

A Sharp Sampling Bound for Primitive Real Toric Arrangements

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19 October 2026

Abstract

Let $A \subset \mathbb{Z}^n \setminus \{0\}$ be a finite spanning collection of primitive normals, and let E_A be the least common multiple of the largest Smith invariant factors of its nonsingular n -row submatrices. If $E_A > 1$, every grid $(m^{-1}\mathbb{Z}^n)/\mathbb{Z}^n$ with $E_A \mid m$ and $m \geq nE_A$ meets every ordinary stratum of the associated real toric arrangement. The multiplier n is sharp in every dimension $n \geq 2$, for every prescribed value $E_A \geq 2$. For arbitrary integral normals, we describe the successful denominators exactly using the maximum codegree of the scaled stratum closures. The proof combines the interior-lattice-point framework of Hanlon and Painter with the standard Smith lcm period. Primitivity excludes the unique extremal lattice polytope, a unimodular simplex, by divisibility of its facet constants. We also give a sufficient bound for realizing all Bondal–Thomsen summands on a smooth toric variety with no torus factor.

1 The sampling problem

Let $A = (u_1, \dots, u_k)$ be a finite list in $\mathbb{Z}^n \setminus \{0\}$ spanning $\mathbb{R}^n$, where $n \geq 1$. Write $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and consider the periodic hyperplanes

$$H_{i,a} = \{x \in \mathbb{R}^n : \langle u_i, x \rangle = a\}, \quad a \in \mathbb{Z}.$$

Their images define a real toric arrangement. An *ordinary stratum* is a connected component of the set on which a specified subset of these toric hyperplanes, and no others, vanishes. Its lifts are relatively open cells of the periodic arrangement. We ask when the grid

$$L_m = (m^{-1}\mathbb{Z}^n)/\mathbb{Z}^n, \quad m \in \mathbb{Z}_{>0},$$

meets every such stratum. Membership in L_m means that the denominator divides m , not that the reduced denominator equals m .

For every sublist of n independent normals, form the square matrix B having those normals as rows. Define

$$e(B) = \min\{e \geq 1 : eB^{-1} \text{ has integral entries}\}, \quad E_A = \text{lcm}_B e(B), \quad D_A = \text{lcm}_B |\det B|. \quad (1)$$

The integer $e(B)$ is the largest Smith invariant factor of B ; in particular $E_A \mid D_A$. The period E_A is the standard lcm period of the integral arrangement, as we verify below.

Theorem 1. *Suppose all normals u_i are primitive and $E_A > 1$. If $E_A \mid m$ and $m \geq nE_A$, then L_m meets every ordinary stratum. The divisibility $E_A \mid m$ is necessary for sampling all ordinary strata. For every $n, e \geq 2$, there is such an arrangement with $E_A = e$ whose successful denominators are exactly*

$$\{ne, (n+1)e, (n+2)e, \dots\}.$$

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Hanlon and Painter prove that multiples of D_A at least $(n + 1)D_A$ suffice [2, Theorem A]. Their proof uses integral points in dilations of closed strata, and their final section asks for sharper bounds. We retain that framework. Replacing determinants by the standard Smith period accounts for the vertex denominators; the additional primitive-normal argument removes the extremal multiplier $n + 1$ when $E_A > 1$. Neither the Smith period nor the interior-lattice-point method is introduced here.

The distinction between ordinary strata and oriented, half-open Bondal strata is important. The latter, studied in [1], may combine ordinary strata. Our necessary divisibility and exact threshold concern the ordinary stratification. They give sufficient, but not asserted necessary, conditions for realizing every Bondal–Thomsen label.

2 Vertex denominators and the Smith period

A lifted stratum is obtained by imposing, for each normal, either an equality $\langle u_i, x \rangle = a_i$ or a strict strip $a_i < \langle u_i, x \rangle < a_i + 1$, with integral a_i . It is the relative interior of its closure Δ . Keeping any n independent normals shows that Δ is bounded. Thus every closure is a rational polytope. There are finitely many strata modulo $\mathbb{Z}^n$, since only finitely many of the periodic hyperplanes meet a compact fundamental cube.

