

A Quotient Obstruction to Strict Character Exponents in Almost Quasisimple Groups

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Abstract

A workshop problem asks for an exponent $0 \leq \alpha(g) < 1$ such that $|\chi(g)| \leq \chi(1)^{\alpha(g)}$ for every irreducible complex character of a finite almost quasisimple group and every noncentral element g . The unrestricted formulation has an elementary quotient obstruction. The almost simple group $\mathrm{P}\Gamma\mathrm{L}_3(4)$ has quotient S_3 by $\mathrm{PSL}_3(4)$. Inflating the irreducible degree-two character of this quotient gives value 2 at a noncentral transvection, so no exponent below 1 exists. More generally, if H is the inverse image of the simple socle in $G/Z(G)$ and $E = H'$, the required exponents exist at all noncentral elements exactly when G/E is abelian. In that case we give the explicit, deliberately weak exponent $1 - 16/(\mathrm{ord}(g)^2|G|\log|G|)$. This is a correction to the all-character formulation, not an improvement on optimal character bounds for quasisimple groups.

1 The all-character formulation

Write $Z(G)$ for the center of a finite group G . Following the formulation in [1, p. 741, Problem 1], call G *almost quasisimple* when

$$S \trianglelefteq G/Z(G) \leq \mathrm{Aut}(S)$$

for a finite nonabelian simple group S , identified with its inner automorphism group. The source asks, for each $g \notin Z(G)$, for an explicit exponent $0 \leq \alpha(g) < 1$, as small as possible, satisfying

$$|\chi(g)| \leq \chi(1)^{\alpha(g)} \quad \text{for every } \chi \in \mathrm{Irr}(G). \quad (1)$$

There is no faithfulness restriction on χ in that statement. The distinction is essential: a nonlinear irreducible character inflated from a quotient equals its degree on the kernel.

Theorem 1. *Let $G = \mathrm{P}\Gamma\mathrm{L}_3(4)$ and $g = [I + E_{12}]$, where brackets denote projective scalar classes. Then G is almost simple, $Z(G) = 1$, and $g \neq 1$ has order two. There is a character $\chi \in \mathrm{Irr}(G)$ with $\chi(g) = \chi(1) = 2$. Consequently (1) fails for this G, g for every $\alpha(g) < 1$.*

Section 2 constructs the quotient and character; Appendix A verifies the almost simple hypothesis without a character table or a classification theorem. Section 3 identifies the exact obstruction to existence at every noncentral element. Its quantitative corollary is only an elementary existence bound. In particular, this note does not contradict the quasisimple theorem quoted immediately after Problem 1 in [1], or the uniform bounds of Larsen and Tiep [2]. It does not determine optimal exponents after imposing the missing restriction. The construction uses classical inflation and projective-kernel arguments; no priority claim is made.

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2 An explicit semilinear counterexample

Let $\mathbb{F}_4 = \mathbb{F}_2[w]/(w^2 + w + 1)$ and let $\sigma(a) = a^2$. Define G to consist of pairs $([A], e)$, where $A \in \mathrm{GL}_3(\mathbb{F}_4)$ and $e \in \{0, 1\}$, with multiplication

$$([A], e)([B], f) = ([A\sigma^e(B)], e + f).$$

The second coordinate is taken modulo two. This is $\mathrm{P}\Gamma\mathrm{L}_3(4)$. Every $\lambda \in \mathbb{F}_4^\times$ has $\lambda^3 = 1$, so $\det(\lambda A) = \det A$. Hence

$$\pi : G \longrightarrow \mathbb{F}_4^\times \rtimes \langle \sigma \rangle, \quad ([A], e) \longmapsto (\det A, e)$$

is well defined. The determinant identity for $A\sigma^e(B)$ shows that it is a homomorphism. Diagonal matrices $\mathrm{diag}(a, 1, 1)$ and the field automorphism show surjectivity. Its kernel is $S = \mathrm{PSL}_3(4)$. Since $\sigma(w) = w^{-1}$, the quotient is

$$G/S \cong C_3 \rtimes C_2 \cong S_3,$$

where the involution acts by inversion. Counting independent columns and then dividing by the scalar group and determinant group gives

$$\begin{aligned} |\mathrm{GL}_3(4)| &= (64 - 1)(64 - 4)(64 - 16) = 181440, \\ |\mathrm{PGL}_3(4)| &= 60480, \quad |S| = 20160, \quad |G| = 120960. \end{aligned}$$

