy (31 runs, 14 minutes; `RERUN_LOG.txt`), including the computation (a) of Section 6 for $n = 25$ and the generator of the certificates (ii) for $n = 19, 20$. All outputs agree with the recorded ones, up to running times and to lines that depend on how the recorded runs were split; the regenerated certificates are identical to those of the verification run. The run of the second round ran the package again from the archive (27 programs, 41 runs), among them the exact computation of `original/` for every $n \leq 24$ and the generator of the certificates (ii) for every $19 \leq n \leq 24$; all outputs agree with the recorded ones up to running times and printed paths, and the six regenerated certificate files have the recorded checksums (`independent_run_2/rerun_package_v1.txt`).

Verification. The results were first obtained, with proofs and with the programs in `original/`. Two rounds of independent verification, all AI-assisted, followed; each run re-derived the proofs and wrote its own programs. In the first round two runs examined the first written version of the results. The first run re-derived all lemmas, the case analysis of Step 1 and the proof of Theorem C line by line; read [HMSB12, HMP13, Don25], the arXiv version of [Don27] and [PP06, Sections 1–3], and checked the hypotheses of Theorem 5.1; recomputed all products of Tables 2 to 4 and checked Step 1 on the 3,567,251 even types of $24 \leq n \leq 50$ and $n = 60, 61, 64, 65$; checked the certificates (i); and computed the supports for $n \leq 24$ with its own character tables, confirming Table 1, $T(C) = E(\mu, n)$ for $12 \leq n \leq 24$, and Lemma 5.3, Remark 5.7 and the proof of Theorem C for odd $n \leq 23$. The second run computed the supports without square roots (Section 3.2), identical to those of the note for every $n \leq 24$ and every class; compared all structure constants with a brute force for $n \leq 10$; ran all programs of `original/` again, with identical outputs; made the computations (a)–(c) of Section 6; produced and checked the certificates (ii); tested its programs with deliberately wrong input; and searched the literature and checked the bibliography. Neither run found a mathematical error. Their required corrections were made: the range $n \geq 5$ and the case A_4 are stated (Problem 1.2, Proposition 1.3); the classes of n -cycles are described as those with the fewest cycles (Remark 5.8); the odd conjugator of one small gadget is noted in Step 1, and the bound $L \geq 6$ of two even long gadgets (an earlier text had $L \geq 7$, which is also true, since they are used only for odd L) is that of Table 2; the journal version of [Don27] is cited. When the note was written, `check_note.py` was added and the package was run again, as described above.

In the second round a further run examined the text of the note line by line, as if nothing had been checked. It read the sources again (the record; [Don25], page 1433 in the published report; Problem 19.12 in versions 48 and 49 of [KM26]; [HMSB12] completely, in the authors’ version; the statements of [Don27] in the arXiv version; and [PP06, Sections 1–3]), re-derived all lemmas and proofs, and checked that the hypotheses of Theorem 5.1 as stated in [PP06] are met. Its programs, written for this run, compute all structure constants of A_n by brute force for $n \leq 10$ and the exact supports for $n \leq 27$, with a perfect matching for every class, the numbers of Table 1 for all 23 values of n , $T(C) = E(\mu, n)$ and the permutation of Remark 4.5 for $12 \leq n \leq 27$, and Lemma 5.3, Remark 5.7 and the proof of Theorem C for odd $n \leq 27$; they recompute Tables 2 to 4 as read from the $\text{\TeX}$ source, check Step 1 of the proof of Theorem B, implemented as written, on all even types of $24 \leq n \leq 52$, Lemma 5.5 for odd $n \leq 79$ and Corollary 5.2 with its converse for $3 \leq n \leq 32$, and check all certificates (i) and (ii), the latter after regenerating the six files from the archive. It also ran the package again (see above) and repeated the literature search. It found no mathematical error. The corrections it required are made here: the order of the authors of [HMSB12], and a citation of it by section instead of a page of the authors’ version; the value at the inverse class in place of the complex conjugate in the modular computation of Section 3.2; the compatibility conditions (1)–(5) of [PP06]; the description of the certificates (ii) in the abstract and in Section 3 (their regeneration, the fallback guide of their generator, the checking programs); the evidence of Section 6, now from two computations up to $n = 27$; and two points of wording.

Scope and priority. The note gives a partial answer to Problem 1.2, the precise question of the record: Theorem A for $5 \leq n \leq 24$, and for all n the classes of Theorems B and C. For $n \geq 25$ and all other classes the question stays open, and so does the case of alternating groups of Problem 19.12 of [KM26]; the other questions of the paragraph of [Don25] are not treated. The question is Dona’s [Don25]; the notion, the conjecture, Lemma 2.2 (with Hall’s theorem [Hal35]), the criterion of Remark 2.3 and Proposition 1.3(a) are from [HMSB12]; Lemma 2.1, Frobenius’ rule and the Murnaghan–Nakayama rule are standard [JK84]. Theorem 5.1 is due to Thom [Tho65] and to Edmonds, Kulkarni and Stong [EKS84]; we quote it, with its attribution, from [PP06] and did not read

the two original papers. Since Theorem C for $n \leq 24$ is part of Theorem A, only its cases $n \geq 25$ depend on Theorem 5.1. We could not read Boccara’s paper [Boc82] on cycles as products of two permutations of given classes, which may contain Corollary 5.2 in combinatorial form: the publisher refused automated access, and the zbMATH record has no review. The certificates (ii) are due to one of the verification runs of the first round.