Lemma 2. *The integer E_A is the least positive integer clearing the coordinates of all zero-dimensional strata. Every lifted closed stratum Δ satisfies that $E_A\Delta$ is a lattice polytope. If L_m meets all strata, then $E_A \mid m$.*

Proof. Every vertex is the intersection of n independent active hyperplanes, so it has the form $B^{-1}b$, with $b \in \mathbb{Z}^n$. Its coordinates become integral after multiplication by E_A . Conversely, for each nonsingular B and every $b \in \mathbb{Z}^n$, the point $B^{-1}b$ lies on n independent arrangement hyperplanes and is a zero-dimensional stratum. Additional hyperplanes through it do not change this fact.

If all such torus points lie in L_m , then $mB^{-1}b \in \mathbb{Z}^n$ for all $b \in \mathbb{Z}^n$. Taking the standard basis vectors as b gives $e(B) \mid m$, hence $E_A \mid m$. The assertion about every Δ follows by applying the vertex calculation to all its vertices. $\square$

For completeness, the definition in (1) agrees with the lcm period formed from all sublists in Kamiya–Takemura–Terao [4, §2]. For a sublist J , write $\Lambda_J = \langle u_i : i \in J \rangle_{\mathbb{Z}}$ and $\text{sat } \Lambda_J = \text{span}_{\mathbb{R}}(\Lambda_J) \cap \mathbb{Z}^n$. The torsion of $\mathbb{Z}^n/\Lambda_J$ is $\text{sat } \Lambda_J/\Lambda_J$; give the trivial group exponent one.

Choose an independent sublist $I \subset J$ with the same real span, and extend I to a full-rank basis sublist B of A . Because the rows of B are independent,

$$\text{span}_{\mathbb{R}}(\Lambda_I) \cap \Lambda_B = \Lambda_I.$$

Consequently $\text{sat } \Lambda_I/\Lambda_I$ injects into $\mathbb{Z}^n/\Lambda_B$, whose exponent is $e(B)$. The group $\text{sat } \Lambda_J/\Lambda_J$ is a quotient of $\text{sat } \Lambda_I/\Lambda_I$. Its exponent therefore divides E_A . Conversely, every basis sublist is among the sublists J , proving equality of the two lcm definitions.

3 Codegrees and the exact threshold

For a nonempty lattice polytope $Q \subset \mathbb{R}^n$, define its codegree by

$$c(Q) = \min\{\ell \geq 1 : \text{relint}(\ell Q) \cap \mathbb{Z}^n \neq \emptyset\}.$$

This definition uses the ambient integral lattice and also applies when the affine span of Q does not contain the origin. It is invariant under integral translation.

Lemma 3. *If Q has dimension d , then $1 \leq c(Q) \leq d + 1$. Moreover, $\text{relint}(\ell Q) \cap \mathbb{Z}^n \neq \emptyset$ holds exactly for the integers $\ell \geq c(Q)$. If Q has full dimension n , then $c(Q) = n + 1$ if and only if Q is a unimodular simplex.*

Proof. If $z \in \text{relint}(\ell Q) \cap \mathbb{Z}^n$ and v is a lattice vertex of Q , then $z + v \in \text{relint}((\ell + 1)Q) \cap \mathbb{Z}^n$. Indeed, $(z + v)/(\ell + 1)$ is a convex combination, with positive coefficient, of the relative interior point z/ℓ and the point $v \in Q$. This proves the upper-interval assertion.

Choose $d + 1$ affinely independent vertices of Q . Their average lies in the relative interior of their d -simplex and hence in $\text{relint } Q$. Their sum is an integral point in $\text{relint}((d + 1)Q)$. This includes $d = 0$.

For the remaining assertion, an integral unimodular affine transformation reduces a unimodular simplex to $\text{conv}(0, e_1, \dots, e_n)$. An interior integral point of its ℓ -th dilation must have positive integral coordinates whose sum is less than ℓ ; thus the first possible value is $n + 1$.

Suppose next that a full-dimensional Q is not a simplex. Triangulate Q using its vertices. There are at least two maximal simplices, and a pair of adjacent maximal simplices shares an interior facet. Such a facet exists because the dual adjacency graph of a triangulation of a convex polytope is connected: a generic segment between interior points crosses only facets. The sum of the n lattice vertices of that interior facet belongs to $\text{int}(nQ)$.