For a concrete representation of the quotient, set

$$C = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These integral matrices satisfy $C^3 = R^2 = I$ and $RCR = C^{-1}$. Thus $(w^j, e) \mapsto C^j R^e$ defines a representation of $\mathbb{F}_4^\times \rtimes \langle \sigma \rangle$ over $\mathbb{C}$. The two eigenvalues of C are the distinct primitive cube roots of unity. The matrix R exchanges the corresponding eigenlines, so the representation has no invariant line and is irreducible. Inflation along the surjection π preserves irreducibility. Its character χ has degree two and is identically two on S .

Take $g = ([I + E_{12}], 0)$. It has determinant one, is not the identity projective class, and has order two in characteristic two. It is noncentral even without identifying $Z(G)$: for $h = ([I + E_{23}], 0)$,

$$(I + E_{12})(I + E_{23}) = I + E_{12} + E_{23} + E_{13}, \quad (I + E_{23})(I + E_{12}) = I + E_{12} + E_{23}.$$

These matrices are not scalar multiples, since their diagonal entries are all one. Thus $gh \neq hg$. As $g \in \ker \pi$,

$$|\chi(g)| = 2 > 2^\alpha = \chi(1)^\alpha \quad (\alpha < 1).$$

Together with the almost simple verification in Appendix A, this proves Theorem 1.

3 The exact obstruction to existence

For an almost quasisimple group G , put $Z = Z(G)$. Let H be the inverse image in G of the nonabelian simple socle S of G/Z , and put $E = H'$.

Lemma 2. *The group E is perfect, $H = EZ$, and $E/(E \cap Z) \cong S$. If an irreducible complex representation ρ of G is nontrivial on E , then*

$$\{x \in G : \rho(x) \text{ is scalar}\} = Z.$$

Proof. Since $H/Z \cong S$ is perfect, $H = H'Z = EZ$. Centrality of Z gives $E = H' = (EZ)' = E'$, and the first claims follow. In particular E is nonabelian.

Every nontrivial normal subgroup M of G/Z contains S . Indeed $M \cap S$ is either 1 or S . In the former case $[M, S] \leq M \cap S = 1$, so M centralizes S . But $C_{\text{Aut}(S)}(S) = 1$: if an automorphism φ commutes with every inner automorphism, then the inner automorphisms induced by $\varphi(s)$ and s agree for every s , whence $\varphi(s) = s$ because $Z(S) = 1$. Thus the former case forces $M = 1$.

Let $P = \rho^{-1}(\mathbb{C}^\times I)$. It is normal and contains Z by Schur's lemma. If $P/Z \neq 1$, the preceding paragraph gives $H \leq P$. Then $\rho(E)$ consists of scalars and is abelian. As E is perfect, this image is trivial, contrary to the hypothesis. Therefore $P = Z$. $\square$

Theorem 3. *For a finite almost quasisimple group G , the following are equivalent:*

1. *For every $g \notin Z(G)$, there is an exponent $0 \leq \alpha(g) < 1$ satisfying (1).*
2. *The quotient G/E is abelian.*

If G is almost simple, the second condition is equivalent to G/S being abelian.

Proof. If G/E is nonabelian, it has a nonlinear irreducible complex character ψ . To recall why, a finite group whose irreducible characters all have degree one has as many conjugacy classes as elements, by the sum-of-squares formula, and is abelian. Inflate ψ to G . Choose $g \in E \setminus Z$, which is possible because $E/(E \cap Z) \cong S$. Then $|\psi(g)| = \psi(1) > 1$, precluding every exponent below one.