What was read: [HMSB12] completely, in the authors’ version dated 1 February 2012 (the journal version was not compared); [HMP13] in the journal’s open file; [Don25] in the published report; [Don27] in the arXiv version v1 (the journal version was not compared); Problem 19.12 in [KM26], version 49; and [PP06] in the arXiv version v3, Sections 1–3. The run of the second round read again [HMSB12] completely, [Don25], the statements of [Don27] in the arXiv version, Problem 19.12 in versions 48 and 49 of [KM26], and [PP06, Sections 1–3]. The web page with the computations of [HMSB12] could not be reached (again on 23 October 2026).

The searches were made on 22 and 23 October 2026: the works citing [HMSB12] in OpenAlex and Semantic Scholar (one, [HMP13]; Crossref counts none); zbMATH (titles with “expansive” or “expansiveness” in group theory, and the review of [HMSB12]); arXiv (phrase searches for conjugacy expansiveness, for normal subsets and products of conjugacy classes of alternating groups, and for the Kurovka problem); an OpenAlex phrase search; and five web searches. The run of the second round repeated them in part on 23 October 2026 (OpenAlex, Crossref, zbMATH, the arXiv API and a sixth web search; Semantic Scholar and some arXiv queries were refused). The related work we found concerns products of large classes or normal subsets of A_n , such as [GM21, Don27], squares of classes and growth of products; none of it counts the classes in SC . No printed answer to Problem 1.2 was found in the sources searched. A search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [Boc82] G. Boccara. Cycles comme produit de deux permutations de classes données. *Discrete Mathematics*, 38(2-3):129–142, 1982. doi:10.1016/0012-365X(82)90282-5.
- [Don25] Daniele Dona. Alternating groups as products of three conjugacy classes. *Oberwolfach Reports*, 22(2):1431–1434, 2025. Report 27/2025, Computational Group Theory (organisers H. Dietrich, B. Eick, G. Nebe, C. Roney-Dougal). doi:10.4171/OWR/2025/27.
- [Don27] Daniele Dona. Products of three conjugacy classes in the alternating group. *Journal of Combinatorial Theory, Series A*, 225:106253, 2027. Paper no. 106253; v1 is the only arXiv version. arXiv:2505.06012, doi:10.1016/j.jcta.2026.106253.
- [EKS84] Allan L. Edmonds, Ravi S. Kulkarni, and Robert E. Stong. Realizability of branched coverings of surfaces. *Transactions of the American Mathematical Society*, 282(2):773–790, 1984. doi:10.1090/S0002-9947-1984-0732119-5.
- [GM21] Martino Garonzi and Attila Maróti. Alternating groups as products of four conjugacy classes. *Archiv der Mathematik*, 116(2):121–130, 2021. doi:10.1007/s00013-020-01531-7.

- [Hal35] P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, s1-10(1):26–30, 1935. doi:10.1112/jlms/s1-10.37.26.
- [HMP13] Zoltán Halasi, Attila Maróti, and F. Petényi. Character expansiveness in finite groups. *International Journal of Group Theory*, 2(2):9–17, 2013. doi:10.22108/ijgt.2013.1660.
- [HMSB12] Zoltán Halasi, Attila Maróti, Said Sidki, and Marcelo Bezerra. Conjugacy expansiveness in finite groups. *Journal of Group Theory*, 15(4):485–496, 2012. doi:10.1515/jgt-2012-0004.
- [JK84] Gordon James and Adalbert Kerber. *The Representation Theory of the Symmetric Group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1984. First published by Addison-Wesley, Reading, MA, 1981. doi:10.1017/CB09781107340732.
- [KM26] E. I. Khukhro and V. D. Mazurov, editors. *Unsolved Problems in Group Theory. The Kourovka Notebook, No. 21*. Sobolev Institute of Mathematics, Novosibirsk, 2026. Version 49 of 20 October 2026. arXiv:1401.0300.
- [PP06] Ekaterina Pervova and Carlo Petronio. On the existence of branched coverings between surfaces with prescribed branch data, I. *Algebraic & Geometric Topology*, 6(4):1957–1985, 2006. arXiv:math/0508434, doi:10.2140/agt.2006.6.1957.
- [Tho65] René Thom. L’équivalence d’une fonction différentiable et d’un polynôme. *Topology*, 3:297–307, 1965. doi:10.1016/0040-9383(65)90079-0.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.8.0; record OWR-14299292-010.