Finally let $Q = \text{conv}(v_0, \dots, v_n)$ be a nonunimodular simplex. The half-open fundamental parallelepiped of the integral vectors $(v_i, 1)$ contains a nonzero integral point

$$(z, j) = \sum_{i=0}^n \lambda_i (v_i, 1), \quad 0 \leq \lambda_i < 1.$$

This follows because the index of the lattice they generate is the normalized volume of Q , which exceeds one. If r coefficients are nonzero, then $1 \leq j = \sum_i \lambda_i \leq r - 1$. Add $(v_i, 1)$ for each zero coefficient. All coefficients are now positive and their integer sum is

$$h = j + n + 1 - r \leq n.$$

The new spatial coordinate is an integral point in $\text{int}(hQ)$. Adding a vertex $n - h$ times yields one in $\text{int}(nQ)$. Thus in either nonunimodular case $c(Q) \leq n$, completing the proof. $\square$

Choose a lift with closure Δ_S for each ordinary toric stratum S , and set

$$C_A = \max_S c(E_A \Delta_S). \quad (2)$$

A different lift is an integral translate, so C_A does not depend on these choices.

Theorem 4. *For arbitrary nonzero integral normals spanning $\mathbb{R}^n$,*

$$L_m \text{ meets every ordinary stratum} \iff m = E_A \ell \text{ for an integer } \ell \geq C_A.$$

In particular, $1 \leq C_A \leq n + 1$.

Proof. Lemma 2 forces $E_A \mid m$. For $m = E_A \ell$, meeting the stratum S is equivalent to

$$\text{relint}(m \Delta_S) \cap \mathbb{Z}^n = \text{relint}(\ell E_A \Delta_S) \cap \mathbb{Z}^n \neq \emptyset.$$

Lemma 3 makes this equivalent to $\ell \geq c(E_A \Delta_S)$. Taking the maximum proves the claim and its dimension bound. $\square$

Formula (2) is a structural characterization, not an efficient formula for computing C_A from the normals alone. It makes explicit the codegree consequence of the approach suggested in [2, §5].

4 Primitive facets and sharpness

Lemma 5. *Let Q be a full-dimensional unimodular simplex, expressed irredundantly using primitive inward facet normals as*

$$Q = \{x : \langle a_j, x \rangle \geq b_j, \ 0 \leq j \leq n\}.$$

Then $\sum_j a_j = 0$ and $\sum_j b_j = -1$.

Proof. For the standard simplex the inward normals are $e_1, \dots, e_n, -(1, \dots, 1)$, with constants $0, \dots, 0, -1$. Every unimodular simplex is $t + U \operatorname{conv}(0, e_1, \dots, e_n)$, where $t \in \mathbb{Z}^n$ and $U \in \operatorname{GL}_n(\mathbb{Z})$. Its normals are $U^{-T}a_j$, which remain primitive, and its constants are $b_j + \langle U^{-T}a_j, t \rangle$. Both asserted sums follow. $\square$

Proof of Theorem 1. A stratum of dimension $d < n$ satisfies $c(E_A \Delta_S) \leq d + 1 \leq n$. Consider a full-dimensional chamber and put $Q = E_A \Delta_S$. Each facet is supported by an original hyperplane $\langle u_i, x \rangle = a$, with integral a . Since u_i is primitive, its primitive inward normal for Q is u_i or $-u_i$, and its primitive support constant is $E_A a$ or $-E_A a$. All these constants are divisible by E_A . If Q were a unimodular simplex, Lemma 5 would make their sum -1 , contrary to $E_A > 1$. Hence Lemma 3 gives $c(Q) \leq n$. We obtain $C_A \leq n$, and Theorem 4 proves the bound and necessary divisibility.

For sharpness, fix $n, e \geq 2$ and take

$$A_{n,e} = \{e_1, \dots, e_n, (1, e, \dots, e)\}.$$

These normals are primitive. Their nonsingular full-rank submatrices have determinants of absolute value 1 or e . For a basis obtained by replacing e_j , $j \geq 2$, by $(1, e, \dots, e)$, the row quotient is cyclic of order e . It follows that $E_{A_{n,e}} = D_{A_{n,e}} = e$.

One chamber is

$$x_i > 0 \ (1 \leq i \leq n), \quad x_1 + e(x_2 + \dots + x_n) < 1.$$

These inequalities also imply $x_i < 1$, so they do specify a chamber of the entire arrangement. For a grid point with denominator dividing $e\ell$, write $x_i = a_i/(e\ell)$ with positive integers a_i . Its weighted coordinate sum is at least $(1 + e(n-1))/(e\ell)$. Thus this chamber can be hit only if

$$1 + e(n-1) < e\ell,$$

which, for integral ℓ , is equivalent to $\ell \geq n$. The upper bound already proved samples all strata when $\ell \geq n$. Hence $C_{A_{n,e}} = n$. $\square$

The sharp family can be made positively spanning by adjoining the negative of every normal; doing so changes neither the arrangement nor its period or threshold.