Conversely, suppose G/E is abelian. Every nonlinear irreducible representation ρ is nontrivial on E , since otherwise it would factor through an abelian group. For $g \notin Z$, Lemma 2 says that $\rho(g)$ is not scalar. Averaging an inner product makes ρ unitary. Its eigenvalues at g have absolute value one and are not all equal, so the strict triangle inequality gives $|\chi(g)| < \chi(1)$. There are finitely many nonlinear irreducible characters. For each with $|\chi(g)| > 1$, the ratio $\log |\chi(g)| / \log \chi(1)$ is strictly below one. The maximum of these ratios and zero is an allowed common exponent. Characters with absolute value at most one satisfy the inequality automatically. Linear characters have absolute value exactly one. This proves existence. For an almost simple group $Z = 1$ and $E = S$, giving the final assertion. $\square$

Remark 4. The condition is G/E abelian, not merely G/H abelian. For a group with nontrivial center, replacing E by H can discard a central-extension obstruction. Alternatively, without assuming G/E abelian, restricting to irreducible representations nontrivial on E removes the scalar-kernel obstruction by Lemma 2.

Corollary 5. *Assume G/E is abelian. If $N = |G|$, $g \notin Z(G)$, and $m = \text{ord}(g)$, then (1) holds with*

$$\alpha(g) = 1 - \frac{16}{m^2 N \log N} \in (0, 1).$$

Proof. We have $N \geq 60$ and $m \geq 2$, so the claimed exponent lies in $(0, 1)$. Let χ have degree $d > 1$. By Lemma 2, the eigenvalues $z_1, \dots, z_d$ of g are m -th roots of unity that are not all equal. An eigenvalue of multiplicity k , where $1 \leq k \leq d - 1$, gives at least $k(d - k) \geq d - 1$ unordered pairs of unequal eigenvalues. For any such pair,

$$1 - \Re(z_i \overline{z_j}) \geq 1 - \cos(2\pi/m) = 2 \sin^2(\pi/m) \geq \frac{8}{m^2}.$$

The last inequality follows from the concavity bound $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$. Consequently

$$|\chi(g)|^2 = d^2 - 2 \sum_{i < j} (1 - \Re(z_i \bar{z}_j)) \leq d^2 - \frac{16(d-1)}{m^2} \leq d^2 \left(1 - \frac{16}{m^2 N}\right),$$

where $d^2 \leq N$ follows from the sum of squared character degrees. Using $\sqrt{1-x} \leq e^{-x/2}$ gives

$$|\chi(g)| \leq d e^{-8/(m^2 N)} \leq d \exp\left(-\frac{16 \log d}{m^2 N \log N}\right) = d^{\alpha(g)}.$$

The second inequality uses $\log d \leq \frac{1}{2} \log N$. Linear characters again have absolute value one. $\square$

This bound is intentionally weak. Its purpose is to make the positive part of Theorem 3 explicit; it provides no comparison with the sharp or uniform estimates in [2, 1].

A The almost simple hypothesis

Here we verify the group-theoretic hypotheses used in Theorem 1. Put $x_{ij}(a) = I + aE_{ij}$.

Lemma 6. *The group $S = \text{PSL}_3(4)$ is nonabelian simple.*

Proof. Elementary matrices generate $\text{SL}_3(4)$ by Gaussian elimination. For the diagonal step, on any two coordinates put

$$w(a) = x_{12}(a)x_{21}(-a^{-1})x_{12}(a) = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}.$$

Then $w(a)w(-1) = \text{diag}(a, a^{-1})$, so the determinant-one diagonal matrices are elementary products as well. For distinct i, j, k , with commutator convention $[u, v] = uvu^{-1}v^{-1}$,

$$[x_{ik}(a), x_{kj}(1)] = x_{ij}(a).$$

It follows that the projective image S is perfect.