Both restrictions in the primitive bound are essential. With normals $e_1, \dots, e_n, (1, \dots, 1)$, one has $E_A = 1$, and the standard unimodular simplex is a chamber closure. Therefore $C_A = n + 1$. Multiplying all of these normals by $e \geq 2$ instead gives $E_A = e$, but a chamber scaled by E_A is again the standard simplex. This nonprimitive arrangement still has $C_A = n + 1$. For $n = 1$, all primitive normals are 1 or -1 , so $E_A > 1$ is impossible; the primitive theorem is nonvacuous starting in dimension two.

As a comparison of sufficient bounds, consider

$$A = \{e_1, e_2, e_3, (1, 1, 1), (1, -1, 1), (1, 1, -1)\}.$$

Its full-rank determinants have absolute values 1, 2, 4. For the basis consisting of the last three vectors, the gcd of the 2-minors is 2, giving Smith factors 1, 2, 2. All other full-rank bases have exponent at most 2. Thus $D_A = 4$ and $E_A = 2$. Theorem 1 supplies $m = 6$, whereas the determinant sufficient bound of [2] supplies $m = 16$. This does not say that 16 is a necessary denominator in their theorem. The accompanying exact enumeration finds 68 ordinary strata and $C_A = 3$ in this example.

5 Toric summands and the scope of the result

Let X be a smooth toric variety of dimension n , with no torus factor, and let A be the primitive generators of its rays in the cocharacter lattice. The no-torus-factor hypothesis ensures that the rays span the ambient real vector space. In the standard toric Frobenius decomposition, described in [2, §4], the summand attached to a grid point is determined by the integral floor, or equivalently ceiling, labels of its pairings with these ray generators. Let the Bondal–Thomsen collection denote the set of isomorphism classes occurring for arbitrary real labels, equivalently for sufficiently divisible toric Frobenius maps.

Corollary 6. *If $E_A > 1$, then for every multiple m of E_A with $m \geq nE_A$, the degree- m toric Frobenius pushforward of $\mathcal{O}_X$ contains every member of the Bondal–Thomsen collection as a direct summand. If $E_A = 1$, the sufficient bound $m \geq n + 1$ applies.*

Proof. The floor and ceiling labels are constant on each lifted ordinary stratum. Integral translations change them by the principal divisor relation, so the corresponding line-bundle class is well defined on the torus. Meeting every ordinary stratum therefore realizes every possible label class. Apply the standard splitting formula and Theorems 1 and 4. $\square$

This is a sufficient bound for summands only. Different ordinary strata can yield the same label or the same line bundle; accordingly, neither the necessity $E_A \mid m$ nor the exact codegree threshold is asserted for this coarser sampling problem. We make no extension of the sheaf statement to singular toric varieties.

The broad question about the structure of Bondal–Thomsen polytopes in Hanlon’s Oberwolfach report [3, pp. 900–903] motivates this application. The present sampling theorem does not classify those polytopes or count their lattice points in general. It also does not resolve the subtorus question in [1, Question 1.6].

6 Exact checks and attribution

The source package contains a self-contained Python checker using integers and rational arithmetic. It tests 89 arrangements and 559 nonsingular bases, examining 2,555,030 grid points and 3,184 strata counted across cases. Reference grids use the sufficient theorem of [2]; 70 planar cases are also reconstructed independently by exact polygon clipping, giving 521 two-dimensional cells. The checks include 18 sharp-family cases, 564 nonmultiple obstructions, and 369 threshold comparisons, as well as the two essential-hypothesis examples. Normal and optimized Python runs agree. These finite regressions supplement, rather than prove, the general theorems.

The Smith invariant is established in [4], and the polytope method comes from [2]. The codegree facts used above are elementary lattice-polytope geometry, for which full proofs were included. The proposed refinement is the primitive facet argument and its sharp family. The cited primary sources and a bounded bibliographic search were compared; this does not establish absolute priority.

AI assistance was used in proof development, source comparison, exact-check implementation, and manuscript preparation. This preprint is self-audited and unrefereed; neither independent peer review nor formal proof-assistant certification is claimed.

References

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