The action of S on the 21 one-dimensional subspaces of $\mathbb{F}_4^3$ is faithful: a linear map fixing every line is scalar, as follows by applying it to basis vectors and their pairwise sums. It is doubly transitive. To send an ordered pair of distinct lines to another, extend representative vectors of each pair to a basis, choose a linear map between the bases, and adjust the image of the third basis vector to make its determinant one. Thus the action is primitive.

Let U be the projective image of $\{I + aE_{12} + bE_{13} : a, b \in \mathbb{F}_4\}$. This is an abelian normal subgroup of the stabilizer S_1 of $\langle e_1 \rangle$. Indeed every matrix stabilizing that line has block upper-triangular form, and its conjugation preserves the matrices $I + e_1 \ell$ with $\ell(e_1) = 0$. Conjugating by coordinate permutations shows that the conjugates of U contain all elementary matrices; these permutations have determinant one in characteristic two. Therefore the conjugates of U generate S .

Let $N \neq 1$ be normal in S . Its orbits form a block system for the primitive action. If they were singletons, faithfulness would give $N = 1$; hence N is transitive and $S = NS_1$. In S/N , all conjugates of U have the same image because S_1 normalizes U . This one abelian image generates S/N , which is therefore abelian. Perfectness of S forces $N = S$. Since $|S| = 20160$ and S is perfect, it is nonabelian. $\square$

Lemma 7. *Conjugation embeds $G = \text{P}\Gamma\text{L}_3(4)$ in $\text{Aut}(S)$. In particular G is almost simple and $Z(G) = 1$.*

Proof. The group S is normal in G . Suppose $([A], e)$ centralizes S . For every $i \neq j$ and $a \in \mathbb{F}_4$, equality of projective classes gives

$$Ax_{ij}(a^{2^e})A^{-1} = \lambda x_{ij}(a)$$

for some $\lambda \in \mathbb{F}_4^\times$. The unscaled matrices on both sides have trace $3 = 1$ in characteristic two, so taking traces forces $\lambda = 1$. Putting $a = 1$ shows that A commutes with every E_{ij} , $i \neq j$, hence A is scalar. For example, these commutation identities first force all off-diagonal entries to vanish and then equate the diagonal entries. With $a = w$, the remaining identity forces $w^{2^e} = w$, so $e = 0$. Thus $C_G(S) = 1$, the conjugation map is injective, and $S \leq G \leq \text{Aut}(S)$. Also $Z(G) \leq C_G(S) = 1$. $\square$

B Reproducibility and attribution

The accompanying standard-library Python checker uses arithmetic in $\mathbb{F}_2[w]/(w^2 + w + 1)$ and integers, with no floating-point computations. It enumerates all 60480 projective invertible matrices and generates all 20160 determinant-one classes using elementary matrices. It verifies inverses, scalar invariance of determinants, the three determinant cosets, and double transitivity on 21 points. It also checks the six-element quotient representation, its character norm, and the displayed noncentral witness. Additional finite multiplicity tests check the unequal-pair inequality used in Corollary 5.

The 1,222,248 diagnostic checks pass both normally and with Python optimization enabled. Explicit exceptions, rather than removable assertions, enforce the checks. These finite computations supplement the proofs above; they are not a formal verification of the general theorem.

The numerical order and outer automorphism information for $\text{PSL}_3(4)$ can also be found in the ATLAS [3]; neither its character tables nor its generator files are required here. The note concerns the literal unrestricted formulation of [1, Problem 1]. It should not be interpreted as a solution to the problem of optimal bounds for characters nontrivial on the quasisimple component. The elementary ingredients are classical, and the novelty of this formulation correction is not certified. AI assistance was used in drafting, mathematical exploration, coding, and consistency checks. This preprint has not undergone independent peer review or proof-assistant verification.

References

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- [3] ATLAS of Finite Group Representations, Linear group $L_3(4)$, accessed 21 October 2026. <https://brauer.maths.qmul.ac.uk/Atlas/lin/L34/>.